Housing and Liquidity*

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Abstract
Coinciding with the start of the US housing boom, there were large increases in home-equity lending and loan-to-equity ratios. We develop a model where housing bears a liquidity premium because it collateralizes consumption loans. Since liquidity depends on beliefs, even with fundamentals constant, prices can display complicated trajectories, including cyclic, chaotic and stochastic paths generated as self-fulfilling prophecies. Some of these resemble bubbles. Based only on fundamentals (financial innovation), we can account for some but not that much of the empirical price boom. This suggests there is room for other factors, including self-fulfilling prophecies. Since the liquidity premium is nonmonotone in loan-to-equity ratios, continuing financial innovation can endogenously generate a price bust. Also, to study the impact of monetary policy, we introduce money and banking.

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1 Introduction

Housing plays multiple roles: (i) it provides direct utility as a consumer durable; (ii) it can yield capital gains or losses as an asset; and (iii) it can facilitate intertemporal transactions when credit markets are imperfect. As regards (iii), in the presence of limited commitment, it can be difficult for consumers to get unsecured loans, and this generates a role for home equity as collateral. The fact that housing is pledgeable – i.e., it can be used to secure consumer credit – implies that equilibrium house prices can bear a liquidity premium. Thus, one may be willing to pay more for a house than its fundamental value, as defined below, because home ownership conveys security in the event that one needs a loan. Since liquidity depends at least partly on beliefs, one might conjecture that in housing markets, even when fundamentals are deterministic and time invariant, equilibrium prices might display complicated patterns, some of which resemble bubbles.

Our goal is to make these ideas precise and study their implications for the aggregate US housing-market experience since the turn of the millennium. It is commonly heard that there was a bubble over this period, that eventually burst, leading to all kinds of problems. It has also been noted that over the period there were important financial developments, perhaps interpreted broadly to include regulatory changes. Holmstrom and Tirole (2011) say that “In the runup to the subprime crisis, securitization of mortgages played a major role ... by making nontradable mortgages tradable [and this] led to a dramatic growth in the US volume of mortgages, home equity loans, and mortgage-backed securities in 2000 to 2008.” Concerning home-equity loans, in particular, Reinhart and Rogoff (2009) claim financial innovation allowed people to “turn their previously illiquid housing assets into ATM machines,” and Ferguson (2008) says they began to “treat their homes as cash machines.”
Figure 1 provides evidence of relative stability prior to the late 1990’s, when some big changes occurred (data sources are detailed in Appendix A). First we show house prices, divided by the CPI to correct for the purely nominal effect of inflation, and divided by a rent index to additionally correct for changes in the demand for shelter relative to other goods, both normalized to 1 in 1991. It is clear from this what people mean by a bubble: dramatic price increases, with collapse a decade later. To indicate what happened to supply, we also show nominal residential investment over nominal GDP, and a real residential investment index over real GDP, normalized to 0.6 in 1991. These are flows, not stocks of existing houses, and differ due to measurement methods, but both move more or less with prices. We also show HEL (home-equity loans) over the CPI, and what we call the LTE (loan-to-equity) ratio, normalized to 0.3 in 1991. These both increase considerably, before falling off
somewhat at the end, with the big increase in HEL due to greater home equity and higher LTE ratios. Higher LTE ratios represent financial innovation: the same loan requires less collateral.

Figure 2: HEL and Refinancing: Loan Size and Cash Available

The HEL data in Figure 1 provide a conservative measure of the use of houses in facilitating consumption loans. A related device is the HELOC (home-equity line of credit), which makes cash available if one needs it. For our purposes HEL and HELOC are similar: they are not distinguished in the model, and they are combined when we look at the data. One can also free up cash by mortgage refinancing. Using data from Greenspan and Kennedy (2013), Figure 2 shows that refinancing is much bigger than home-equity borrowing (left panel), but the cash made available for consumption from the latter is the same or bigger than the former (right panel). Most cash acquired by refinancing is used to pay off previous mortgages, with only about 15% available for consumption. In principle, our theory accommodates refinancing for consumption purposes, even if most of the discussion uses the HEL label. The bigger point is that the sum (see the right panel of Figure 2) goes from 0.65% of
GDP in 1991 to over 4% in 2005. We conclude that there has been a major increase in the use of housing assets to generate liquidity.¹

Summarizing, in the late 1990’s house prices, quantities, HEL and LTE ratios all began to increase; then prices and quantities fall, as do HEL and LTE ratios, but the latter fall later and less. Once one takes liquidity into account, is it possible to generate equilibria consistent with this experience? Our idea is that housing has a certain *moneyness*, in that like currency it ameliorates intertemporal frictions. Yet houses are different, and ought to be modeled explicitly. On the supply side, houses unlike currency are produced by the private sector. On the demand side, houses unlike cash yield direct utility, which rules out equilibria where the price is 0 or goes to 0, and this makes it more challenging to construct interesting dynamics.² We also show how houses differ more generally from financial assets. To further connect with monetary economics, we present an extension where sometimes agents need cash, which they get from banks using home-equity loans, allowing us to analyze the effects of monetary policy on housing markets and the interaction between public and private supplies of liquidity.

The paper is partly an exercise in theory – we want to know how liquidity con-

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¹Here are some related facts: Consumers withdrew $9 trillion from home equity over the period, financing 3% of consumption and contributing about 1/5 of the growth in debt (Ferguson 2008; Disney and Gathergood 2011; Greenspan and Kennedy 2013). Also, Mian and Sufi (2011) report the following: from 2002 to 2008, homeowners extracted 1/4 of the increase in home equity; this added $1.25 trillion to debt; these loans were used mainly for consumption rather than paying off credit card debt or buying financial assets; and they were used more by the young and those with low credit scores. This is all broadly consistent with our theory. As another interpretation of the facts, suppose a house were only a consumer durable. Then the rent-price ratio should be roughly the sum of discount and depreciation rates. There are other considerations (e.g., taxes), but unless they changed over the period, there should not be a rise in the price-rent ratio like the one in Figure 1. Harding et al. (2007) estimate depreciation rates of around 2.5%, so for reasonable discount rates, the rent-price ratio should be about 5. In data from Campbell et al. (2009), before 1995 it is about 5, then drops to 3.7 by 2007. Again, this is consistent with our theory.

²Steady state is unique here, and while it is possible to generate dynamics in this case, it is harder. Most monetary models, e.g., have a steady state where *M* is valued and another one where it is not, and it is easy to construct equilibria that transit between the former and the latter. That does not work here, because house prices are bounded away from 0.
siderations affect the market when housing and home-equity lending are modeled explicitly – but we also investigate some issues quantitatively: (i) We show numerically how deterministic cyclic and chaotic dynamics can emerge as self-fulfilling prophecies, illustrating an inherent excess volatility, relative to fundamentals, in housing markets. However, those equilibria do not display the classic bubble pattern of an extended price runup followed by collapse. (ii) We then describe a deterministic equilibrium with exactly this pattern. However, price increases each period are bounded by the rate of time preference, which is arguably violated in the evidence. (iii) Therefore we demonstrate how stochastic (sunspot) equilibria can generate larger capital gains several periods in a row, before collapsing, at a random date, due to nothing more than beliefs. (iv) We also ask how much of the empirical price boom can be explained based only on changing fundamentals – i.e., increasing LTE ratios. With a bit of effort we can generate some, and maybe as much as $1/2$, of the boom. One can interpret this as saying even when one bends over backwards to make financial innovation do the job, there is evidently room for other factors.

To understand some of our results it is useful to recognize how financial improvements can generate not only a boom but also a bust in prices. When borrowing constraints are tight, if houses become more pledgeable, and hence better in terms of relaxing these constraints, there will be an increase in the demand for housing. This increases the price as long as supply is not perfectly elastic. But continued increases in pledgeability eventually cause demand and price to go back down, because higher LTE ratios lower the marginal value of liquidity as they further relax borrowing limits. In particular, housing must be priced fundamentally in two cases: when it cannot be used at all to relax borrowing constraints; and when it can be used to secure loans big enough to render borrowing constraints slack. In between, when home-equity ameliorates but does not overcome these constraints, housing
commands a liquidity premium. This nonmonotonicity is inescapable: as the LTE
together increases, house prices first go up and then go back down.

Our use of the term bubble seems consistent with standard usage – prices above
fundamental values – but we do not want to get bogged down in semantics and
do not mind if others call it “merely” a liquidity premium. In any case, since this
premium/bubble is affected by beliefs, there can be complicated dynamics even with
stationary fundamentals. There is too much related work to go into it all, so we
refer to Farhi and Tirole (2012) for a general discussion of dynamics and bubbles.
Our focus on credit frictions follows a large literature summarized in Gertler and
Kiyotaki (2010). A direct antecedent of the model is Kiyotaki and Moore (1997),
although they and their followers usually impose debt limits on firms, while we
impose them on households to study consumption rather than investment loans. As
for research on housing, generally, there again is too much to go into detail, although
we can say much of it asks how the market responds to preference, technology or
policy shocks, while we study responses to financial innovation, and show how the
market can have complicated dynamics even with no shocks.\(^3\)

Other features further distinguish our approach: (i) In contrast to some recent
papers on housing (e.g., Brunnermeier and Julliard 2008 or Burnside et al. 2011),
\(^3\)In addition to work mentioned elsewhere, housing papers that influenced the project include
Aoki et al. (2004), Hurst and Stafford (2004), Campbell and Hercowitz (2005), Iacoviello (2005),
Glaeser et al. (2008), Ngai and Tenreyro (2009), Novy-Marx (2009), Piazzesi and Schneider (2009),
(2012), Piazzesi et al. (2013) and Iacoviello and Pavan (2013). Especially relevant are Liu et
who also have real estate used as collateral, but by producers, not consumers. Also relevant
Arce and Lopez-Salido (2011), which is similar in spirit but different technically – e.g., they use an
OLG model, while we have infinitely-lived agents and hence a transversality condition; they have
some agents holding houses that give them no utility, while housing here is always valued for
the utility flow, and this puts a lower bound on prices; they have multiple steady states, while
we get complicated dynamics with a unique steady state, including recurrent booms and busts.
Rocheteau and Petrosky-Nadeau (2012) also have home-equity consumption loans, but do not
study endogenous dynamics. Also relevant is Kocherlakota (2011), which we discuss at the end of
Section 4.

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our agents are fully rational, with homogeneous beliefs and often perfect foresight. (ii) In addition to exogenous debt limits, as in Kiyotaki and Moore (1997), we also consider endogenous debt limits, as in Kehoe and Levine (1993). (iii) We examine various mechanisms for determining the terms of trade, including bargaining as well as price taking. (iv) In addition to a price boom, we can get an endogenous bust from the above-mentioned nonmonotonicity. (v) We study the effects of monetary policy on housing in a setting with microfoundations for money and banking. This is important because some research (e.g., Glaeser et al. 2010) suggests lower nominal rates must increase house prices, while when we model money and banking explicitly, we find nonmonotonic effects over an empirically relevant range.

We also want to emphasize up front how our results differ technically from two recent papers. In a setup that is similar in some ways, Rocheteau and Wright (2013) study price dynamics with a financial asset – i.e., a Lucas “tree” in fixed supply with a constant dividend in terms of “fruit.” Housing is different in several respects. First, it enters the utility function, not just the budget equation, which means it can have a liquidity premium or bubble component when the supply is either high or low, depending on preferences (i.e., demand elasticity), while in the model with financial assets this only happens when supply is low. The results are also different because Rocheteau and Wright (2013) have multiple steady states, while we have a unique steady state, which makes it more interesting and challenging to construct interesting dynamics, as we said. Additionally, we have construction and not a fixed asset supply, which is important for housing, and matters for results. Importantly, endogenous dynamics emerge here even when supply is endogenous (at least, as long as the production function is not linear).

Our results differ from Gu et al. (2014), too. First, that model also has multiple steady states. Moreover, dynamics in that model depend on the nonmonotonicity
of borrowers’ payoffs in debt limits, because of the way these limits affect the terms of trade, which is technically very different. For one thing, it means cycles in that model may exist with generalized Nash bargaining, but not with take-it-or-leave-it offers, nor with Kalai bargaining. In this paper cycles exist for all of these (and other) solution concepts, because the results are driven by completely different economic forces – the inescapable nonmonotonicity in the marginal value of liquidity mentioned above, rather than terms-of-trade effects. Moreover, Gu et al. (2014) only study unsecured (Kehoe-Levine) credit, not collateralized (Kiyotaki-Moore) credit; we focus on the latter, but also show how to combine the two. So the economics is very different, even if the math is similar in these and many other papers (see Azariadis 1993, Farhi and Tirole 2012 and references therein).

While the ingredients in our model, or at least the ways in which they are combined, are novel, the basic model builds on an environment that is standard in monetary economics: see Williamson and Wright (2010a,b) or Nosal and Rocheteau (2011) for recent surveys. We do not expect everyone to be familiar with this approach, but readers should be aware that it is an established and well-understood framework that has previously proved fruitful in a variety of applications concerning liquidity, if not housing. If some features look unusual to those outside the field, two decades of research in monetary theory leads us to believe that this class of models is useful and natural for the issues at hand.

The paper is organized as follows. Section 2 presents the environment. Sections 3 and 4 discuss steady states and dynamics with fixed supply. Section 5 and 6 endogenize unsecured lending and supply. Section 7 analyzes recent housing market experience. Section 8 incorporates money and banking. Section 9 concludes.\footnote{To conclude this introduction, we emphasize the project is not about imperfect housing markets: houses are traded in frictionless markets, like capital in standard growth theory. For a sample of work on frictional housing markets, see Head and Lloyd-Ellis (2012) and references therein.}
2 Environment

One way to model credit is to have a desire to smooth consumption when income is fluctuating; another is to have random desires to consume. We choose the latter, mainly for convenience, but it is worth mentioning that this is the same as standard Kiyotaki-Moore models, except instead of producers having random opportunities to invest, households have random opportunities or needs to consume. As in those models, one cannot always meet such opportunities or needs using current resources, so there is a motive for borrowing. The essential friction is limited commitment/enforcement: agents are free to renege on promised payments. Therefore, some form of punishment for opportunistic behavior is necessary for lending to be viable. Punishment in Kehoe-Levine models involves taking away defaulters’ access to future credit; in Kiyotaki-Moore models it involves taking away some of their assets. Kiyotaki-Moore punishments are used here as a benchmark, but Section 5 endogenizes unsecured borrowing limits as in Kehoe-Levine.

A now standard way to formalize the asynchronization of resources and expenditures is to assume that in discrete time agents interact in two alternating settings: a frictionless centralized market, as in Arrow-Debreu, labeled the AD market; and a market with credit frictions, as in Kiyotaki-Moore, labeled the KM market. At each $t$ there are two nonstorable consumption goods $x_t$ and $y_t$, plus labor $\ell_t$ and housing $h_t$, where $\ell_t$, $x_t$ and $h_t$ are traded in the AD market and $y_t$ in the KM market. In terms of modeling strategy, $x_t$ mainly plays the role of numeraire, and can be eliminated for most of what we do, but we prefer to stay close to the standard specification in monetary economics where both $x_t$ and $y_t$ are needed to match certain facts (e.g., Aruoba et al. 2012). The more interesting action involves agents borrowing to get $y_t$, using $h_t$ as collateral, with $\ell_t$ allowing them to pay off KM debt.
by working in AD. For now, households have access to no payment instrument in the KM market except promises to deliver numeraire (we introduce currency below). This deferred settlement generates a role for $h_t$ as collateral.

Period utility is generally $U(x_t, y_t, h_t, \ell_t)$; for simplicity we use $U = U(x_t, h_t) + u(y_t) - \ell_t$, where $U(\cdot)$ and $u(\cdot)$ satisfy the usual monotonicity and curvature assumptions, plus $u(0) = 0$ (this can be relaxed). Having $U$ linear in $\ell_t$ keeps the analysis tractable.\(^5\) Lifetime utility is $\sum \beta^t U(x_t, h_t, \ell_t)$, where $\beta = (1 + r)^{-1}$ and $r > 0$ is the discount rate between the AD market and the next KM market. There is no explicit discounting between KM and AD, without loss of generality, since we can always rescale AD utility by $\beta$ for any $\beta$. Hence, the length of time between KM and AD can be arbitrarily long. Figure 3 shows the timing, although note that one cannot say the AD market convenes after the KM market, any more than one can say day follows night rather than night follows day.

\[\text{Figure 3: Time Line with Utility}\]

In the KM market, with probability $\alpha$ each household realizes a shock, meaning there is an opportunity or desire to consume $y_t$. With probability $1 - \alpha$, they realize no such shock that period. One can set $\alpha = 1$, but $\alpha < 1$ allows housing

\(^5\)Basically, it reduces dimensionality of the state space by making the wealth distribution degenerate at the end of every period, as in Lagos and Wright (2005). Wong (2012) shows the same simplification obtains for a much wider class of preferences. Or, as Rocheteau et al. (2008) show, one can use any $U$ if one assumes indivisible labor à la Rogerson (1987), which is part of many standard macro models anyway (e.g., the literature following Hansen 1985). In any case, since our main goal is to establish that certain types of outcomes are possible, often by way of example, focusing on a particular class of preferences is not a big limitation.
to be valued as collateral even if it is not used as collateral every period. What is more important is that households generally have insufficient resources to get \( y_t \) without using credit. For now there are no storable assets other than housing, and all household income accrues in the AD market. Thus, households purchase \( y_t \) in KM in exchange for a debt obligation coming due in the AD market. Although they have quasi-linear preferences, since \( u(y_t) \) is strictly concave, with probability \( \alpha \) households need \( y_t \) now — they are not indifferent between KM and AD goods.

Let \( W_t(d_t, h_t) \) be a household’s value function entering the AD market with debt \( d_t \) and house \( h_t \). Note that \( d_t \) is paid off in AD, since with our preference assumptions, households are indifferent between short- and long-term loans if credit constraints are slack, and strictly prefer short-term debt if they are binding. Hence, households start debt free in KM, where \( V_{t+1}(h_{t+1}) \) is the value function, next period. Then the AD problem is

\[
W_t(d_t, h_t) = \max_{x_t, \ell_t, h_{t+1}} \{U(x_t, h_t) - \ell_t + \beta V_{t+1}(h_{t+1})\}
\]

subject to

\[
x_t + \psi_t h_{t+1} = \ell_t + \psi_1 h_t - d_t + T_t,
\]

where \( \psi_t \) is the price of \( h_t \), the real wage is 1 because we assume \( x_t \) is produced one-for-one with \( \ell_t \), and \( T_t \) denotes other funds (in Section 8 it will be cash transfers).

There are nonnegativity constraints, \( x_t \geq 0, h_{t+1} \geq 0 \) and \( \ell_t \in [0, 1] \), but they are assumed not to bind. Eliminating \( \ell_t \) using the budget equation, we get

\[
W_t(d_t, h_t) = \psi_t h_t + T_t - d_t + \max_{x_t} \{U(x_t, h_t) - x_t\} + \max_{h_{t+1}} \{\beta V_{t+1}(h_{t+1}) - \psi_1 h_{t+1}\}.
\]

This implies \( h_{t+1} \) is independent of \( (d_t, h_t) \), and so we do not have to track the wealth distribution in the KM market, which is the big simplification that comes

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6In other words, there is a precautionary demand for pledgeability, akin to the precautionary demand for liquidity that Keynes (1936) defined in terms of providing “for contingencies requiring sudden expenditure and for unforeseen opportunities of advantage.” This is exactly what our KM shocks capture (see also Telyukova and Visschers 2012). Alternatively, one can interpret \( \alpha \) as the probability of meeting a trading partner in a bilateral random-matching process. In any case, while \( \alpha = 1 \) works in theory, for the quantitative work we want to allow \( \alpha \leq 1 \).
from our preference assumptions (recall fn. 5). The FOC’s are

\[ U_1(x_t, h_t) = 1 \text{ and } \psi_t = \beta \partial V_{t+1}/\partial h_{t+1}. \]  

(1)

The envelope conditions are

\[ \partial W_t/\partial d_t = -1 \text{ and } \partial W_t/\partial h_t = U_2(x_t, h_t) + \psi_t. \]  

(2)

Hence, \( W \) is linear in debt, but not in housing, because \( h_t \) affects \( U(\cdot) \) directly as well as through the budget constraint.

Now consider the KM market. Limited commitment implies a borrowing limit that depends on one’s pledgeable assets. Here this means \( d_t \leq D_t = D(e_t) \), with \( e_t = \psi_t h_t \), where again \( \psi_t \) is the price of \( h_t \) in terms of numeraire. Often we focus on \( D(e_t) = D_0 + D_1 e_t \), with \( D_0 \geq 0 \) and \( D_1 \in [0, 1] \), although linearity is not especially important. It makes sense to allow \( D_1 < 1 \), since it is possible that only a fraction of one’s assets can be seized, for whatever reason. It also makes sense to consider \( D_0 > 0 \) and hence to potentially have \( D(e_t) > e_t \) if there are punishments available beyond asset confiscation, like exclusion from future credit. For now \( D_0 \) is exogenous; again, it is endogenized in Section 5. Also, for now \( D_1 \) is constant; later it varies over time.

For AD goods, as mentioned \( x_t \) is produced one-for-one with \( \ell_t \), and for now the aggregate stock of \( h_t \) is fixed at \( H \). For KM goods, \( y_t \) is produced by retailers using a technology summarized by cost function \( v(y_t) \). Various ways are used in the related literature to specify this technology; the simplest has \( y_t \) produced by individuals at utility cost \( v(y_t) \), although sometimes retailers are modeled in more detail. Here, to focus on essentials, the retailers’ problem is made trivial by assuming they trade with probability 1 and their cost function is \( v(y_t) = y_t \). Having simplified this part of the specification, we now want the KM market to nest search-based and competitive theories. For the former, one can imagine that households meet retailers bilaterally.
at random, with arrival rate \( \alpha \), and bargain over the terms of trade. For the latter, one can assume households realize a demand shock with probability \( \alpha \), in which case they enter a multilateral Walrasian market.

It is now standard in monetary economics to nest bargaining, price taking and other solution concepts by positing that households receive a quantity \( y \) in exchange for a payment in terms of a debt obligation \( d \) determined by a general trading mechanism that gives \( y = y(D) \) and \( d = d(D) \) for any debt limit \( D \). To proceed, let the unconstrained efficient quantity \( y^* \) solve \( u'(y^*) = 1 \), given the cost function \( v(y) = y \). Then consider mechanisms of the form

\[
y(D) = \begin{cases} \frac{f(D)}{y^*} & \text{if } D < d^* \\
 y^* & \text{otherwise} \end{cases} \quad \text{and} \quad d(D) = \begin{cases} D & \text{if } D < d^* \\
 d^* & \text{otherwise} \end{cases}
\]

(3)

where \( f \) is a strictly increasing function with \( f(0) = 0 \), and \( d^* \) is the debt limit that renders the constraint slack, \( f(d^*) = y^* \). According to (3), if \( d^* \leq D \) households get \( y^* \) and assume debt \( d^* \), but if \( d^* > D \) they go to the limit \( d = D \) and get \( y = f(D) < y^* \). Thus, \( f(D) \) tells us how much constrained households get. Inversely, if \( g = f^{-1} \) then \( g(y) \) tells us the debt required to get \( y \).

Gu et al. (2013) prove any mechanism satisfying some natural axioms must take the form in (3); here we motivate it by way of examples. Walrasian pricing reduces to (3) with \( g(y) = y \), corresponding to a unit price of \( y \), which is the competitive price with cost function \( v(y) = y \). For a simple bilateral mechanism, consider Kalai’s proportional bargaining solution, which maximizes household surplus subject to that surplus being a fixed fraction \( \theta \) of the total surplus. It is easy to check Kalai’s solution in this context reduces to (3) with \( g(y) = \theta y + (1 - \theta) u(y) \). One can also use Nash bargaining.\(^7\) Or one can use the extensive-form bargaining game in

\(^7\)Generalized Nash reduces to (3) with \( g(y) = [\theta u'(y) y + (1 - \theta) u(y)] / [\theta u'(y) + 1 - \theta] \), which differs from Kalai when \( u'' < 0 \) and \( \theta \in (0, 1) \), except at \( y = y^* \). When \( \theta = 1 \), however, Nash and Kalai both give \( g(y) = y \), which in this setup is the same as Walras.
Appendix B, which we think is important because some people question the use of axiomatic bargaining in nonstationary models (e.g., Coles and Muthoo 2003).

We give numerical results below for various mechanisms, but for the theory, we can use the general notation in (3), assuming only that \( \gamma \) is twice continuously differentiable on \((0, \varphi^*)\). Then the KM value function is

\[
V_t(h_t) = (1 - \alpha) W(0, h_t) + \alpha [u(y) + W(d, h_t)] = W(0, h_t) + \alpha [u(y) - d] ,
\]

where it is understood that \( y \) and \( d \) are given by (3) with \( D = D(\psi_t h_t) \). This says that if households do not trade in KM they go to AD debt free, but with probability \( \alpha \) they trade, and get surplus \( u(y) - d \) using the fact that \( W \) is linear in debt by (2). It is now easy to derive

\[
\partial V_t / \partial h_t = U_2(x_t, h_t) + \psi_t + \alpha D_1 \psi_t \lambda(y) ,
\]

where \( \lambda(y) \) is the liquidity premium, or equivalently the Lagrange multiplier on \( d \leq D \), satisfying:

\[
\lambda(y) \equiv \begin{cases} u'(y) / g'(y) - 1 & \text{if } y < y^* \\ 0 & \text{otherwise} \end{cases}
\]  

(4)

Inserting \( \partial V_t / \partial h_t \) into the FOC (1), we arrive at the housing Euler equation,

\[
r \psi_t = U_2(x_{t+1}, h_{t+1}) + (\psi_{t+1} - \psi_t) + \alpha D_1 \psi_{t+1} \lambda(y) .
\]  

(5)

The three terms on the RHS describe the three effects of home ownership mentioned earlier: (i) it provides utility; (ii) it can yield capital gains; and (iii) it has liquidity value because it can relax credit restrictions. For instance, with Walrasian pricing, or bargaining with \( \theta = 1 \), \( g'(y) = 1 \), so if \( d \leq D \) binds then \( \lambda(y) = u'(y) - 1 \). Thus, bigger \( h_t \) increases equity next period by \( \psi_{t+1} \), which relaxes the constraint by \( D_1 \), which is worth \( u'(y_t) - 1 \), since households value KM consumption at \( u' \) but also
have to repay the loan. With other mechanisms, like bargaining with $\theta < 1$, we have $\lambda(y) = u'(y)/g'(y) - 1$, as pricing is generally nonlinear.

Now set $h_t = H$, use (1) to write $x_t = X(H)$, and note that $y_t$ depends on $D = D(e_t)$, as described by (3), say $y_t = Y(e_t) \equiv y \circ D(e_t)$, where $y \circ D(e) = y[D(e)]$. Then (5) is a univariate difference equation in prices,

$$
\psi_t = \Psi(\psi_{t+1}) \equiv \beta U_2[X(H), H] + \beta \psi_{t+1} + \beta \alpha D_1 \psi_{t+1} \lambda \left[ Y(\psi_{t+1}H) \right].
$$

An equilibrium is given by any nonnegative and bounded sequence $\{\psi_t\}$ solving $\psi_t = \Psi(\psi_{t+1})$, where boundedness is required to satisfy a standard transversality condition. From any path for $\psi_t$ one easily recovers $e_t = \psi_t H$, $y_t = Y(e_t)$, etc.

## 3 Steady State

A steady state $\psi^*$ solves $\psi^* = \Psi(\psi^*)$, or

$$
\rho \psi^* = U_2[X(H), H] + \alpha D_1 \psi^* \lambda \left[ Y(\psi^*H) \right].
$$

We interpret this as the long-run demand for housing. Suppose the last term vanishes, so there is no liquidity effect, because $\alpha = 0$ (the KM market is closed), $D_1 = 0$ (housing is not pledgeable) or $\lambda = 0$ (the debt limit is not binding). Then (7) implies $\partial \psi/\partial H < 0$, and there is a unique steady state, $\psi^* = \psi^* \equiv U_2[X(H), H]/\rho$.

We call $\psi^*$ the fundamental value of $H$. When there is no liquidity effect, $\psi^* = \psi^*$ occurs at the unique intersection of downward-sloping demand and vertical supply.

When the liquidity effect is operative, let us assume $\lambda'(y) \leq 0$. This is a mild assumption for the following reasons. First, its main role is to allow a simple proof that steady state is generically unique, which makes our job harder, because it is harder to generate interesting dynamics with a unique steady state (recall fn. 2). Second, for many mechanisms, including Walrasian pricing, Kalai bargaining, and
Nash bargaining at least if \( \theta \) is not too small, \( \lambda' (y) \leq 0 \) holds automatically. Third, one can prove steady state is unique without \( \lambda' (y) \leq 0 \) using the method in Wright (2010), but that entails a more involved argument. Hence, for present purposes, it makes sense to impose \( \lambda' (y) \leq 0 \), even if it is not strictly necessary.

**Proposition 1** There is a unique steady state \( \psi^* > 0 \), and a critical \( e^* \) such that \( \psi^* H > e^* \) implies \( \lambda = 0 \) and \( \psi^* = \psi^* \), while \( \psi^* H < e^* \) implies \( \lambda > 0 \) and \( \psi^* > \psi^* \).

**Proof:** From (7) one derives

\[
\frac{\partial \psi}{\partial H} = \psi \frac{U_{11} U_{22} - U_{12}^2 + U_{11} \alpha \psi^2 \lambda' y'}{U_{11} (U_2 - \alpha \psi^2 H \lambda' y')}. 
\]

This implies \( \partial \psi / \partial H < 0 \). Therefore there is a unique solution to (7) for fixed \( H \). Obviously \( \psi^* > 0 \) since \( \psi^* \geq \psi^* \). The critical \( e^* \) is the equity required to get the unconstrained quantity, \( D (e^*) = g (y^*) \), where \( g = f^{-1} \) and \( f \) is defined by (3). Since \( D \) is increasing and \( f \) is strictly increasing, there is a unique such \( e^* \). ■

The critical \( e \) at which the debt limit becomes slack is \( e^* = [g (y^*) - D_0] / D_1 \), assuming \( g (y^*) > D_0 \), so that collateral has a role in the first place. When \( \psi^* H < e^* \) houses bear a liquidity premium that we call a bubble, \( \psi^* > \psi^* \). This is similar to other liquid assets, including fiat money, of course. Yet \( H \) is different. In monetary models, there is a steady state where \( M \) is valued, plus another one where it is not valued; our steady state is unique. Further, consider replacing \( H \) with a financial asset, say a Lucas tree, in fixed supply \( S \), with dividend \( \kappa > 0 \) and fundamental price \( \kappa / r \). One can show there can be a liquidity premium iff \( S \) is low (e.g., Lester et al. 2012). With housing there is a liquidity premium iff \( e \) is low, but \( e = \psi H \) can be low either when \( H \) is low or when \( H \) is high depending on the elasticity of demand.\(^8\)

\(^8\)For \( U = \tilde{U} (x) + h^{1-\sigma} / (1 - \sigma) \), e.g., \( \sigma < 1 \) implies \( e \) is low when \( H \) is low and \( \sigma > 1 \) implies
Relatedly, notice that $\lambda = 0$ when $D_1 = 0$ and when $D_1 \geq [g(y^*) - D_0] / e^*$, but $\lambda > 0$ for intermediate $D_1$. This is the inescapable nonmonotonicity mentioned in the Introduction. We summarize this as follows (the proof follows from the observations made above):

**Proposition 2** We can have $e < e^*$ either when $H$ is low or when $H$ is high, depending on $U$. We have $\partial \psi^*/\partial H < 0$ but $\partial W/\partial H$ is ambiguous, where $W$ is welfare. Also, $\psi^*$ increases with $D_1$ for low $D_1$ and decreases for high $D_1$.

## 4 Dynamics

First, it is easy to establish that any interesting dynamics must emerge from liquidity considerations.

**Proposition 3** If $\alpha D_1 = 0$ then the unique equilibrium is $\psi_t = \psi^* \forall t$.

**Proof:** If $\alpha D_1 = 0$ then (6) defines a linear difference equation that can be rewritten

$$\psi_{t+1} = -U_2 [X(H), H] + (1 + r) \psi_t.$$  

The steady state is the fundamental price, $\psi^* = \psi^*$. Any solution to this difference equation other than $\psi_t = \psi^* \forall t$ implies $\psi_t$ is unbounded or goes negative. Hence $\psi_t = \psi^* \forall t$ is the unique equilibrium. \(\blacksquare\)

When $\alpha D_1 > 0$, the liquidity channel is operative, and $H \psi_{t+1} < e^*$ implies $\lambda [Y(\psi_{t+1}H)] > 0$, so nonlinearity kicks in. In general, it is natural to think of $\psi_t$ as a function of $\psi_{t+1}$, because given $\psi_{t+1}$ demand for $h_t$ is single-valued, and so market clearing uniquely pins down $\psi_t$. However, since there can be multiple values of $\psi_{t+1}$ that yield the same $\psi_t$, the inverse $\psi_{t+1} = \Psi^{-1}(\psi_t)$ may be a correspondence. This

---

$e$ is low when $H$ is high. One can similarly show that bigger $H$ may tighten the debt limit by enough to make welfare fall, which cannot happen it the model with financial assets. Another difference between $H$ and assets like $M$ or $S$ is this: if $M$ or $S$ bears a premium because some agents value it for liquidity, anyone who is not liquidity constrained has demand 0 for the asset; this is typically not true for $H$ (an exception is noted in fn. 13 below). To be clear, we are not saying these differences between $H$ and other assets are of major significance – it’s just good to be precise about them.
is true in many models, although the economics here is different: in OLG models, e.g., a backward-bending $\Psi^{-1}$ is often interpreted in terms of a large wealth effect making savings or supply nonmonotone, while here it is due to nonmonotonicity in the liquidity premium $\lambda$. Of course $\Psi$ and $\Psi^{-1}$ cross on the $45^o$ line at $\psi^*$. Textbook methods (e.g., Azariadis 1993) imply that when $\Psi$ has a slope less than $-1$ on the $45^o$ line, $\Psi^{-1}$ and $\Psi$ also cross off the $45^o$ line, and there is a cycle of period $2$ – i.e., a solution $(\psi^1, \psi^2)$ to $\psi^2 = \Psi(\psi^1)$ and $\psi^1 = \Psi(\psi^2)$, or a fixed point of $\Psi^2$, that is nondegenerate in the sense that $\psi^1 \neq \psi^2$.

In a 2-cycle, $\psi_t$ oscillates between $\psi^1$ and $\psi^2$ as a self-fulfilling prophecy. More generally, $n$-period cycles are nondegenerate solutions to $\psi = \Psi^n(\psi)$. We first show that $n$-cycles exist, for different $n$, numerically, then provide general conditions and discuss the economic content. All examples use $D(e) = e$, along with

$$U(x, h) = \bar{U}(x) + A_h \frac{h^{1-\sigma}}{1-\sigma} \text{ and } u(y) = A_y \frac{(y + \varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}}{1-\gamma}. \quad (9)$$

Different cases in terms of parameters and mechanisms are listed in Table 1. These examples show what is possible; more reasonable parameter values are discussed below, including a calibration, using standard methods, in Section 7.9.

<table>
<thead>
<tr>
<th>Example</th>
<th>Mechanism</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\theta$</th>
<th>$A_h$</th>
<th>$\sigma$</th>
<th>$A_y$</th>
<th>$\gamma$</th>
<th>$\varepsilon$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Walras</td>
<td>0.50</td>
<td>0.80</td>
<td>n/a</td>
<td>0.125</td>
<td>n/a</td>
<td>1.5125</td>
<td>2.0</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Walras</td>
<td>0.50</td>
<td>0.60</td>
<td>n/a</td>
<td>0.333</td>
<td>n/a</td>
<td>3.2479</td>
<td>7.0</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>Kalai</td>
<td>0.90</td>
<td>0.60</td>
<td>0.90</td>
<td>0.10</td>
<td>n/a</td>
<td>0.5882</td>
<td>9.0</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>Game</td>
<td>0.50</td>
<td>0.80</td>
<td>0.60</td>
<td>0.125</td>
<td>n/a</td>
<td>3.0368</td>
<td>8.0</td>
<td>0.1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>Walras</td>
<td>0.99</td>
<td>0.95</td>
<td>n/a</td>
<td>0.10</td>
<td>4.0</td>
<td>1.027</td>
<td>8.0</td>
<td>0.0001</td>
<td>1.9</td>
</tr>
<tr>
<td>6</td>
<td>Walras</td>
<td>0.99</td>
<td>0.95</td>
<td>n/a</td>
<td>0.10</td>
<td>4.0</td>
<td>1.028</td>
<td>16.0</td>
<td>0.0001</td>
<td>1.9</td>
</tr>
</tbody>
</table>

Some parameters are irrelevant for some results – e.g., we report n/a for $\sigma$ when we have a fixed supply $h = 1$ since it is obvious from (9) that $\sigma$ is does not matter in this case. Similarly, $\bar{U}$ is irrelevant for all results. The function $u(y)$ in (9) is from Lagos and Wright (2005), and the role of $\varepsilon$ is to force $u(0) = 0$; for the more relevant examples $\varepsilon$ is small so $u(y)$ is close to CRRA. Given the linearity of $\bar{U}$ in $\ell$, the classic RBC model of Hansen (1985) is a special case of our $U(x, h) - \ell + u(y)$, without $h$ or $y$, since he was interested in neither housing nor liquidity. These remarks are meant to suggest the specification is fairly standard, or at least by no means pathological.
The left panel of Figure 4 shows $\Psi(\psi)$ and its inverse for Example 1, using Walrasian pricing. Notice the kink at $\psi = y^*$, above which the liquidity constraint is slack and $\Psi$ is linear. For Example 1, the steady state is the unique equilibrium: any other solution to the dynamical system is unbounded. Example 2 changes parameters, and in addition to steady state, there is a 2-cycle, as shown in the right panel of Figure 4. In this example $y^* = 1.0833$, and the constraint binds in alternate periods. Examples 3 and 4 (not shown) display similar 2-cycles with Kalai and strategic bargaining, and are included to verify the results are robust to changing the mechanism. Going back to Example 2, we verified that it also has a 3-cycle, with $\psi^1 = 0.8680 < y^*$, $\psi^2 = 1.5223 > y^*$ and $\psi^3 = 1.1134 > y^*$. As is standard, if a 3-cycle exists then there are $n$-cycles for all $n$ by the Sarkovskii theorem, plus chaotic dynamics by the Li-Yorke theorem (again see Azariadis 1993).

This illustrates how volatility in prices can emerge because housing conveys liquidity, which can involve self-fulfilling prophecies, as we now explain. Consider
a 2-cycle (n-cycles are similar but more complicated). Suppose at \( t \) agents expect \( \psi_{t+1} \) will be high. Then home equity and hence liquidity will be relatively plentiful at \( t + 1 \), which lowers the amount people are willing to pay for it at \( t \). Thus, low \( \psi_t \) can be consistent with market clearing given high \( \psi_{t+1} \). By similar logic, high \( \psi_{t+1} \) is consistent with low \( \psi_{t+2} \), and so on. Agents are willing to pay more for \( H \) when they know the price is about to fall, precisely because the price decrease means liquidity will soon be scarce, which makes it currently dear. Hence, the prices of liquid assets have an inherent tendency to oscillate. Of course, for this to work, the liquidity effect has to dominate the more standard effect that says low \( \psi_{t+1} \) acts to reduce \( \psi_t \) – and hence it obviously will not work, e.g., for all \( \beta \). But these examples show it is possible.\(^{10}\)

We now provide some more general conditions for cycles, normalizing \( H = 1 \) to reduce notation.

**Proposition 4** Cyclic and chaotic equilibria exist for some parameters. In particular, consider any \( u(y) \), \( D(e) = D_1e \) and Walrasian pricing. Assume \( 0 < u'(D_1) - 1 < \frac{r}{\alpha D_1} \). Then we can normalize \( U(x, h) \) so that \( y^s = 1 < y^* \), and cycles exist if

\[
u'(D_1) + D_1 u''(D_1) < 1 - \frac{2 + r}{\alpha D_1}.
\]

**Proof:** That \( n \)-cycles and chaos exist for some parameters was demonstrated above. For the general conditions, first, assume that \( y \leq D_1 \psi \) binds in steady state. Then (7) implies

\[
r \psi^s = U_2 + \alpha D_1 \psi^s [u'(D_1 \psi^s) - 1].
\]

\(^{10}\) Although one can use a similar intuition in a variety of contexts, it is important to recall from the Introduction that the economic mechanism here is quite different from those in ostensibly related papers – e.g., Rocheteau and Wright (2013) or Gu et al. (2014). Obviously any paper on nonlinear dynamics will make use of standard tools, like the Sarkovskii and Li-Yorke theorems, just like any paper on comparative statics uses the implicit function theorem, but that does not mean all such papers are the same in terms of economic content.
Now normalize utility by setting $U_2 = \tilde{U}_2 \equiv r - \alpha D_1 [u'(D_1) - 1]$, where $\tilde{U}_2 > 0$ as long as $r > \alpha D_1 [u'(D_1) - 1]$. This normalization means $\psi^* = 1$. And as long as $u'(D_1) > 1$ we satisfy the assumption that $y \leq D_1 e$ binds. The condition given in the Proposition, $0 < u'(D_1) - 1 < r/\alpha D_1$, thus allows us to set $\psi^* = 1$ and ensure $y^* = D_1 < y^*$. Given this, it is a simple calculation to show the standard sufficient condition for 2-cycles,

$$
\Psi' (\psi^*) = \beta + \beta \alpha D_1 [u' (D_1 \psi^*) - 1] + \beta \alpha D_1^2 \psi^* u'' (D_1 \psi^*) < -1,
$$

holds iff (10) is satisfied.

**Proposition 5** If $u(y)$ is given by (9) then: (i) the assumption in Proposition 4, $0 < u'(D_1) - 1 < r/\alpha D_1$, reduces to $(D_1 + \varepsilon)^\gamma < \eta < (D_1 + \varepsilon)^\gamma (1 + r/\alpha D_1)$; and (ii) assuming $D_1$ is not too small and $\varepsilon$ is not too big, cycles exist $\forall \gamma > \bar{\gamma} (D_1, \varepsilon)$ for some $\bar{\gamma} (D_1, \varepsilon)$, where

$$
\bar{\gamma} (1, 0) \equiv \frac{2 + r + \alpha (\eta - 1)}{\alpha \eta}.
$$

**Proof:** Given $u(y)$ in (9), part (i) is a simple calculation. For part (ii), for this $u(y)$, the LHS of (10) is $\eta (D_1 + \varepsilon)^{-\gamma - 1} [D_1 (1 - \gamma) + \varepsilon]$. When $D_1 = 1$ and $\varepsilon = 0$, this means (10) holds iff $\gamma > \bar{\gamma} (1, 0)$ as given in (11). By continuity, for $(D_1, \varepsilon)$ near $(1, 0)$, there is a $\bar{\gamma} (D_1, \varepsilon)$ near $\bar{\gamma} (1, 0)$ such that (10) holds $\forall \gamma > \bar{\gamma} (D_1, \varepsilon)$. 

Intuitively, Propositions 4 and 5 say that we are more likely to get cycles with higher risk aversion, in the sense of curvature in $u(y)$, although some people (e.g., Swanson 2012) argue that risk aversion is subtle in this context: with our preferences, agents are risk neutral with respect to wealth lotteries in the AD market, even if they are risk averse when it comes to the KM good $y$. Still, bigger $\gamma$ helps, as indicated by Proposition 5. Moreover, the results are robust in the following sense:
Proposition 6 Propositions 4 and 5 hold with bargaining instead of price taking at least if $\theta$ is not too small.

Proof: When $\theta = 1$, Kalai, Nash and the strategic bargaining solution in Appendix B are all yield the same difference equation as Walrasian pricing. The result follows for $\theta$ close to 1 by continuity.

These findings indicate that housing markets can exhibit excess volatility relative to fundamentals. However, $\psi_t$ tends to go up and down rather too regularly, compared to a typical bubble with prolonged price increases followed by collapse. So consider Example 5, which has an equilibrium with exactly that pattern. Figure 5 depicts $\psi_{t+1} = \Psi^{-1}(\psi_t)$, zooming in around $\psi^*$. Note $\psi^*$ corresponds to two values of $\psi_{t+1}$, $\psi^*$ and $\psi_1$. One equilibrium is $\psi_t = \psi^* \forall t$. Another is this: start at $\psi^*$; then at some arbitrary date $t$, jump to $\psi_1$ and set off on the trajectory shown by the dashed lines. Along this path $\psi_t$ is increasing, but that cannot go on forever without violating transversality. As shown, after 5 periods, the price crashes from $\psi_2$ to $\psi_3 < \psi^*$, before recovering in an oscillatory orbit back to $\psi^*$.

This looks the classic bubble, first growing then bursting. And it uses reasonable parameters, including $\beta = 0.95$, and $\gamma = 8$. However, $\psi_t$ cannot increase too fast.

Proposition 7 With perfect foresight, the capital gain $(\psi_{t+1} - \psi_t) / \psi_t$ is bounded by $r$.

Proof: Rearrange the difference equation (6) as

$$\psi_{t+1} = (1 + r) \psi_t - U_2 [X (H), H] - \alpha \psi_{t+1} D_1 \lambda \left[ y(\psi_{t+1} H) \right].$$

Since $U_2 > 0$ and $\alpha \psi_{t+1} D_1 \lambda \geq 0$, we have $(\psi_{t+1} - \psi_t) / \psi_t < r$. ■

To the extent that one sees capital gains above reasonable values for $r$ in the data, this is a problem for perfect foresight. Therefore, consider sunspot equilibria, where $\psi_t$ fluctuates stochastically, even when fundamentals are deterministic.
Proposition 8  Sunspot equilibria exist for some parameters, and in particular they exist under the conditions given in Propositions 4-6.

Proof: Suppose that when the price is $\psi^1$ it jumps to $\psi^2 > \psi^1$ with probability $\zeta^1$, and stays at $\psi^1$ with probability $1 - \zeta^1$. When it is $\psi^2$ it falls back to $\psi^1$ with probability $\zeta^2$. When $\zeta^1 = \zeta^2 = 1$ this equilibrium reduces to a 2-cycle, the existence of which is guaranteed by the previous Propositions. By continuity, there are equilibria with $\zeta^1$ and $\zeta^2$ close to but less than 1. ■

The method of proof used here is standard in OLG models of sunspots going back to Azariadis and Guesnerie (1986). In our context, we conclude that housing markets can exhibit stochastic volatility, not only deterministic cycles. However, the sunspots guaranteed by Proposition 8 look similar to the 2-cycles in Figure 4. So we now construct a sunspot equilibrium more along the lines of Figure 5. Example 6 has an equilibrium where prior to $\hat{t}$, $\psi_t = \psi^* = 0.5255$. From $t = \hat{t}$ to $\hat{t} + 4$, each period there is a probability (over which agents have rational expectations) of
jumping to a deterministic path from $\psi = 0.5350$ back to $\psi^*$. After $t + 4$, everything is deterministic again. One realization has $\psi_t$ growing at 9% per year for 5 years, then collapsing and oscillating back to $\psi^*$. This is not shown, but looks like Figure 5 with bigger capital gains.

The next thing to show in this Section is Figure 6, which plots $\psi$ as a function of $D_1$ for Example 6, except that we change $\gamma$ to 0.16, to better illustrate the point.\footnote{This is the value used in the calibration exercise in Section 7. A similar picture obtains with $\gamma$ set as in Example 6, except the curve is very nonlinear and peaks at a very low value of $D_1$. Also, note that picture represents a comparison across steady states; Section 7 presents transitions in response to a dynamically changing $D_1$.} If $D_1 = 0$ then $\psi = \psi^*$. As $D_1$ increases, first demand for $h$ and hence the price $\psi$ rise, as does $y$. As $D_1$ increases further, $y$ rises more, and soon the liquidity premium $\lambda$ and price $\psi$ begin to fall. Here $\psi$ peaks just to the left of around $D_1 = 0.2$. If we keep increasing $D_1$ the debt constraint becomes slack, which happens here just to the left of $D_1 = 1$. At that point, $\psi$ falls all the way back to the fundamental value $\psi^*$, and $y = y^*$. This shows clearly the nonmonotonicity in the liquidity premium and hence the price $\psi$ as a function of the LTE ratio.

We close this Section with an important point: there can be recurrent episodes with bubble patterns like Example 5 or 6. By contrast, consider any model of fiat money $M$, or the model of housing $H$ (he calls it land) in Kocherlakota (2011). Those models have multiple steady states, including one where the asset has a 0 price, which makes it is easy to construct equilibria where the value of $M$ or $H$ goes to 0, but then \textit{it never comes back}. Note that $M$ in fiat money models and $H$ in Kocherlakota’s model have 0 intrinsic value. We give $H$ intrinsic value, and get a unique steady state, yet bubbles can grow, burst, take an arbitrarily long hiatus, and start again. Of course, big capital gains must occur with low probability, so in sunspot equilibrium realizations where $\psi_t$ increases a lot several periods in a row are rare events. In Example 6, e.g., they happen on average about once a century.
That seems right: even if bubbles are recurrent, as one learns from Reinhart and Rogoff (2009), they do not happen all that often.

5 Unsecured Debt

At this point we digress slightly to consider what happens when unsecured lending is endogenized as in Kehoe and Levine (1993).\textsuperscript{12} Let $D_0 = \bar{D}_t$ be the unsecured debt limit at $t$, so the total debt constraint is $d_t \leq \bar{D}_t + D_1 \psi_t h_t$. This has two endogenous components: $\bar{D}_t$ is determined as shown below; and while $D_1$ is fixed, it multiplies the endogenous $\psi_t h_t$. Agents always honor secured obligations, given $D_1 \leq 1$, but may in principle renege on unsecured obligations. One can consider at least two types of punishment for defaulters: they are denied access to future unsecured credit; or they are denied access to all future credit. Here we use the latter, taking for granted that defaulters can be excluded from the KM market. Also, we normalize $H = 1$ and $\max\{\bar{\mathcal{U}}(x) - x\} = 0$, and consider $U(x, h) = \bar{\mathcal{U}}(x) + h$.\textsuperscript{13}

\textsuperscript{12}Although it clearly seems interesting to integrate Kehoe-Levine and Kiyotaki-Moore theories of credit, this Section could be skipped without loss of continuity.

\textsuperscript{13}It is not necessary to have $h$ enter linearly, but a slight advantage of this specification is that it implies we can set $h_{t+1} = 0$ for defaulters. Suppose $\psi > \psi^*$, where $\psi^* = 1/r$ given $U_2 = 1,
Following the procedure in Gu et al. (2014), which is an extension of Alvarez and Jermann (2000), one can characterize the unsecured debt limit as follows: First, choose an arbitrary sequence of unsecured debt limits \( \{ D_t \} \). Obviously this choice generally affects payoffs at any \( t \). An endogenous debt limit at \( t \) is given by the maximum obligation agents are willing to honor at \( t \), which is the one that equates his equilibrium continuation payoff to the punishment payoff. Since the former depends on the debt limit next period, as shown in Appendix C, one can express an unsecured debt limit sequence recursively as

\[
D_{t-1} = \beta D_t + \beta + \beta \alpha [u \circ Y(D_t, \psi_t) - Y(D_t, \psi_t)] + \beta \psi_t - \psi_{t-1},
\]

where now, with a slight abuse of notation, \( Y(D_t, \psi_t) = \min \{ y^*, \bar{D}_t + D_t \psi_t \} \) is consumption, and also debt \( d_t = Y(\bar{D}_t, \psi_t) \), assuming Walrasian pricing or bargaining with \( \theta = 1 \). Also, as usual, the housing Euler equation is

\[
\psi_{t-1} = \beta \psi_t + \beta + \beta \alpha \psi_t D_t \left[ u' \circ Y(\bar{D}_t, \psi_t) - 1 \right].
\]

Nonnegative and bounded solutions \( \{ \bar{D}_t, \psi_t \} \) to (12) and (13) constitute equilibria with endogenous unsecured debt limits. Notice there is feedback from the Kehoe-Levine limit \( \bar{D}_t \) to the asset price \( \psi_{t-1} \), and hence to Kiyotaki-Moore credit, as expressed by (13). At the same time, there is feedback from asset prices to \( \bar{D}_{t-1} \), as expressed by (12) through the continuation value. The two types of credit are not orthogonal. It was demonstrated above that there can be endogenous dynamics in \( \psi_t \) and hence in Kiyotaki-Moore credit. As in Gu et al. (2014), one can also show there are belief-based dynamics in \( \bar{D}_t \) and hence in Kehoe-Levine credit. We leave a detailed analysis of the joint dynamics for future work, and here focus on at steady

so that those excluded from KM strictly prefer \( h_{t+1} = 0 \) since they get no benefit from liquidity. Suppose \( \psi = \psi^* \). Then they are indifferent, so we can set \( h_{t+1} = 0 \). Note this is an exception to the observation in fn. 8, where by typically we meant a utility function with curvature in \( h \).
states, which are interesting enough in their own right, and satisfy

\[ r\bar{D} = 1 + \alpha \left[ u \circ Y(\bar{D}, \psi) - Y(\bar{D}, \psi) \right] - r\psi \]  \hspace{1cm} (14)

\[ r\psi = 1 + \alpha \psi D_1 \left[ u' \circ Y(\bar{D}, \psi) - 1 \right]. \] \hspace{1cm} (15)

Equation (15) defines a relationship \( \psi = \Gamma(\bar{D}) \) that is nonincreasing, intuitively, because higher unsecured debt limits reduce the collateral value and hence the price of housing. Notice \( \Gamma(\bar{D}) = \psi^* \) when \( \bar{D} \) is large, and \( \Gamma(0) = \psi^0 \geq \psi^* \). By standard arguments, one can show (14) defines a continuous relationship between \( \bar{D} \) and \( \psi \). Substitution implies

\[ 1 + \alpha u \circ Y[\bar{D}, \Gamma(\bar{D})] - \alpha Y[\bar{D}, \Gamma(\bar{D})] - r\Gamma(\bar{D}) - r\bar{D} = 0. \] \hspace{1cm} (16)

If there is a solution \( \bar{D} \) to (16), an equilibrium with endogenous unsecured debt exists. If \( \bar{D} \) is large the LHS is negative and if \( \bar{D} = 0 \) it is positive. By continuity there exists a solution \( \bar{D} > 0 \). One can also check that \( u(y^*) < (1 + r)y^* - D_1 \) implies \( \psi > \psi^* \).

This integrated credit model seems worthy of further study, because of the feedback from secured to unsecured lending, and in the opposite direction. It also provides a nice way to endogenize LTE ratios. For present purposes, however, we return to a fixed \( D_0 \). The objective of this digression was to show that there is no reason in principle why one cannot combine secured and unsecured credit.

### 6 Construction

To go beyond fixed \( H \), we now present a model with a resemblance to a classic paper by Poterba (1991), even if our microfoundations and applications are different. One reason to endogenize \( H \) is that the behavior of construction is an integral part of the episode under study. Another is that it allows us to further distinguish \( H \).
from \( M \): housing is produced by private agents; outside money is not. Of course, inside money is produced by private agents – which is exactly what we are modeling when we endogenize \( H \). Housing is a form of inside money, since it is a privately created asset that not only yields return and stores value, but also helps in the payment/settlement process.

The technology for home building is summarized by a cost function: increasing the stock by \( \Delta h_t \) requires an input of \( c(\Delta h_t) \) in numeraire, where \( c'(\Delta h) > 0 \) and \( c''(\Delta h) \geq 0 \forall \Delta h > 0 \). Also, \( c(0) = c'(0) = 0 \) and \( c'(\infty) = \infty \), and houses depreciate at rate \( \delta \). Construction is carried out by a price-taking representative firm. Hence, price equals marginal cost:

\[
\psi_t = c'[h_{t+1} - (1 - \delta) h_t].
\]

The households' problem is unchanged except for depreciation, which implies \( e_t = (1 - \delta) \psi_t h_t \). Now, letting \( \Lambda(e) \equiv \lambda \circ Y(e) \) and \( \Omega(h) = U_2[X(h), h] \), to keep the notation manageable, we write (6) as

\[
\psi_t = \beta \Omega(h_{t+1}) + \beta \psi_{t+1} (1 - \delta) + \beta \alpha D_1 (1 - \delta) \psi_{t+1} \Lambda[(1 - \delta) \psi_{t+1} h_{t+1}].
\]

Equilibrium is a nonnegative and bounded path \( \{h_t, \psi_t\} \) satisfying (17)-(18) for a given initial condition \( h_0 \).

**Proposition 9** Versions of Propositions 1-8 hold with \( H \) endogenous.

**Proof:** In steady state (17)-(18) reduce to

\[
\psi = c'(\delta h)
\]

\[
(r + \delta) \psi = \Omega(h) + \alpha D_1 (1 - \delta) \psi \Lambda[(1 - \delta) \psi h],
\]

where (20) is again long-run demand, while (19) is long-run supply. These can be combined into

\[
r + \delta = \frac{U_2[x(h), h]}{c'(\delta h)} + \alpha D_1 (1 - \delta) \Lambda[(1 - \delta) c'(\delta h) h].
\]
The RHS goes to $\infty$ as $h$ goes to 0, and vice-versa, and it is strictly decreasing. Hence, steady state $(h^*, \psi^*)$ exists and is unique. By a similar argument to the one used earlier, we can have $\lambda > 0$ when the (now endogenous) supply is high or when it is low, depending on elasticities. This establishes versions of Propositions 1-2 that apply to steady state with $H$ endogenous.

As regards dynamics, consider first Proposition 3. The dynamical system has one predetermined variable $h_t$, and one jump variable $\psi_t$. When $\alpha D_1 = 0$ one can check (details available on request) by routine methods that $(h^*, \psi^*)$ is a saddle point. Hence, given $h_0$ there is a unique $\psi_0$ such that the path for $\{h_t, \psi_t\}$ starting at $(h_0, \psi_0)$ converges to $(h^*, \psi^*)$, while any path starting at $(h_0, \psi')$ for $\psi' \neq \psi_0$ is unbounded or goes negative. This establishes the uniqueness of equilibria, not only steady states, when the liquidity channel is inoperative.

For $\alpha D_1 > 0$, simply note the following: As a special limiting case, the model with $H$ fixed is recovered by making supply vertical. Propositions 4-6 concern examples and parameter conditions that admit periodic equilibria with $H$ fixed. Similar results hold here, as special cases when supply is vertical, and by continuity when it is not vertical but very steep. The same is true for the stochastic equilibria in Proposition 8.

The remaining result is Proposition 7. Since that follows directly from the Euler equation, capital gains are still bounded, although the relevant expression now takes depreciation into account: $(1 - \delta) \psi_{t+1}/\psi_t$ is bounded above by $1 + r$. ■

7 Experiments

We now use the model with construction to organize a narrative concerning recent events. As the story goes, at the start of the episode, financial innovation gave households easier access to home-equity loans – that’s what people mean when
they say consumers started using previously illiquid housing assets as ATM’s. This
stimulated housing demand, and hence prices and construction, for a while; then the
market reversed. We now show how this can happen in the model, in three distinct
ways. In what follows, $\mathcal{U} = A_x \log x + A_h \log h - \ell + u(y)$, with $u(y)$ given by (9),
and the construction cost function is $c(\Delta h) = (\Delta h)^{1+\xi} / (1 + \xi)$. Also, we use Kalai
bargaining, but recall that $\theta = 1$ makes this the same as Walrasian pricing. We
consider two specifications, or experiments, as described in Table 2 (note that our
choice for the important variable $\alpha$ is explained below).

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\theta$</th>
<th>$A_x$</th>
<th>$A_h$</th>
<th>$A_y$</th>
<th>$\gamma$</th>
<th>$\varepsilon$</th>
<th>$\xi$</th>
</tr>
</thead>
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<td>0.60</td>
<td>1.00</td>
<td>1.00</td>
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<td>3.80</td>
<td>11.0</td>
<td>0.10</td>
<td>5.00</td>
</tr>
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<td>0.25</td>
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<td>0.85</td>
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<td>0.05</td>
<td>2.18</td>
<td>0.16</td>
<td>0.00</td>
<td>3.80</td>
</tr>
</tbody>
</table>

Table 2: Parameter Values for Experiments 1-2

In Experiment 1, the parameters yield a multiplicity of dynamic equilibria, and
we select one for which the transition after a financial innovation, captured by a
surprise increase in $D_1$, resembles the data. The logic is based on the univariate
model in Section 4, as shown in Figures 4 and 5: when there are multiple $\psi_{t+1}$
consistent with a given $\psi_t$, it is possible for price to jump at some $\hat{t}$. After a $D_1$
shock, given parameters imply multiple equilibrium transitions to the new steady
state in the bivariate system (17)-(18), we can start off on one path (the beginning
of a boom) and then, at some $\hat{t}$, jump to another (the bust), although this is a
temporary setback since in the long run the new $(h^*, \psi^*)$ is higher. Agents have
perfect foresight: they see the price drop coming, which in itself is bad for demand,
but after the drop $e = \psi h$ will be low and hence liquidity scarce, which keeps demand
and price up just before the drop. As in Section 4, we have to be sure that this
liquidity effect is big enough, relative to the usual asset effect that implies a low
price tomorrow tends to make the price lower today.
In particular, start in steady state where $D(e) = D_1 e$, with $D_1$ small, and consider an unexpected one-time increase. For some parameters, as discussed in the proof of Proposition 9, the steady state of system (17)-(18) is a saddle point, and there is a unique equilibrium transition where $\psi$ jumps on impact from the $D_1$ shock then monotonically declines as supply increases. For other parameters, the system displays a classic indeterminacy, where $(h^*, \psi^*)$ is a sink. In this case there are many perfect-foresight paths leading to the new steady state, allowing us to pick one that looks something like the episode in question. One such path is shown in Figure 7, constructed with $\theta = 1$, or equivalently, Walrasian pricing.\footnote{We use parameters such that $y \leq D(e)$ is binding, and verify numerically that both eigenvalues are real and less than 1 in absolute value at the new steady state. Although the methods for computing equilibria here are standard (going back to, e.g., Huggett 1997), we sketch our algorithms in Appendix E.}

Figure 7: Experiment 1, A Transition After a One-Time Surprise Increase in $D_1$

Figure 7 shows paths for the price (blue solid), HEL (red dash) and construction (green dot). We claim this “looks like” Figure 1. Of course, they are not exactly
alike, but Figures 7 and 1 “look alike” in the following sense: (i) the real house price first rises then tumbles, whether we measure it relative to numeraire (shown) or by the price-rent ratio $\psi/U_2$ (not shown but similar); (ii) home-equity loans go up, and stay up after some wiggles; and (iii) construction rises, fluctuates, then drops as we approach the higher new steady state. Home-equity lending rises quickly, even though $h_t$ adjusts slowly because $c''(\Delta h) > 0$. The initial increase is due to the exogenous jump in $D_1$ plus the endogenous price jump that makes $e_t = \psi_t h_t$ rise before $h_t$. Not shown is welfare, but $W$ also increases over the period as credit constraints are relaxed. Experiment 1 is meant to be illustrative, not realistic, and in particular it uses a low $\beta = 0.6$, which we found necessary in this specification for $(h^*, \psi^*)$ to be a sink. Still, the point is that a boom and bust in house prices and quantities can in principle be a self-fulfilling prophecy.

For Experiment 2, instead of using arbitrary parameters, we follow standard practice in calibrating the model. It turns out parameters are such that the equilibrium transition is unique. In this case, we try to capture the data by having $D_1$ change gradually over time. First, set $\beta = 0.95$ and $\delta = 2.5\%$ to generate realistic interest and depreciation rates. Then set the construction cost parameter to $\xi = 3.8$ to match the relative volatility of housing investment and prices. Specifically, we want the model to produce a 68% increase in construction in response to a 51% change in houses price. This means that if we match prices we will match construction, at least on average, and so that should not be considered a success – rather, the goal is to see how well we can do at matching prices given we have calibrated so that construction moves with prices. This puts discipline on Experiment 2 by imposing that quantities move with prices in a realistic way. In particular, there is no sense in which prices increase a lot because we are restricting supply to behave in some artificial fashion.

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Next, set bargaining power $\theta = 0.85$ so that in steady state with $D_1 = 0.22$ we generate a 30% markup in the KM market to match data reported in the Annual Retail Trade Survey (as discussed, e.g., in Aruoba et al. 2011). Further in terms of preferences, set $\varepsilon = 0$, so $u(y)$ is CRRA, with $\gamma = 0.16$, as in Lagos and Wright (2005). We also tried to fit $\gamma$ to the data, but identification is weak: it is clear from the evidence that $\gamma = 0.16$ fits much better than, say, $\gamma = 0.5$ or $\gamma = 0.9$, but any value around $\gamma = 0.16$ fits about as well. Hence, we simply use the value in Lagos and Wright (2005), where it was set to match the elasticity of real balances wrt nominal interest rates.

This leaves a vector of constants from the utility function $\mathbf{A} = (A_x, A_h, A_y)$. We choose $\mathbf{A}$ to match data on the ratio of the value of the housing stock to household final consumption. To be clear, in choosing $\mathbf{A}$ we are not using only the sample average; we are minimizing the sum of squared residuals between the model and data time series. However, for this we only use data from 1996 to 2000. The idea to to pin down $\mathbf{A}$ from data prior to financial innovation in the 2000’s. Note that if we included the data from the 2000’s in setting $\mathbf{A}$ we can explain more of the price runup, but that is not the nature of the exercise – we rather want to see how much we can account for in the recent price data assuming preferences are given by the pre-2000 data, and the only change was in LTE ratios since then.

We now input a series of LTE ratios corresponding to the data in Figure 1 and see what the model says about the paths for HEL, prices and construction, although as we said above, to the extent that we match prices, we approximately match construction by construction, since we calibrated the supply elasticity. To produce an empirical series for $D_1$, for each year 1996-2010, divide HEL by the value of home equity, then divide by $\alpha = 0.25$, the approximate proportion of home owners with a HEL during the episode (Jacobe 2007). The resulting series has $D_1$ increasing
gradually from 0.15 in 1996 to 0.25 in 2005, then jumping to 0.3 in 2007 and 0.33 in 2008, where it stays for two years before dropping to 0.3 in 2010.\textsuperscript{15}

For any such experiment, one has to take a stand on what agents knew and when they knew it. In Experiment 2a, we suppose that in 1996 they predict that $D_1$ will increase gradually to 2006, then stay constant. Then in 2005 they update their beliefs to predict $D_1$ will rise even higher, to 0.33 in 2008, then stay put. Then in 2007 they pessimistically update again to predict that $D_1$ will eventually revert permanently to its 2006 value. Of course these beliefs are to a large extent arbitrary, but alternatives – including extremes, like assuming perfect foresight, or assuming every change was a complete surprise – do not seem all that reasonable either. One way to rationalize our choice is that this belief process generates a rather large effect that one can interpret as an upper bound on the amount we can explain. In other words, we searched over belief processes to see how much a changing $D_1$ can account for the price runup, and to the extent that we find it only accounts for a fraction, this constitutes evidence that there was something else going on, perhaps including some type of self-fulfilling prophecy, as analyzed above.

The results are depicted in Figure 8 in terms of prices, HEL and construction, with the horizontal axis now displaying real time, as we are inputting the year-by-year series for $D_1$, with all variables normalized in 1996 as in Figure 1. We claim this looks a lot like the data, although it is not as volatile. One reason the simulated data is less volatile is that we have only one driving process, $D_1$, and presumably there were other shocks over the period. More to the point, the simulated real price $\psi_t$

\textsuperscript{15}We want to clarify these numbers. In reality, the average individual with good credit can borrow a lot more against home equity than the 33% we use in 2008, often as much as 80%. But the marginal agent with a home-equity loan does not necessary have the best credit or own the best house (recall the results of Mian and Sufi 2011 mentioned in fn.1). Without going to an explicit heterogeneous-agent model, which is clearly desirable but beyond the scope of the paper, we capture this by having the representative agent subject to tighter credit constraints than someone would face if they had better credit and owned a bigger share of aggregate housing, and other, wealth.
peaks at 26% above its 1996 level. This means the model accounts for about 1/2 of the price runup – 52% to be exact – which is obviously relevant but still leaves room for other factors, including sunspots or other endogenous dynamics. We also do well accounting for HEL, and slightly less well for construction even though we calibrated to the average elasticity of construction (in particular, we do not match the big drop at the end of the episode). Still, on the whole, the model does well matching the data qualitatively, but only explains about half of the pattern in house prices. It is also worth noting that KM consumption is only about 11% of total consumption on average.\footnote{This is important because it means that generating sizable price movements in response to changes in LTE ratios does not require that collateralized loans are used for the lion’s share of consumption. It works because, although the KM market is not large in an accounting sense, the marginal utility of KM consumption is high, due to several factors, including bargaining with }$	heta < 1$. Still, 11% is perhaps too large, but related to fn. 15, this is again to some extent due to our use of a representative agent model.

The drop in prices at the end of the episode in Experiment 2a is due to several factors. First, prices naturally fall as construction gradually catches up with de-
mand. Second, households have pessimistic forecasts for future LTE ratios at the end of the sample – but this effect is subtle. Recall that as LTE ratios increase prices react nonmonotonically. Thus, as LTE ratios fall, credit constraints tighten and the liquidity premium increases. To say it another way, we can in principle get a price bust even if agents, instead of pessimistically forecasting a long-run fall in LTE, optimistically forecast that after a minor setback it will continue to rise. In Experiment 2b we illustrate that possibility. If we are free to assume agents are arbitrarily optimistic about future increases in $D_1$, it is possible to generate an outcome that looks almost identical to Figure 8, but for very different reasons.17

Indeed, to take an extreme case, suppose that at the end of 2006 agents all of a sudden realize that the liquidity constraint will no longer bind, say because $D_1$ will go up by enough that $\lambda = 0$ in the future (one could do something similar with an increase in $D_0$). At that point, the bubble – or the liquidity premium or the

\[ \text{Figure 9: Experiment 2b, Gradual Increase in } D_1, \text{ Optimistic 2008 Forecast} \]

---

17In terms of Figure 6, in Experiment 2a, $\psi$ goes down as $D_1$ is forecast to fall, while in Experiment 2b, $\psi$ goes down as $D_1$ is forecast to increase, on either side of the hump (this only as a qualitative description, since Figure 6 and Experiment 2 use different parameter values).
Lagrange multiplier or whatever one wants to call it – bursts. As shown in Figure 9, the result is that prices and construction tank. This is meant to be extreme, involving as it does a major change in credit markets, but it shows dramatically what can happen due to liquidity effects.

Again, welfare goes up as $D_1$ increases. We do not provide exact numbers, since that would require fleshing out details of the model – e.g., adding capital and more general production functions for $x$ and $h$ – not needed for the rest of the analysis. But a “back-of-the-envelope” calculation may be informative. In Experiment 2a, the increase in $D_1$ generates about a 50% increase in $y$. Given that home-equity loans finance around 11% of consumption in the model, this translates to an increase of around 5% of total consumption for the representative agent. Although 5% is not a huge increase in consumptions, it is worth a fair amount in terms of welfare because the marginal utility if $y$ is high due to the search, bargaining and credit frictions in our KM market. Of course, these kind of welfare numbers miss potentially important distributional effects, like those as discussed in Kiyotaki et al. (2011), that the theory was not designed to capture. Still, financial innovation is beneficial for the representative agent, even if some individuals had the misfortune of buying high and selling low over the boom-and-bust cycle.

To summarize, we can generate a boom-and-bust pattern in house prices, along with construction and HEL, in three ways. (i) Parameters may be such that there are multiple transitions after an increase in $D_1$. In Experiment 1, the economy sets out on one path, then jumps at some $\hat{t}$ to another, with a lower $\psi_{\hat{t}}$, before converging to the new steady state. However, this required fairly extreme parameters. (ii) One
can calibrate the model to more reasonable parameter values, as in Experiment 2, and ask what happens during the (unique) transition as \( D_1 \) gradually changes. Prices rise initially, then fall, due to a variety of effects, but a key factor is beliefs about the path of \( D_1 \) at the end of the period. If agents believe \( D_1 \) will go down, this can cause \( \psi \) to fall, as in Experiment 2a. (iii) With the same taste and technology parameters, if they believe \( D_1 \) will go up instead of down, \( \psi \) can again fall, as in Experiment 2b, although that requires a fairly big change in credit conditions.

We do not propose a definitive answer to the question “what happened?” The objective was to demonstrate that there are different ways to generate a boom-and-bust housing cycle. But if one wants something more concrete, one thing we can say is that it is difficult to explain more than half the price runup using only changes in LTE ratios. One can even say that we tried hard to generate a big boom based on financial innovation, but only delivered 50% of the empirical house-price runup. This suggests that self-fulfilling prophecies might well be piece of the puzzle, too, which we think is an interesting conclusion.

8 Money and Banking

We now introduce fiat money.\(^{19}\) In reality, people do not usually use home equity to buy on credit directly from retailers, but use it to borrow from banks. In fact, they usually borrow money from banks, and use that to buy goods, at least if one interprets this broadly: your banker credits your account for the amount of the loan, from which you can withdraw cash, write a check, or swipe a debit card to make

\(^{19}\)Aruoba et al. (2013) also have fiat money in a housing model – but the economics is completely different for several reasons: (i) that paper does not have home equity lending; (ii) it focuses on the period before the boom and bust of the 2000’s; (iii) housing and money in that model are substitutes, heuristically speaking, rather than complements, as they are here. As regards this last point, everything in that model is driven by agents substituting between home and market activity with changes in inflation, which taxes the latter and not the former. Despite superficial similarities in the models, that effect has nothing whatsoever to do with the results here.
purchases. Of course you may also be able to use a credit card, or unsecured debt, but this has a limit as captured in the model by \( D_0 \). Cash may be less desirable than using a check or debit card for a variety of reasons, including exposure to theft (as in He et al. 2008 or Sanches and Williamson 2010), that are not modeled here. What is modeled is that households get money, broadly defined, from banks and use that for retail trade. For simplicity, we consider for now only the case where \( H \) and \( D_0 \) are fixed.

First, as seems reasonable, assume retailers do not know the identities of customers asking for store credit. Also assume that customers can produce claims on nonexistent houses, ones belonging to someone else, or ones that are under water. Then retailers are not inclined to accept such claims, either as a means of payment or as collateral for store credit.\(^{20}\) Assume further that bankers are good, or at least have a comparative advantage over retailers, at checking the authenticity and desirability of real estate claims. This allows home equity to serve as collateral for cash loans from bank to households, who then use the cash to pay retailers. With cash serving as the retail means of payment, we change the label of that market from KM to KW (Kiyotaki-Wright replacing Kiyotaki-Moore). More significantly, the economy now has both inside and outside liquidity, \( H \) and \( M \), and we can now discuss monetary policy.

The timing works as follows. Before trading in the KW market, but after the \( \alpha \) shocks are known, households have access to cash on hand, and can access a new market labeled DD for Diamond and Dybvig (1983). In DD they can withdraw or deposit cash. Banking is competitive, but there is still limited commitment: households can renege on bank loans, and so home equity is required as collateral. It is not important, and cannot be determined, who actually carries money out of

\(^{20}\)See Lester et al. (2012) and references therein for formalizations of an information-based theory of acceptability, including versions where information acquisition is endogenous.
AD, as it can always be reallocated in DD. Hence, assume all cash is deposited in banks at the end of the AD market. Then in DD, those that want to consume in KW withdraw, generally more than their deposits, while those that do not want to consume leave their deposits alone.

The AD value function is now \( W_t (d_t, h_t, m_t) \), where the state lists debt, housing, and money in the bank. The DD value function \( J_{t+1} (h_{t+1}, m_{t+1}) \) depends on housing and cash only, since debt is settled in AD. Then

\[
W_t (d_t, h_t, m_t) = \max_{x_t, \ell_t, h_{t+1}, m_{t+1}} \{ U_t (x_t, h_t) - \ell_t + \beta J_{t+1} (h_{t+1}, m_{t+1}) \}
\]

\[
\text{st } x_t + \psi_t h_{t+1} + \phi_t m_{t+1} = \ell_t + \psi_t h_t + T_t + \phi_t m_t - d_t,
\]

where \( \phi_t \) is the value of a dollar in terms of numeraire. The FOC’s are

\[
U_t (x_t, h_t) = 1, \quad \psi_t = \beta \partial J_{t+1} / \partial h_{t+1} \quad \text{and} \quad \phi_t = \beta \partial J_{t+1} / \partial m_{t+1},
\]

so \((x_t, h_{t+1}, m_{t+1})\) is independent of \((d_t, h_t, m_t)\), and \(W_t\) is linear in wealth.

The DD value function satisfies

\[
J_t (h_t, m_t) = \alpha \max_{\hat{m}_t} V_t [(1 + \rho_t) (\hat{m}_t - m_t) \phi_t, h_t, \hat{m}_t] + (1 - \alpha) W_t [- (1 + \rho_t) m_t \phi_t, h_t, 0]
\]

\[
\text{st } (1 + \rho_t) (\hat{m}_t - m_t) \phi_t \leq D (\psi_t h_t),
\]

where \( \rho_t \) is the bank interest rate, \( D (\psi_t h_t) \) is the limit on borrowing from the bank, and \( V_t \) is the KW value function conditional on wanting to consume. In words, with probability \( \alpha \) households increase \( m_t \) to \( \hat{m}_t \), spend it in KW, and incur a real obligation \((1 + \rho_t) (\hat{m}_t - m_t) \phi_t \); and with probability \( 1 - \alpha \) they leave their money in the bank, skip the KW market, and enter the next AD market with a negative obligation (a positive bank balance), \(-(1 + \rho_t) m_t \phi_t \).

The outcome depends on whether the DD debt limit, \((1 + \rho_t) (\hat{m}_t - m_t) \phi_t \leq D (\psi_t h_t)\), binds. In Case 1 it does not bind, and the FOC for \( \hat{m} \) is \( \partial V_t / \partial \hat{m}_t = \ldots \)
$(1 + \rho_t) \phi_t$, which reduces to $\lambda (y_t) = \rho_t$. The Euler equation for $m$ is

$$(1 + r) \phi_t = (1 + \rho_{t+1}) \phi_{t+1}. \quad (22)$$

Let $i_t$ be the nominal interest rate that makes an agent willing to give up a dollar in AD at $t$ for $1 + i_t$ dollars in AD at $t + 1$. The Fisher equation is $1 + i_t = \phi_t / \beta \phi_{t+1}$, since $\phi_t / \phi_{t+1} = 1 + \pi_t$ is inflation and $1 / \beta = 1 + r_t$ is the real interest rate across AD markets. Then (22) says $\rho_{t+1} = i$. The Euler equation for $h$ is

$$(1 + r) \psi_t = U_2 (x_{t+1}, h_{t+1}) + \psi_{t+1}. \quad (23)$$

As in the baseline model, when the debt limit is slack, housing is priced fundamentally.

In Case 2 the DD debt limit binds, so borrowers go to the limit, and

$$J_t (h_t, m_t) = \alpha V_t \left[ D (\psi_t h_t), h_t, m_t + \frac{D (\psi_t h_t)}{(1 + \rho_t) \phi_t} \right] + (1 - \alpha) W_t [- (1 + \rho_t) \phi_t m_t, h_t, 0]$$

In this case the Euler equations for $m$ and $h$ are

$$(1 + r) \phi_t = \alpha [\lambda (y_{t+1}) + 1] \phi_{t+1} + (1 - \alpha) (1 + \rho_{t+1}) \phi_{t+1} \quad (24)$$

$$(1 + r) \psi_t = U_2 (x_{t+1}, h_{t+1}) + \psi_{t+1} + \alpha \frac{D_1 \psi_{t+1}}{1 + \rho_{t+1}} [\lambda (y_{t+1}) - \rho_{t+1}] . \quad (25)$$

Compared to (23), the liquidity premium appears in (24). There are two subcases.

In Case 2a, bank lending exhausts deposits, and $\rho_{t+1} > 0$. This yields $g (y_{t+1}) = D (\psi_{t+1} h_{t+1}) / (1 - \alpha)$. In Case 2b, when all borrowers go to the limit there is vault cash left over, and $\rho_{t+1} = 0$. Then the Euler equation for $h$ is

$$(1 + r) \psi_t = \psi_{t+1} + U_2 (x_{t+1}, h_{t+1}) + \alpha D_1 \psi_{t+1} i. \quad (26)$$

Two conditions determine which case obtains. One is the individual debt limit, with liquidity bearing a premium iff it binds. The other is an aggregate condition:
if there are more deposits than borrowers can borrow, given the debt limit, then \( \rho = 0 \); otherwise \( \rho > 0 \). Hence, there are three possibilities: in Case 1, aggregate and individual limits are slack; in Case 2a, the individual limit binds but the aggregate is slack; and in Case 2b, both bind (the fourth case cannot happen). For a given inflation rate, Figure 10 partitions \((D_0, D_1)\) space into regions using two curves, \( D_1 = B_1(D_0) \) and \( D_1 = B_2(D_0) \), derived in Appendix D. For large \( D_0 \) and \( D_1 \) debt limits do not bind. As \( D_0 \) and \( D_1 \) decrease, Case 2a emerges with \( \rho \in (0, i) \). As \( D_0 \) and \( D_1 \) decrease further, Case 2b emerges, where debt restrictions are so tight banks cannot lend all their deposits, and \( \rho = 0 \).

![Figure 10: Regions of Different Monetary Equilibria](image)

Going beyond steady state, it seems clear that one can generate interesting dynamics in this model, given the results for the nonmonetary economy. We defer this to future work, and for now focus on the impact on steady state of monetary policy in terms inflation. The next result is proved in Appendix D.

**Proposition 10** There is a unique steady state, which can be one of three regimes:

1. \( B_2(D_0) < D_1 \Rightarrow \rho = i, \psi = \psi^* \) and \( \partial \psi / \partial \pi = 0 \);
2a. $B_1(D_0) < D_1 < B_2(D_0) \Rightarrow \rho \in (0, i), \psi > \psi^* \text{ and } \partial \psi / \partial \pi < 0 \iff F'(y) > 0,$ where $F(y) \equiv \frac{u'(y) g(y)}{g'(y)}$;

2b. $B_1(D_0) > D_1 \Rightarrow \rho = 0, \psi = U_2 / (r - iD_1) \text{ and } \partial \psi / \partial \pi > 0.$

With Walrasian pricing, or bargaining with $\theta \approx 1$, the condition $F'(y) > 0$ in Case 2a holds iff $R(y) \equiv -yu''(y)/u'(y) < 1$. Given the functional form for $u(y)$ in (9), e.g., with $\varepsilon > 0$ $R(y)$ is not constant, and Case 2a works like this: $\gamma < 1$ implies $\psi$ decreases with $\pi$; and $\gamma > 1$ implies $\psi$ first increases then decreases with $\pi$. An example with $\gamma < 1$ is shown Figure 11. In addition to $\psi$ (solid blue), it shows $\rho$ (dash red) and $y$ (dashdot green). Notice the kinks in the curves when we move across regimes: for high $\pi$ we get Case 1; for low $\pi$ we get Case 2b; and for intermediate $\pi$ we get Case 2a. Therefore, when debt limits are tight or inflation is low, in Case 2b, the model unambiguously generates a positive relationship between $\pi$ and home values. And in Case 2a, for slightly looser debt limits or higher inflation, it can generate a positive relationship at least for some parameters. This is all relevant because just such a relationship has been documented in the data.\(^{21}\)

Also notice how the nominal interest rate on bank deposits and loans, $\rho$, increases with $\pi$, but not one-for-one.\(^{22}\) Also notice that KW consumption is unambiguously decreasing in $\pi$ (one can show this analytically, in general, not only in examples). Finally, we think this exercise also sheds light on the following issue. Some people, e.g., Glaeser et al. (2010), investigate how much lower interest rates account for the

\(^{21}\)Aruoba et al. (2013) use various data sources for different time periods and countries, and conclude that a preponderance of evidence indicates appropriately deflated home values are increasing in inflation or nominal interest rates. A previous literature, including Kearl (1979), Follain (1982), Poterba (1991), Brunnermeier and Julliard (2008) and Piazessi and Schneider (2010), focuses on mortgages, money illusion, tax effects and the impact of inflation on the tradeoff between market and home production – ingredients that are absent here, and hence not driving our results.

\(^{22}\)Fisher’s theory of interest rates does not apply to deposits because they convey liquidity – they are spendable in the KW market. It does apply to illiquid assets, like claims to numeraire in the next AD market, traded in the current AD market, that are not spendable in KW.
house price boom, and find a relatively small effect. Presumably, it is taken for granted that lower interest rates stimulate demand. As Figure 11 indicates, when we model money and banking explicitly, it is not at all clear that lower nominal rates should raise house prices. The horizontal axis in Figure 11 shows inflation $\pi$, but one can add the real interest rate $r$ to convert this into the nominal rate $i$. This implies that $\psi$ is quite nonmonotone as $i$ varies over an empirically relevant range. In particular, for deflation near the Friedman rule, and hence nominal interest rates close to 0, our theory predicts house prices should decline with lower nominal interest rates.

9 Conclusion

Housing markets have long been thought by some to be prone to volatility (e.g., Case and Shiller 2003; Shiller 2011). We asked how far we could get trying to explain this volatility, without deviating from classical assumptions, like rational expectations and maximizing agents, by introducing one new twist: liquidity. Housing conveys
liquidity because it relaxes intertemporal trading frictions. As in some other models with liquid assets, there can be equilibria where prices follow cyclic, chaotic and stochastic trajectories even with constant fundamentals. We also constructed deterministic and stochastic equilibria that resemble bubbles in housing markets, with several periods of price increase followed by collapse, and emphasized that there is no problem having recurrent episodes with such patterns. And we considered various generalizations, including different mechanisms for determining the terms of trade, adding money and banking, and combining unsecured plus secured credit.

In terms of substantive results, we considered several controlled experiments. Experiment 1 showed that something that looks qualitatively somewhat like the data in Figure 1 can be generated as a self-fulfilling prophecy based on multiple equilibrium transition paths, although that required unrealistic parameter values. Experiments 2a and 2b showed we can get something that qualitatively looks even more like the data based on gradual financial innovation changing LTE ratios, but quantitatively we explain no more than about 1/2 of the price run up. Should we be disappointed that the theory cannot explain more? Not at all. These types of experiments are designed to answer questions like “how much of the observations can be accounted for by this factor” without “hoping” the answer is close to “all of it.” The fact that we were able to generate no more than 1/2 of the price runup in the data is a strong positive result. It strongly suggests that, while financial innovation is quantitatively relevant, there is something else going on, too. We think it is likely not self-fulfilling prophecies based on multiple equilibria, since we had trouble getting that to work without unreasonable parameters. But it could have to do with sunspots and other endogenous dynamics, such as those presented in Section 4.

In terms of extensions, one could incorporate other pledgeable assets, but that
would not change the basic message. Suppose agents have portfolios \((h,m,k,...)\) of houses, money, capital ... that they choose based on liquidity, return, risk ... Such a model can have endogenous dynamics, for the same reasons discussed above. Importantly, when it becomes easier to use \(h\) as collateral, at the margin, the demand for \(h\) and hence the price of \(h\) is still affected (Venkateswarany and Wright 2013).

Another extension is to allow heterogeneous households. In particular, it may be interesting to develop a life-cycle version of the model, which would be more appropriate for analyzing mortgages, and to incorporate other frictions in the housing market. Also, while there is no default on the equilibrium path in baseline Kiyotaki-Moore or Kehoe-Levine models, one can add informational frictions to change that (as in, e.g., Williamson 2011). Future work could also consider how the model responds to other shocks, perhaps at higher frequencies, as in standard business-cycle analysis. The goal here was to study responses to financial developments, and to demonstrate how housing markets can generate interesting dynamics even with no exogenous shocks.
Appendix A: In Figure 1, Home Prices are given by the FHFA Purchase Only price index, which starts in 1991. To turn it into a real variable, we divide by the CPI, or by the Owners’ Equivalent Rent of Residence index from BLS, with the real series normalized to 1 in 1991. The GDP and Residential Investment data are from BEA. The Current-Dollar Investment in Residential Fixed Assets is divided by nominal GDP. The Chain-Type Quantity Indexes for Investment in Residential Fixed Assets, which uses relevant price indexes for individual components to deflate the nominal series, is divided by real GDP. Since these price indexes are in general different from the GDP deflator, the two series can look quite different. Loan data are from the Federal Reserve Flow of Funds Accounts. Home-equity loans are divided by nominal GDP and by Home Equity, with the resulting series normalized to 0.3 in 1991. Home Equity is measured by first subtracting home-equity loans from mortgage loans to get closed-end mortgages, and then subtracting that from Market Value of Homes, which includes the value of land, as constructed by Davis and Heathcote (2007).

Appendix B (not necessarily for publication): Consider the following game from Wright and Wong (2013), with a general convex cost function \( v(y) \), since the results may be of interest in other applications where \( v(y) = y \) is less natural.

Initial Offer Stage: The seller offers \( (y_t, d_t) \in [0, \infty) \times [0, D_t] \), and then:

- if the buyer accepts, they trade at these terms;
- if the buyer rejects, they go to the next stage.

Final Offer Stage: There is a coin toss such that:

- with probability \( \theta \) the buyer makes a take-it-or-leave-it offer;
- with probability \( 1 - \theta \) the seller makes a take-it-or-leave-it offer.

There is a unique SPE characterized by acceptance of the initial offer. The initial offer is

\[
(y_t, d_t) = \arg \max \ S_{st} \ st \ S_{st} = \theta \left[ u(\bar{y}_t) - \bar{d}_t \right] \ \text{and} \ d \leq D_t,
\]

where \( (\bar{y}_t, \bar{d}_t) \) is the offer a buyer would make off the equilibrium path at the final stage if he won the coin flip. It is easy to see that:

\[
\bar{y} = \begin{cases} 
  v^{-1}(D_t) & \text{if } D_t < v(y^*) \\
  y^* & \text{if } D_t \geq v(y^*)
\end{cases}
\]

and

\[
\bar{d} = \begin{cases} 
  D_t & \text{if } D_t < v(y^*) \\
  v(y^*) & \text{if } D_t \geq v(y^*)
\end{cases}
\]
There are three possibilities: in Case 1 the constraint \( d \leq D \) is slack at both the initial and the final offer; in Case 2 it binds in the initial but not final offer; and in Case 3 it binds in both (it is east to see that it cannot bind in the final but not the initial offer).

Case 1: In the final offer stage, if the buyer proposes, his solves \( \max_y \{ u (y) - d \} \) \( \text{st } d = v (y) \), with solution \( y = y^* \) and \( d = v (y^*) \). If the seller proposes the buyer gets no surplus. So the buyer’s expected surplus before the coin flip is \( \theta [u (y^*) - v (y^*)] \). In the initial offer stage, the seller’s problem is \( \max_y \{ d - v (y) \} \) \( \text{st } u (y) - d = \theta [u (y^*) - v (y^*)] \), with solution \( y = y^* \) and \( d = d^* = (1 - \theta) u (y^*) + \theta v (y^*) \). Since \( d^* > v (y^*) \), this case occurs iff \( D > d^* \).

Case 2: The buyer’s expected payoff before the coin flip is again \( \theta [u (y^*) - v (y^*)] \), but at the initial offer stage the constraint binds, so the seller solves \( \max_y \{ D - v (y) \} \) \( \text{st } u (y) - D = \theta [u (y^*) - v (y^*)] \). This implies \( u (y) = D + \theta [u (y^*) - v (y^*)] \) and \( d = D \). This case occurs iff \( v (y^*) < D < d^* \).

Case 3: At the final offer stage, if the buyer proposes he solves \( \max_y \{ u (y) - D \} \) \( \text{st } D = v (y) \). This implies \( y = v^{-1} (D) \), so his expected surplus before the coin flip is \( \theta [u \circ v^{-1} (D) - D] \). At the initial offer stage, the seller’s problem is \( \max_y \{ D - v (y) \} \) \( \text{st } u (y) - D = \theta [u \circ v^{-1} (D) - D] \). The solution satisfies \( u (y) = \theta u \circ v^{-1} (D) + (1 - \theta) D \) and \( d = D \). This case occurs iff \( D < v (y^*) \) and \( D < u (y^*) - \theta u \circ v^{-1} (D) + \theta D \), the last inequality coming from the observation that, at the first stage, if the constraint is slack the buyer pays \( u (y^*) - \theta u \circ v^{-1} (D) + \theta D \) to get \( y^* \). This last inequality is equivalent to \( (1 - \theta) D < u (y^*) - \theta u \circ v^{-1} (D) \), which always holds if \( D < v (y^*) \).

To summarize, \( d = D \) if \( D < d^* \) and \( d = d^* \) if \( d^* \leq D \); and \( y \) is given by

\[
y = \begin{cases} 
  u^{-1} [\theta u \circ v^{-1} (D) + (1 - \theta) D] & \text{if } D < v (y^*) \\
  u^{-1} [D + \theta [u (y^*) - v (y^*)]] & \text{if } v (y^*) < D < d^* \\
  y^* & \text{if } D > d^*. 
\end{cases}
\]

One can check \( y = g^{-1} (D) \) is differentiable and strictly increasing for \( D < d^* \).

**Appendix C** (not necessarily for publication): Here we analyze the Kehoe-Levine secured debt limit when defaulters are denied access to all future credit. As in the text, set \( U (x, h) = \tilde{U} (x) + h \), so defaulters choose \( h_{t+1} = 0 \), and normalize \( \max \{ \tilde{U} (x) - x \} = 0 \). Then

\[
W_t^E (h_t, d_t) = -d_t + h_t + \psi_t (h_{t+1}^E - h_t) + \beta V_{t+1}^E (h_{t+1}^E),
\]

(28)
where superscript $E$ denotes equilibrium values. Assume Walrasian pricing, and let $Y(\bar{D}_{t+1}, \psi_{t+1}h_{t+1}) = \max \{y^*, \bar{D}_{t+1} + D_t \psi_{t+1}h_{t+1}\}$. Then

$$V^E_{t+1}(h_{t+1}) = \alpha [u \circ Y(\bar{D}_{t+1}, \psi_{t+1}h_{t+1}) - Y(\bar{D}_{t+1}, \psi_{t+1}h_{t+1})] + W^E_{t+1}(h_{t+1}, 0). \quad (29)$$

The punishment payoffs are $V^P_t = W^P_t = 0$. The repayment constraint for an agent in AD with $(h_t, d_t)$ is therefore

$$- d_t - \psi_t (h^E_{t+1} - h_t) + \beta V^E_{t+1}(h^E_{t+1}) \geq \psi_t h_t (1 - D_t). \quad (30)$$

The LHS is the value of paying off $d_t$, adjusting $h_t$, and continuing in good standing; the RHS is the value of the assets that cannot be seized $(1 - D_t) h_t$, plus continuing with the punishment payoff. To find the maximum unsecured debt an agent would honor at $t$, substitute $d_t = \bar{D}_t + D_t \psi_t h_t$ into (30) with equality, and rearrange to get

$$\beta V^E_{t+1}(h^E_{t+1}) = \bar{D}_t + \psi_t h^E_{t+1}. \quad (31)$$

As in Gu et al. (2012), combine (28) and (29) to get a recursive equation in $V^E_t(h_t)$,

$$V^E_t(h_t) = \alpha [u \circ Y(\bar{D}_t, \psi_t h_t) - Y(\bar{D}_t, \psi_t h_t)] + h_t - \psi_t (h^E_{t+1} - h_t) + \beta V^E_{t+1}(h^E_{t+1}).$$

Now substitute (31), use market clearing $h^E_t = 1$, and rearrange to get a recursive expression in $\bar{D}_t$,

$$\bar{D}_{t-1} = \beta \bar{D}_t + \beta \alpha [u \circ Y(\bar{D}_t, \psi_t) - Y(\bar{D}_t, \psi_t)] + \beta + \beta \psi_t - \psi_{t-1},$$

which is (12) in the text. And, as usual, the Euler equation yields (13). ■

**Appendix D** (not necessarily for publication): Here we verify the results in Proposition 10 and derive

$$B_1(D_0) = \begin{cases} 
\frac{r [g(\tilde{\gamma})(1-\alpha) - D_0]}{ig(\bar{\gamma})(1-\alpha) - iD_0 + HU_2}, & \text{if } D_0 < g(\bar{\gamma})(1-\alpha) \\
0, & \text{if } D_0 > g(\bar{\gamma})(1-\alpha)
\end{cases} \quad (32)$$

$$B_2(D_0) = \max \left\{ \frac{r [g(\tilde{\gamma})(1-\alpha)(1+i) - D_0]}{HU_2}, 0 \right\} \quad (33)$$

where $\tilde{\gamma}$ and $\bar{\gamma}$ satisfy $\lambda(\tilde{\gamma}) = i/\alpha$ and $\lambda(\bar{\gamma}) = i$.

In Case 1, when $d \leq D(e)$ is not binding, $i = \rho = \lambda(y)$, $r\psi = U_2[X(H), H]$ and

$$g(y) < \frac{D_0 + D_1 \psi H}{(1+\rho)(1-\alpha)}.$$
The inequality comes from two observations: when \( \rho > 0 \), to clear the DD market we need \( g(y) = \phi_t M_t / \alpha \); and when the debt limit is slack, we have \( (1 - \alpha) \phi_t M_t < \alpha (D_0 + D_1 \psi H) / (1 + \rho) \). This equilibrium exists if

\[
g(y) < \frac{D_0 + D_1 \psi H}{(1 - \alpha)(1 + \rho)},
\]

or \( D_1 > r[g(y)(1 + i)(1 - \alpha) - D_0] / H U_2 \). This can be written \( D_1 > B_2(D_0) \) with \( B_2(D_0) \) given by (33). Uniqueness, \( \partial \psi / \partial i = 0 \) and \( \partial y / \partial i < 0 \) follow immediately.

In Case 2, when the constraint is binding,

\[
i = \alpha \lambda(y) + (1 - \alpha) \rho \quad (34)
\]
\[
r\psi = \alpha [\lambda(y) - \rho] \frac{\psi D_1}{1 + \rho} + U_2 [X(H), H] \quad (35)
\]
\[
g(y) = \phi_t M_t + \frac{D_0 + D_1 \psi H}{1 + \rho}. \quad (36)
\]

In Case 2a, market clearing and a binding debt constraint imply

\[
\phi_t M_t = \frac{\alpha (D_0 + D_1 \psi h)}{(1 - \alpha)(1 + \rho)}. \quad (37)
\]

Using (34), we get \( \rho = (i - \alpha \lambda) / (1 - \alpha) \). This, (37) and (36) yield

\[
\psi = \frac{g(y)[1 + i - \alpha - \alpha \lambda(y)] - D_0}{D_1 H}.
\]

Substituting these into (35), we get

\[
\frac{r}{D_1} = \frac{\alpha[\lambda(y) - i]}{1 + i - \alpha - \alpha \lambda(y)} + \frac{H U_2 [X(H), H]}{g(y)[1 + i - \alpha - \alpha \lambda(y)] - D_0} \equiv \Phi(y). \quad (38)
\]

The RHS is decreasing in both \( i \) and \( y \), so there is only one solution for \( y \) and \( \partial y / \partial i < 0 \). Note that \( \rho < \lambda(y) \). This and (34) imply \( 0 < \rho < i \) and \( i < \lambda(y) \). Also,

\[
\frac{\partial \rho}{\partial i} \simeq -\frac{\alpha D_1 \psi^2 H \lambda(y)}{(1 + \rho)^2 (1 - \alpha)^2} + U_2 g'(y) > 0
\]
\[
\frac{\partial \psi}{\partial i} \simeq -\frac{\partial}{\partial y} \frac{u'(y)g(y)}{g'(y)},
\]

where \( A \simeq B \) means \( A \) and \( B \) have the same sign. Therefore, \( \partial \psi / \partial i < 0 \) iff \( u'(y)g(y) / g'(y) \) is increasing, which depends only on \( yu''(u) / u'(y) \) with Walrasian pricing or bargaining with \( \theta \approx 1 \). This equilibrium exists iff (38) has a solution in
(\bar{y}, \bar{y})$, where \( \lambda(\bar{y}) = i/\alpha \), which happens when \( \rho = 0 \) and \( \lambda(\bar{y}) = i \), which happens when the credit constraint is slack. Since \( \Phi(y) \) is decreasing in \( y \), this requires \( \Phi(\bar{y}) > r/D_1 \geq \Phi(\tilde{y}) \), where

\[
\frac{r}{D_1} > \frac{H U_2 [X(H), H]}{g(\bar{y})(1-\alpha)(1+i) - D_0} \equiv \Phi(\bar{y})
\]

\[
\frac{r}{D_1} < i + \frac{H U_2 [X(H), H]}{g(\bar{y})(1-\alpha) - D_0} \equiv \Phi(\bar{y})
\]

Summarizing, Case 2a obtains iff \( B_1(D_0) < D_1 < B_2(D_0) \). Finally, in Case 2b,

\[
i/\alpha = \lambda(y) \tag{39}
\]

\[
r \psi = i \psi D_1 + U_2 [X(H), H] \tag{40}
\]

\[
g(y) = \phi_t M_t + D_0 + D_1 \psi H > \frac{D(\psi H)}{1-\alpha} \tag{41}
\]

The last inequality holds because there idle cash in banks, \((1-\alpha)\phi_t M_t > \alpha(D_0 + D_1 \psi H) \). Now (39) determines \( y \) and (40) determines \( \psi \). This equilibrium exists iff (41) holds, which reduces to \( B_1(D_0) > D_1 \). It is obvious in this case that \( \partial y/\partial i < 0 \) and \( \partial \psi/\partial i > 0 \).

**Appendix E** (not for publication): Here we sketch very briefly the algorithms used for the results in Section 8, beginning with Experiment 1:

Step 1: Compute steady state when \( D_1 = 0 \) and \( D_1 = 1 \), say \((\psi^0, h^0)\) and \((\psi^1, h^1)\).

Step 2: When the \( D_1 \) shock hits, set \( \psi_t \) to some \( \psi > 0 \) and \( h_{t+1} = (e')^{-1}(\psi) + (1-\delta)h_0 \).

Step 3: Solve for the transition path with these initial conditions. Any path converging to \((\psi^1, h^1)\) is an equilibrium. If there is no convergent path, go back to Step 2.

The transition paths computed in this way are not unique when \((\psi^1, h^1)\) is a sink. We specify parameters such that under Walrasian pricing \( y \leq D(e) \), and both eigenvalues are real and less than 1, at \((\psi^1, h^1)\). Then we compute a number of such paths, and for the sake of illustration, pick one that resembles the data.

For Experiments 2a and 2b:
Step 1: Compute the starting $h$ and the ending $(\psi^*, h^*)$.

Step 2: Assume the economy converges after, say, 100 years.

Step 3: Guess a transition path for $\psi$, and from this compute the path for $h_{t+1}$.

Step 4: Update the transition path for $\psi_t$ using the computed $h_{t+1}$ sequence.

Step 5: Repeat Steps 3-4 until the transition paths converge.

We apply the algorithm, starting anew after each change in beliefs, and splice together the resulting paths. While obviously there is an issue constructing perfect-foresight equilibria with unanticipated changes in parameters, for the sake of illustration, this a reasonable approximation to a situation where there are some expected and some unexpected changes.
References


