Idiosyncratic risk and the equity premium*

A. Carvajal† H. Zhou‡

February 11, 2023

In an economy with aggregate risk, the equity premium is the difference between the expected return of a dollar invested in an asset bearing the same (distribution of) risk as the whole economy and the risk-free interest rate. The equity premium puzzle is the observation, first put forward by Mehra and Prescott (1985), that standard macroeconomic models with homogeneous agents and no idiosyncratic risk fail to explain the equity premia typically observed in the data.[1]

Shortly after, Mankiw (1986) presented a setting in which the presence of uninsurable idiosyncratic risk increases the equity premium predicted by homogeneous agent models. Furthermore, Constantinides and Duffie (1996) observed that this is indeed the case for CRRA preferences and counter-cyclically heteroskedastic idiosyncratic risk.[2] However, Krueger and Lustig (2010) showed that the same result does not hold when the representative agent has CRRA preferences and the distribution of idiosyncratic risk follows a particular form of pro-cyclical heteroskedasticity in a two-period economy. Under these assumptions, the equity premium is not affected by the presence of idiosyncratic risk. For the same class of preferences, Storesletten et al. (2007) had shown that in an OLG economy, the effect of idiosyncratic shocks in the equity premium is significantly smaller than in the two-period case considered by Constantinides and Duffie.

* This is *very* preliminary work. All comments and observations are welcome!

† University of California, Davis, and EPGE-FGV; acarvajal@ucdavis.edu

‡ Shanghai University of Finance and Economics; zhouhang@mail.sufe.edu.cn

[1] See also Mehra (2003).

[2] See also the empirical results in Cogley (2002).

This paper aims to further our understanding of the effect of idiosyncratic risk on the equity premium. We consider different classes of preferences and different co-variations between the idiosyncratic shocks' variance and the economy's aggregate income. For short-lived assets, such as those considered in Constantinides and Duffie (1996) and Krueger and Lustig (2010), we offer a complete characterization of the effect, relying on the cross-moments of different derivatives of the utility function and the aggregate income of the economy. For long-lived assets, such as those in Storesletten et al. (2007), a full characterization is elusive, but we present sufficient conditions for the reversal of the effect found by Constantinides and Duffie.

1 A Two-Period Setting

In a two-period economy, let non-degenerate random variable W represent the future wealth of an economy were all agents are identical, and let function $u : \mathbb{R}_+ \to \mathbb{R}$ be the agents' Bernoulli utility index and $\beta > 0$ their discount factor. Suppose that the support of W is a subset $\mathcal{W}$ of $\mathbb{R}_{++}$ and that u is $\mathbf{C}^3(\mathbb{R}_{++})$, strictly increasing and strictly concave and has non-negative third derivative.

Only one asset can be traded: "equity" in the economy, namely the asset that pays W in the second period.

1.1 *Benchmark: only aggregate risk*

In the absence of any other shocks, the present and future consumption of an individual in this economy are, respectively, $c = \bar{\omega} - q \cdot y$ and $C = W + W \cdot y$, where $\bar{\omega} > 0$ is each individual's wealth in the first period, q denotes the price of the asset and y is the quantity of the asset demanded by the individuals. The portfolio problem of the agents in this economy is, hence,

$$\max_y \left\{ u(\bar{\omega} - q \cdot y) + \beta \mathbb{E}[u(W + W \cdot y)] \right\}.$$

Since all agents are identical, only a no-trade equilibrium is possible and

$$q = \frac{\beta \mathbb{E}[u'(W) \cdot W]}{u'(\bar{\omega})}. \tag{1}$$

If we define the function $m : \mathbb{R}_{++} \to \mathbb{R}$, as $m(w) = \beta u'(w)/u'(\bar{w})$, this economy's *stochastic discount factor* is the random variable $m(W)$ and we can re-write Eq. (1) as $q = \mathbb{E}[m(W) \cdot W]$.

Using m for pricing other income flows, note that if the agents could also trade a risk-less asset with payoff $\mathbb{E}(W)$, its price would equal

$$\mathbb{E}[m(W) \cdot \mathbb{E}(W)] = \mathbb{E}[m(W)] \cdot \mathbb{E}(W) = \frac{\beta \mathbb{E}[u'(W)] \cdot \mathbb{E}(W)}{u'(\bar{w})}.$$

The *equity premium* measures how much more expensive this risk-less asset would be, namely the relative price of the risk-less to the risky asset (minus one). In the absence of any other risk, thus, the equity premium is

$$\bar{p} = \frac{\mathbb{E}[m(W) \cdot \mathbb{E}(W)]}{\mathbb{E}[m(W) \cdot W]} - 1 = \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(W) \cdot W\right]} - 1 = -\frac{\mathbb{C}\mathrm{ov}[u'(W), W]}{\mathbb{E}[u'(W) \cdot W]}, \tag{2}$$

where $\mathbb{C}\mathrm{ov}$ is the *covariance* operator.

Remark 1. *It is clear from Eq.* (2) *that the equity premium is independent of the discount factor β and the present marginal utility $u'(\bar{w})$, and does not depend on the assumption that the Bernoulli function is the same in both periods.*

Remark 2 (Selden preferences). *In fact, suppose that the preferences of the individual over present consumption, c, and future risky consumption, C, are represented by*

$$v(c) + \beta v(u^{-1}(\mathbb{E}[u(C)])), \tag{3}$$

where v and β capture the agent's impatience and u her attitude towards risk.

If we re-write the portfolio program, the first-order condition that replaces Eq. (1) *is that*

$$q = \frac{\beta v'(u^{-1}(\mathbb{E}[u(W)]))}{v'(\bar{w})} \frac{\mathbb{E}[u'(W) \cdot W]}{u'(u^{-1}(\mathbb{E}[u(W)]))}.$$

The stochastic discount factor now takes the form

$$m(w) = \frac{\beta v'(u^{-1}(\mathbb{E}[u(W)]))}{v'(\bar{w})} \frac{u'(w)}{u'(u^{-1}(\mathbb{E}[u(W)]))},$$

while the prices of the economy's equity and the risk-less bond are still given by $\mathbb{E}[m(W) \cdot W]$ and $\mathbb{E}[m(W)] \cdot E(W)$. Importantly, the constant

$$\frac{\beta v'(u^{-1}(\mathbb{E}[u(W)]))}{v'(\bar{w})} \frac{1}{u'(u^{-1}(\mathbb{E}[u(W)]))}$$

cancels out in the computation the relative price $E[m(W)] \cdot E(W)/\mathbb{E}[m(W) \cdot W]$, *so the equity premium is still given by Eq.* (2).

Conceptually, this observation simply says that the equity premium depends on the (homogeneous) traders' attitude towards risk and not in their impatience: while the individual's tastes on inter-temporal smoothing affect the prices of both the risky and the risk-less assets, their relative price depends *only* on the individual's attitude towards risk. For the purposes of this paper, it follows that all our results apply when the individuals preferences are of the form (3), where u is the Bernoulli function capturing their risk attitude.

1.2 *Idiosyncratic risk*

Let random variable S, with $\mathbb{E}(S \mid W) = 0$, be the agents' uninsurable, future idiosyncratic risk. The agents' future consumption is $C = W + S + W \cdot Y$ and, following the same logic as above, the equity premium is

$$p = \frac{\mathbb{E}\left[m(C)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[m(C) \cdot W\right]} - 1 = \frac{\mathbb{E}\left[u'(C)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(C) \cdot W\right]} - 1. \tag{4}$$

Iterating expectations, this is

$$p = \frac{\mathbb{E}\left\{\mathbb{E}\left[u'(W+S) \mid W\right]\right\} \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\mathbb{E}\left[u'(W+S) \mid W\right] \cdot W\right\}} - 1. \tag{5}$$

Remark 3. *Note from Eq.* (5) *that if the economy displays idiosyncratic risk, using Eq.* (2) *instead of Eq.* (4) *misspecifies the equity premium, as it amounts to assuming the equality*

$$\mathbb{E}\left[u'(W+S) \mid W\right] = u'\left(\mathbb{E}(W+S \mid W)\right),$$

which in general requires that the Bernoulli function be quadratic.

1.3 *Idiosyncratic risk and the equity premium*

Using the second-order expansion

$$u'(w+s) \approx u'(w) + u''(w) \cdot s + \tfrac{1}{2} \cdot u'''(w) \cdot s^2, \tag{6}$$

we get that

$$\mathbb{E}\left[u'(W+S) \mid W\right] \approx u'(W) + \tfrac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W), \tag{7}$$

where $\mathbb{V}$ is the variance operator. This allows us to approximate Eq. (5) by

$$\hat{p} = \frac{\mathbb{E}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \mathbb{V}(S \mid W)\right] \cdot W\right\}} - 1. \tag{8}$$

2 Two Important Examples: CRRA and CARA Preferences

Theorem 1 (Irrelevance, 1). *If one of the following two conditions holds, idiosyncratic risk does not affect the equity premium, in the sense that $\hat{p} = \bar{p}$:*

1. *the Bernoulli function is quadratic, or*
2. *the first derivative of the Bernoulli function is homogeneous and $\mathbb{V}(S \mid W) = \sigma W^2$ for some constant $\sigma \geq 0$, almost surely.*

Proof. That the first condition suffices is straightforward, as mentioned in Remark 3: under a quadratic function, $u'''(w) = 0$ for all w and the equality follows.

To see that the second condition also suffices, suppose that $u'(w) = u'(1)w^{-\rho}$, for some $\rho > 0$, and hence that $u'''(w) = \rho(\rho+1)u(1)w^{-(\rho+2)}$. Substituting this and the functional form of the conditional variance of S, we get

$$\begin{aligned}
\hat{p} &= \frac{\mathbb{E}\left[W^{-\rho} + \rho(\rho+1)\frac{\sigma}{2} \cdot W^{-(\rho+2)} \cdot W^2\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[W^{-\rho} + \rho(\rho+1)\frac{\sigma}{2} \cdot W^{-(\rho+2)} \cdot W^2\right] \cdot W\right\}} - 1 \\
&= \frac{\mathbb{E}\left\{\left[1 + \rho(\rho+1)\frac{\sigma}{2}\right] \cdot W^{-\rho}\right\} \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[1 + \rho(\rho+1)\frac{\sigma}{2}\right] \cdot W^{-(\rho-1)}\right\}} - 1 \\
&= \frac{\mathbb{E}[W^{-\rho}) \cdot \mathbb{E}(W)}{\mathbb{E}(W^{-(\rho-1)}]} - 1 \\
&= \bar{p}.
\end{aligned} \tag{9}$$

□

Remark 4. *The first case in the theorem generalizes Proposition 1 in Mankiw (1986). The second case encompasses the whole class of CRRA Bernoulli functions: all homogeneous functions (the case explicitly solved in Krueger and Lustig, 2010) as well as all the logarithmic functions.*

These two cases where one obtains irrelevance, however, appear rather limiting. Quadratic preferences are the least prudent version of prudent preferences, while the assumption that the volatility of idiosyncratic shocks is pro-cyclical and follows that specific functional may be unsatisfactory, in that it requires the assumption that the relative shock S/W be independent of W.

On the other hand:

THEOREM 2 (Relevance, 1). *Suppose that the Bernoulli function is exponential, $u(w) = -e^{-\alpha w}$ for some $\alpha > 0$, and that for constants $\sigma \geq 0$ and γ, $\mathbb{V}(S \mid W) = \sigma W^{\kappa}$ almost surely. The equity premium is larger, equal, or smaller in the presence of idiosyncratic risk depending on whether idiosyncratic is is counter-, a-, or pro-cyclical. That is, $\hat{p} \gtreqless \bar{p} \Leftrightarrow \gamma \lesseqgtr 0$.*

Proof. By direct computation, for this Bernoulli function

$$\hat{p} = \frac{\mathbb{E}\left\{e^{-\alpha W} \cdot \left[1 + \frac{1}{2}\alpha^2 \cdot \mathbb{V}(S \mid W)\right]\right\} \cdot \mathbb{E}(W)}{\mathbb{E}\left\{e^{-\alpha W} \cdot \left[1 + \frac{1}{2}\alpha^2 \cdot \mathbb{V}(S \mid W)\right] \cdot W\right\}} - 1$$

and

$$\bar{p} = \frac{\mathbb{E}\left(e^{-\alpha W}\right) \cdot \mathbb{E}(W)}{\mathbb{E}\left(e^{-\alpha W} \cdot W\right)} - 1.$$

Whether $\hat{p}$ is larger or smaller than $\bar{p}$ depends thus on the sign of

$$\mathbb{E}\left[e^{-\alpha W} \cdot \mathbb{V}(S \mid W)\right] \cdot \mathbb{E}\left(e^{-\alpha W} \cdot W\right) - \mathbb{E}\left(e^{-\alpha W}\right) \cdot \mathbb{E}\left[e^{-\alpha W} \cdot \mathbb{V}(S \mid W) \cdot W\right]. \tag{10}$$

Substituting $\mathbb{V}(S \mid W) = \sigma W^{\gamma}$ into Eq. (10), we need to determine the sign of

$$\mathbb{E}\left(e^{-\alpha W} \cdot W^{\gamma}\right) \cdot \mathbb{E}\left(e^{-\alpha W} \cdot W\right) - \mathbb{E}\left(e^{-\alpha W}\right) \mathbb{E}\left(e^{-\alpha W} \cdot W^{\gamma+1}\right).$$

If we let V be an (ancillary) random variable distributed identically to W and independent from it, we can rewrite the latter expression as $\mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^{\gamma} \cdot (V - W)\right]$, which is proportional, by a factor of $1/2 \Pr(V \neq W) > 0$, to

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^{\gamma} \cdot (V - W) \mid V > W\right] + \mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^{\gamma} \cdot (V - W) \mid V < W\right].$$

This expression is equivalent to

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot W^{\gamma} \cdot (V - W) \mid V > W\right] + \mathbb{E}\left[e^{-\alpha(V+W)} \cdot V^{\gamma} \cdot (W - V) \mid W < V\right],$$

which, by direct computation, is

$$\mathbb{E}\left[e^{-\alpha(W+V)} \cdot (W^{\gamma} - V^{\gamma}) \cdot (V - W) \mid V > W\right].$$

This number is positive, null, or negative, depending on whether γ is negative, null, or positive. □

Remark 5. *This theorem covers all CARA Bernoulli functions.*

A comparison of the previous two theorems suggests a connection between the behavior of the coefficients of risk aversion, the behavior of the conditional variance of the idiosyncratic shocks, and the effect of the latter on the equity premium. As is well known, the relative risk aversion coefficient approximates the agents' willingness to pay to insure against *multiplicative* shocks of variance 2. The second statement in Theorem 1 proves that when such willingness to pay is constant on the agents' wealth, if the variance of the multiplicative idiosyncratic shock is also constant, then the shock has no effect on the premium.

The absolute risk aversion coefficient, on the other hand, approximates the willingness to pay to insure against *additive* shocks of variance 2, and Theorem 2 suggests that when such willingness to pay is constant, the equity premium changes with the presence of additive idiosyncratic risk unless the conditional variance of such risk is constant.

Whether, conditional on aggregate wealth, the idiosyncratic risk is homoskedastic or heteroskedastic is an empirical question and we take no position about it. The irrelevance result of Krueger and Lustig (and our minimal extension in Theorem 1) requires a very particular form of heteroskedasticity, as the following theorem shows.

Theorem 3 (Relevance, 2). *Suppose that the first derivative of the Bernoulli function is homogeneous, and that for some constants $\sigma \geq 0$ and γ, $\mathbb{V}(S \mid W) = \sigma W^{\gamma}$ almost surely. Whether the equity premium is larger or smaller in the presence of idiosyncratic risk, depends on whether γ is smaller or larger than 2. That is, $\hat{p} \gtreqless \bar{p} \Leftrightarrow \gamma \lesseqgtr 2$.*

Proof. As in the proof of Theorem 1, assume that $u'(w) = u'(1)w^{-\rho}$, for some $\rho > 0$. Substituting the functional form of the conditional variance of S, we get, instead of Eq. (9),

$$\hat{p} = \frac{\mathbb{E}\left\{\left[1+\rho(\rho+1)\frac{\sigma}{2}\cdot W^{\gamma-2}\right]\cdot W^{-\rho}\right\}\cdot\mathbb{E}(W)}{\mathbb{E}\left\{\left[1+\rho(\rho+1)\frac{\sigma}{2}\cdot W^{\gamma-2}\right]\cdot W^{-(\rho-1)}\right\}} - 1,$$

while

$$\bar{p} = \frac{\mathbb{E}(W^{-\rho})\cdot\mathbb{E}(W)}{\mathbb{E}[W^{-(\rho-1)}]} - 1.$$

We need to show that $\gamma < 2$ is necessary and sufficient for

$$\mathbb{E}\left\{[1+\Delta\cdot W^{\gamma-2}]\cdot W^{-\rho}\right\}\cdot\mathbb{E}[W^{-(\rho-1)}] > \mathbb{E}(W^{-\rho})\cdot\mathbb{E}\left\{[1+\Delta\cdot W^{\gamma-2}]\cdot W^{-(\rho-1)}\right\},$$

where $\Delta = \rho(\rho+1)\sigma/2 > 0$.

Let us define random variable V as in the proof of Theorem 2. By direct computation, we need to argue that

$$\mathbb{E}\left[\Delta\cdot W^{-\rho}\cdot V^{-\rho+1}\cdot(W^{\gamma-2}-V^{\gamma-2})\right] > 0.$$

Using the same technique as in the proof of Theorem 2, the left-hand side of this expression is directly proportional, by a factor of $\Pr(W > V)$, to

$$\mathbb{E}\left[\Delta\cdot W^{-\rho}\cdot V^{-\rho}\cdot(V-W)\cdot(W^{\gamma-2}-V^{\gamma-2}) \mid W > V\right].$$

This expression is positive if, and only if, $\gamma < 2$. □

For CRRA Bernoulli functions, the power functional form of the conditional variance is mathematically useful in Theorem 1, and we conjecture that the irrelevance cannot be guaranteed without it. More importantly, however, the assumption that the conditional variance is pro-cyclical implies that idiosyncratic risk is not as significant when an economy is in a recession as when it is in a boom, which may be untenable.

For CARA functions, when the idiosyncratic risk is counter-cyclical (and, again, of a particular functional form), its presence increases the equity premium. If in this case the heteroskedasticity is in the direction assumed by Krueger and Lustig (2010), the equity premium is *lower* in the presence of idiosyncratic risk.

3 General Preferences

If we concentrate in the case where the idiosyncratic shock is homoskedastic, we get the following general result:

Theorem 4 (Relevance, 3). *Suppose that $u''' > 0$, and that for some constant $\sigma > 0$, $\mathbb{V}(S \mid W) = \sigma^2$ almost surely. The equity premium $\hat{p}$ ranges monotonically from*

$$\lim_{\sigma \to 0} \hat{p} = \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(W) \cdot W\right]} - 1 = \bar{p},$$

to

$$\lim_{\sigma \to \infty} \hat{p} = \frac{\mathbb{E}\left[u'''(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'''(W) \cdot W\right]} - 1.$$

This monotonicity is increasing if, and only if,

$$\frac{\mathbb{C}\text{ov}[u'(W), W]}{\mathbb{E}[u'(W)]} > \frac{\mathbb{C}\text{ov}[u'''(W), W]}{\mathbb{E}[u'''(W)]}. \tag{11}$$

Proof. Under the assumptions of the theorem, Eq. (8) rewrites as

$$\hat{p} = \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W) + \frac{1}{2} \cdot \mathbb{E}\left[u'''(W)\right] \cdot \mathbb{E}(W) \cdot \sigma^2}{\mathbb{E}\left[u'(W) \cdot W\right] + \frac{1}{2} \cdot \mathbb{E}\left[u'''(W) \cdot W\right] \cdot \sigma^2} - 1.$$

The two limits follow, thus, by direct computation.

Since $\sigma > 0$, note that, $\hat{p}$ is increasing in σ if, and only if,

$$\mathbb{E}[u'''(W)] \cdot \mathbb{E}(W) \cdot \mathbb{E}\left[u'(W) \cdot W\right] > \mathbb{E}[u'(W)] \cdot \mathbb{E}(W) \cdot \mathbb{E}\left[u'''(W) \cdot W\right].$$

By monotonicity and (strict) prudence, and since W takes only positive values, and we can rewrite the expression as

$$\frac{\mathbb{E}\left[u'''(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'''(W) \cdot W\right]} - 1 > \frac{\mathbb{E}\left[u'(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u'(W) \cdot W\right]} - 1 = \bar{p},$$

which is equivalent to Eq. (11). □

The denominators on both sides of Eq. (11) are positive, and risk aversion implies that the numerator on the left-hand side is negative, so the ratio on the left-hand side is negative. None of our assumptions pins down the sign of the numerator on the right-hand side, though. If the covariance between the economy's aggregate income and the third derivative of the Bernoulli function is positive, then the variance of the idiosyncratic shock *decreases* the equity premium.

Corollary 1. *Suppose that* $\mathbb{V}(S \mid W) = \sigma^2 > 0$ *almost surely.*

1. *If the first derivative of the Bernoulli function is homogeneous,* $\hat{p}$ *is increasing in* σ.
2. *If the Bernoulli function is exponential,* $\hat{p}$ *does not depend on* σ.

As before, the first claim applies to the class of CRRA Bernoulli functions. It confirms that the irrelevance result in Krueger and Lustig (2010) depends on the specific type of heteroskedasticity that they assume for the idiosyncratic shock. If S is homoskedastic conditional on W, its presence implies a higher equity premium. The second claim provides another irrelevance result which covers the class of CARA functions for the homoskedastic case, as expected.

4 Higher-Order Moments

The empirical work of Guvenen (2016) highlights the importance of higher moments of the distribution of idiosyncratic shocks, in particular its negative skewness and high kurtosis. To introduce these considerations, suppose that in addition to the assumptions on the first two moments, namely that $\mathbb{E}(S \mid W) = 0$ and $\mathbb{V}(S \mid W) = \sigma^2 > 0$ almost surely, one further knows that $\mathbb{E}(S^3 \mid W) = \sigma^3 \cdot \delta < 0$ and $\mathbb{E}(S^4 \mid W) = \sigma^4 \cdot \kappa > 0$. In words, supposed that the idiosyncratic risk is homoskedastic and negatively skewed, and has constant kurtosis.

To consider, first, the case of the third moment, we can replace Eq. (6) with the more accurate, third-order approximation

$$u'(w+s) \approx u'(w) + u''(w) \cdot s + \tfrac{1}{2} \cdot u'''(w)s^2 + \tfrac{1}{6} \cdot u^{[4]}(w) \cdot s^3,$$

and Eq. (7) with

$$\mathbb{E}\left[u'(W+S) \mid W\right] \approx u'(W) + \tfrac{1}{2} \cdot u'''(W) \cdot \sigma^2 + \tfrac{1}{6} \cdot u^{[4]}(W) \cdot \sigma^3 \cdot \delta.$$

This results in the equity premium being approximated better by

$$\hat{p}_3 = \frac{\mathbb{E}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \sigma^2 + \frac{1}{6} \cdot u^{[4]}(W) \cdot \sigma^3 \cdot \delta\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \sigma^2 + \frac{1}{6} \cdot u^{[4]}(W) \cdot \sigma^3 \cdot \delta\right] \cdot W\right\}} - 1,$$

where we use the sub-index to denote the order of the approximation.[3]

Using the same technique as in the proof of Theorem 4, it is not difficult to show that if $u^{[4]} < 0$,[4] then $\hat{p}_3$ decreases monotonically in $\delta < 0$ between

$$\lim_{\delta \to -\infty} \hat{p}_3 = \frac{\mathbb{E}\left[u^{[4]}(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u^{[4]}(W) \cdot W\right]} - 1.$$

and $\lim_{\delta \to 0} \hat{p}_3 = \hat{p}_2$ if, and only if,

$$\frac{\mathbb{C}\text{ov}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \sigma^2, W\right]}{\mathbb{E}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \sigma^2\right]} > \frac{\mathbb{C}\text{ov}[u^{[4]}(W), W]}{\mathbb{E}[u^{[4]}(W)]}.$$

Under this condition, the negative skewness of idiosyncratic shocks help to explain larger equity premia in the same way in which their variance does.

In order to consider the effect of high kurtosis, a fourth-order approximation yields the premium

$$\hat{p}_4 = \frac{\mathbb{E}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \sigma^2 + \frac{1}{6} \cdot u^{[4]}(W) \cdot \sigma^3 \cdot \delta + \frac{1}{24} \cdot u^{[5]}(W) \cdot \sigma^4 \cdot \kappa\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left\{\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \sigma^2 + \frac{1}{6} \cdot u^{[4]}(W) \cdot \sigma^3 \cdot \delta + \frac{1}{24} \cdot u^{[5]}(W) \cdot \sigma^4 \cdot \kappa\right] \cdot W\right\}} - 1.$$

If we continue to assume that $u^{[4]} > 0$ and further suppose that $u^{[5]} < 0$.[5], then, as a function of $\kappa > 0$, premium $\hat{p}_4$ increases monotonically between $\lim_{\kappa \to 0} \hat{p}_4 = \hat{p}_3$ and

$$\lim_{\kappa \to \infty} \hat{p}_4 = \frac{\mathbb{E}\left[u^{[5]}(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u^{[5]}(W) \cdot W\right]} - 1$$

if, and only if,

$$\frac{\mathbb{C}\text{ov}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \sigma^2 + \frac{1}{6} \cdot u^{[4]}(W) \cdot \sigma^3 \cdot \delta, W\right]}{\mathbb{E}\left[u'(W) + \frac{1}{2} \cdot u'''(W) \cdot \sigma^2 + \frac{1}{6} \cdot u^{[4]}(W) \cdot \sigma^3 \cdot \delta\right]} > \frac{\mathbb{C}\text{ov}[u^{[5]}(W), W]}{\mathbb{E}[u^{[5]}(W)]}.$$

Again, under this condition, a significant kurtosis in the distribution of idiosyncratic shocks ameliorates the equity premium puzzle.

---

[3] This means that, for the purposes of this section, we refer to the premium defined in Eq. (8) as $\hat{p}_2$.

[4] That is, if the Bernoulli function displays "temperance". See Ekern (1980) and Eeckhoudt et al (1995).

[5] That is, that the Bernoulli function exhibits "edginess".

These observations can be generalized as follows. Suppose that for any $k \geq 3$, the k^{th} standardized moment of the conditional distribution of S is the constant μ_k almost surely.[6] The K^{th}-order approximation to the marginal utility yields the equity premium

$$\hat{p}_K = \frac{\mathbb{E}\left[\sum_{k=1}^{K} \frac{1}{k!} \cdot u^{[k]}(W) \cdot \sigma^k \cdot \mu_k\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[\sum_{k=1}^{K} \frac{1}{k!} \cdot u^{[k]}(W) \cdot \sigma^k \cdot \mu_k \cdot W\right]} - 1$$

Theorem 5. *Suppose that for all $k = 3, \ldots, K$, $(-1)^k u^{[k]} < 0$ and $(-1)^k \mu_k > 0$.*[7] *The equity premium $\hat{p}_K$ ranges monotonically between $\lim_{\mu_K \to 0} \hat{p}_K = \hat{p}_{K-1}$ and*

$$\lim_{\mu_K \to (-1)^{K+1}\infty} \hat{p}_K = \frac{\mathbb{E}\left[u^{[K]}(W)\right] \cdot \mathbb{E}(W)}{\mathbb{E}\left[u^{[K]}(W) \cdot W\right]} - 1.$$

Such monotonicity is increasing, namely

$$\frac{\partial \hat{p}_K}{\partial \mu_K} > 0,$$

if, and only if,

$$(-1)^K \frac{\mathbb{C}\mathrm{ov}\left[\sum_{k=1}^{K-1} \frac{1}{k!} \cdot u^{[k]}(W) \cdot \sigma^k \cdot \mu_k, W\right]}{\mathbb{E}\left[\sum_{k=1}^{K-1} \frac{1}{k!} \cdot u^{[k]}(W) \cdot \sigma^k \cdot \mu_k \cdot W\right]} > (-1)^K \frac{\mathbb{C}\mathrm{ov}\left[u^{[K]}(W), W\right]}{\mathbb{E}\left[u^{[K]}(W) \cdot W\right]}. \qquad (12)$$

The proof of this theorem is very similar to the argument for Theorem 4, so we defer it, along with the remaining proofs in the paper, to an appendix. For the canonical families of Bernoulli functions, the following is a consequence of the theorem:

Corollary 2. *Under the assumptions of this section:*

1. *If the first derivative of the Bernoulli function is homogeneous, $\check{p}$ is increasing in κ.*
2. *If the Bernoulli function is exponential, $\check{p}$ does not depend on κ.*

5 Long-Lived Assets

Consider, for a specific application, the case of a stationary overlapping generations economy where individuals live for two periods and the only asset in the economy

[6] That is, that $E(S^k \mid W) = \sigma^k \cdot \mu_k$ W-a.s.

[7] The first condition is the assumption that the Bernoulli function is K^{th}-degree risk averse.

pays W, i.i.d., every period. If the risk is traded using a long-lived asset, its return must be adjusted to include the price of the asset.

5.1 *Benchmark: only aggregate risk*

In the absence of any other risk, the problem of the young generation is

$$\max_y \left\{ v(\bar{w} - q \cdot y) + \mathbb{E}[u((W+q)\cdot y)] \right\},$$

and its first-order condition is that

$$v'(\bar{w} - q \cdot y) \cdot q = \mathbb{E}[u'((W+q)\cdot y) \cdot (W+q)].$$

Market clearing requires that q be the solution to

$$q = \frac{\mathbb{E}[u'((W+q)) \cdot (W+q)]}{v'(\bar{w} - q)},$$

while a risk-less asset with the same expected return should be priced at

$$\frac{\mathbb{E}[u'(W+q)] \cdot [\mathbb{E}(W) + q]}{v'(\bar{w} - q)}.$$

The relative price of the risk-less asset to the risky asset (minus 1) is again the *equity premium*:

$$\bar{p} = \frac{\mathbb{E}\left[u'(W+q)\right] \cdot [\mathbb{E}(W) + q]}{\mathbb{E}\left[u'(W+q) \cdot (W+q)\right]} - 1. \tag{13}$$

5.2 *Idiosyncratic risk*

Under idiosyncratic risk, the premium is

$$p = \frac{\mathbb{E}\left[u'(W+q+S)\right] \cdot [\mathbb{E}(W) + q]}{\mathbb{E}\left[u'(W+q+S) \cdot (W+q)\right]} - 1. \tag{14}$$

Using the quadratic expansion

$$u'(w+q+s) \approx u'(w+q) + u''(w+q) \cdot s + \tfrac{1}{2} \cdot u'''(w+q) \cdot s^2,$$

we get the approximation

$$\hat{p} = \frac{\mathbb{E}\left[u'(W+q) + \frac{1}{2} \cdot u'''(W+q) \cdot \mathbb{V}(S \mid W)\right] \cdot [\mathbb{E}(W) + q]}{\mathbb{E}\left\{\left[u'(W+q) + \frac{1}{2} \cdot u'''(W+q) \cdot \mathbb{V}(S \mid W)\right] \cdot (W+q)\right\}} - 1 \tag{15}$$

to the equity premium.

The problem would be a trivial extension of the previous results, were it not for the dependence of q on the distribution of S. The purpose of this paper is not to develop the general comparative statics of this dependence, but to determine how that dependence affects the effect of the distribution of S on the equity premium.

5.3 *Idiosyncratic risk and the equity premium*

Sometimes it will be convenient to write $\mathbb{V}(S \mid W = w) = \Sigma(w, \theta)$, for some differentiable function $\Sigma : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}_{+}$, where constant θ is a parameter of the conditional distribution of S such that $\partial\Sigma/\partial\theta > 0$.

THEOREM 6 (Relevance, 5). *Suppose that premium $\hat{p}$ is decreasing in asset price q, keeping θ fixed, and that q is increasing in θ.*[8] *Then, a necessary condition for $\hat{p}$ to be non-decreasing in θ is that*

$$\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')} > \frac{\mathbb{C}\mathrm{ov}(U''' \cdot V, W)}{\mathbb{E}(U''' \cdot V)}, \tag{16}$$

where $U^{[n]} = u^{[n]}(W + q)$ and $V = \Sigma(W, \theta)$.

If, on the other hand, given θ premium $\hat{p}$ is non-decreasing in q, then Eq. (16) *suffices for $\hat{p}$ to be increasing in θ.*

Note that the tension between the direct effect of q and the direct effect of θ arises when the former is negative. A complication when trying to determine the sign of the latter effect is that it involves the response of the third derivative of the Bernoulli function, since the asset is traded. Instead of attempting a full characterization, we find sufficient conditions:

THEOREM 7. *Suppose that the Bernoulli function is differentiable four times, and that $u^{[4]} < 0$. Premium $\hat{p}$ is decreasing in asset price q if*

$$\mathbb{C}\mathrm{ov}(U''' \cdot V, W) \leq 0 \tag{17}$$

[8] As would be the case in Aiyagari (1994), for instance.

and

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')},\frac{\mathbb{C}\mathrm{ov}(U'''\cdot V,W)}{\mathbb{E}(U'''\cdot V)}\right\}\geq\max\left\{\frac{\mathbb{C}\mathrm{ov}(U'',W)}{\mathbb{E}(U'')},\frac{\mathbb{C}\mathrm{ov}(U^{[4]}\cdot V,W)}{\mathbb{E}(U^{[4]}\cdot V)}\right\}, \quad (18)$$

where $U^{[n]}=u^{[n]}(W+q)$ *and* $V=\mathbb{V}(S\mid W)$.

5.4 *Homoskedastic risk*

Considering the case of homoskedastic idiosyncratic risk, with $\mathbb{V}(S\mid W)=\sigma^2$ almost surely on W, for the rest of this subsection we assume that q depends differentiably on σ^2, with first derivative q'.

It is useful to re-write Eq. (15) as

$$\hat{p}=\frac{\mathbb{E}[u'(W+q)]\cdot[\mathbb{E}(W)+q]+\frac{1}{2}\cdot\mathbb{E}[u'''(W+q)]\cdot[\mathbb{E}(W)+q]\cdot\sigma^2}{\mathbb{E}[u'(W+q)\cdot(W+q)]+\frac{1}{2}\cdot\mathbb{E}[u'''(W+q)\cdot(W+q)]\cdot\sigma^2}-1. \quad (19)$$

With multiple instances of σ and q, a full characterization of the total differential of $\hat{p}$ with respect to σ is possible, but cumbersome and rather uninformative. Instead, we derive independent necessary and sufficient conditions, focusing on the case when idiosyncratic risk increases the premium.

Theorems 6 and 7 immediately imply the following two results for this case:

Corollary 3 (Relevance, 6). *Suppose that premium $\hat{p}$ is decreasing in asset price q, keeping σ fixed, and that q is increasing in σ. Then, a necessary condition for $\hat{p}$ to be non-decreasing in σ is that*

$$\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')}>\frac{\mathbb{C}\mathrm{ov}(U''',W)}{\mathbb{E}(U''')}. \quad (20)$$

Corollary 4. *Suppose that the Bernoulli function is differentiable four times and $u^{[4]}<0$. Premium $\hat{p}$ is decreasing in asset price q, given variance σ, if*

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')},\frac{\mathbb{C}\mathrm{ov}(U''',W)}{\mathbb{E}(U''')}\right\}\geq\max\left\{\frac{\mathbb{C}\mathrm{ov}(U'',W)}{\mathbb{E}(U'')},\frac{\mathbb{C}\mathrm{ov}(U^{[4]},W)}{\mathbb{E}(U^{[4]})}\right\}. \quad (21)$$

To be sure, note that the condition that $\mathbb{C}\mathrm{ov}[U''',W]\leq 0$, which specializes Eq. (17) to the case at hand, does not need to be assumed explicitly, as it is implied by the assumption that $u^{[4]}<0$.

5.5 *Example: CARA preferences and homoskedastic risk*

Unfortunately, the previous two corollaries fail to provide a full characterization of the sign of the effect of σ on p. Still,

THEOREM 8 (Relevance, 6). *If $v' = 1$ and the Bernoulli function u is exponential, then the equity premium $\hat{p}$ is decreasing in σ.*

5.6 *Example: CRRA preferences and heteroskedastic risk*

Considering now the case where $\mathbb{V}(S \mid W = w) = \sigma w^2$, with $\sigma \geq 0$.[9] As before, we a assume that q depends differentiably on σ, with first derivative q'.

THEOREM 9 (Relevance, 7). *Suppose that $v' = 1$ and the Bernoulli function u is homogeneous of degree $-\rho < 0$. If*

$$q \leq \frac{\min\left\{\rho, 1\right\}}{2} \cdot \inf \mathcal{W},$$

then the equity premium $\hat{p}$ is decreasing in σ.

9 The result can be extended to any power $1 < \beta < 3$, at the expense of some analytical complications.

References

Acharya, S. and K. Dogra, "Understanding HANK: insights from a PRANK", *unpublished mimeo*, 2020.

Aiyagari, S.R., "Uninsured idiosyncratic risk and aggregate saving", *Quarterly Journal of Economics* 109 (1994) 659–684.

Bilbiie, F., "Monetary policy and heterogeneity: an analytical framework", *unpublished mimeo*, 2019.

Cogley, T., "Idiosyncratic risk and the equity premium: evidence from the consumer expenditure survey", *Journal of Monetary Economics* 49 (2002) 309–334.

Constantinides, G.M. and D. Duffie, "Asset pricing with heterogeneous consumers", *Journal of Political Economy* 104 (1996) 219–240.

Debortoli, D. and J. Galí., " Idiosyncratic income risk and aggregate fluctuations.", *National Bureau of Economic Research* w29704 (2022).

Eeckhoudt, L., C. Gollier and T. Schneider, "Risk-aversion, prudence and temperance: a unified approach", *Economics Letters* 48 (1995) 331–336.

Ekern, S., "Increasing n-th degree risk", *Economics Letters* 6 (1980) 329–333.

Guvenen, F. , "Income risk over the life cycle and the business cycle," *NBER Reporter* 4 (2016) 16-20.

Krueger, D. and H. Lustig, "When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)?", *Journal of Economic Theory* 145 (2010) 1–41.

Mankiw, N.G., "The equity premium and the concentration of aggregate shocks", *Journal of Financial Economics* 17 (1986) 211–219.

Mehra, R., and E.C. Prescott, "The equity premium: a puzzle", *Journal of monetary Economics* 15 (1985), 145–161.

Mehra, R., "The equity premium: why is it a puzzle?", *Financial Analysts Journal* 59 (2003) 54–69.

Storesletten, K., C.I. Telmer and A. Yaron, "Asset pricing with idiosyncratic risk and overlapping generations", *Review of Economic Dynamics* 10 (2007) 519–548.

Werning, I., "Incomplete markets and aggregate demand", *National Bureau of Economic Research* w21448 (2015).

Appendix: proofs

Proof of Corollary 1: For the first statement, by Theorem 4 it suffices to show that Eq. (11) holds strictly. Assume, as in the proof of the second part of Theorem 1, that $u'(w) = u'(1)w^{-\rho}$, where $\rho > 0$ since u is strictly concave. By direct computation, it suffices to show that

$$\mathbb{E}(W^{-\rho+1}) \cdot \mathbb{E}(W^{-\rho-2}) - \mathbb{E}(W^{-\rho-1}) \cdot \mathbb{E}(W^{-\rho}) > 0.$$

Letting V be an (ancillary) random variable distributed identically to W and independent from it, the latter expression is equivalent to

$$\mathbb{E}(V^{-\rho+1} \cdot W^{-\rho-2} - V^{-\rho} \cdot W^{-\rho-1}) > 0.$$

Now, this expectation equals

$$\mathbb{E}\left[(V \cdot W)^{-\rho} \cdot \left(\tfrac{V}{W} - 1\right) \cdot \left(\tfrac{1}{W} - \tfrac{1}{V}\right)\right] + \mathbb{E}\left[(V \cdot W)^{-\rho} \cdot \left(\tfrac{1}{W} - \tfrac{1}{V}\right)\right].$$

Note that the first summand of the last expression is strictly positive, since both random variables are non-degenerate and take only strictly positive values, so the integrand is positive. The second summand is null, since they are identically distributed. It follows that the sum is strictly positive, as needed.

To prove the second claim, suppose that $u(w) = -e^{-\alpha w}$, where $\alpha > 0$. Note that

$$\frac{\mathbb{E}\left[u'(W) \cdot W\right]}{\mathbb{E}\left[u'(W)\right]} = \frac{\mathbb{E}(e^{-\alpha W} \cdot W)}{\mathbb{E}(e^{-\alpha W})} = \frac{\mathbb{E}\left[u'''(W) \cdot W\right]}{\mathbb{E}\left[u'''(W)\right]}.$$

It follows from Theorem 4 that $\hat{p} = \bar{p}$, regardless of the variance of the idiosyncratic risk. □

Proof of Theorem 5: The computation of the two limits is straightforward. Similarly, by direct computation, $\partial \hat{p}_K / \partial \mu_K > 0$ if, and only if,

$$\mu_K \mathbb{E}[u^{[K]}(W)] \cdot \mathbb{E}\left[\sum_{k=1}^{K} \frac{\sigma^k \mu_k}{k!} \cdot u^{[k]}(W) \cdot W\right] > \mu_K \mathbb{E}[u^{[K]}(W) \cdot W] \cdot \mathbb{E}\left[\sum_{k=1}^{K} \frac{\sigma^k \mu_k}{k!} \cdot u^{[k]}(W)\right].$$

Since $(-1)^K \mu_K > 0$, the latter is equivalent to

$$(-1)^K \mathbb{E}[u^{[K]}(W)] \cdot \mathbb{E}\left[\sum_{k=1}^{K} \frac{\sigma^k \mu_k}{k!} \cdot u^{[k]}(W) \cdot W\right] > (-1)^K \mathbb{E}[u^{[K]}(W) \cdot W] \cdot \mathbb{E}\left[\sum_{k=1}^{K} \frac{\sigma^k \mu_k}{k!} \cdot u^{[k]}(W)\right].$$

This is

$$(-1)^K \frac{\mathbb{E}\left[\sum_{k=1}^{K} \frac{\sigma^k \mu_k}{k!} \cdot u^{[k]}(W)\right]}{\mathbb{E}\left[\sum_{k=1}^{K} \frac{\sigma^k \mu_k}{k!} \cdot u^{[k]}(W) \cdot W\right]} < (-1)^K \frac{\mathbb{E}[u^{[K]} \cdot W])}{\mathbb{E}[u^{[K]}]},$$

which can be re-writtem as Eq. (12). □

Proof of Corollary 2: For the first claim, note that the proof of the first claim in Corollary 1 consists of arguing that

$$\frac{\mathbb{E}(W^{-\rho+1})}{\mathbb{E}(W^{-\rho})} > \frac{\mathbb{E}(W^{-\rho-1})}{\mathbb{E}(W^{-\rho-2})}.$$

By the same argument, note also that

$$\frac{\mathbb{E}(W^{-\rho-1})}{\mathbb{E}(W^{-\rho-2})} > \frac{\mathbb{E}(W^{-\rho-3})}{\mathbb{E}(W^{-\rho-4})}.$$

It follows that when $u'(w) = u'(1)w^{-\rho}$, with $\rho > 0$,

$$\frac{\mathbb{E}\left[u'(W)\cdot W\right]}{\mathbb{E}\left[u'(W)\right]} > \frac{\mathbb{E}\left[u'''(W)\cdot W\right]}{\mathbb{E}\left[u'''(W)\right]} > \frac{\mathbb{E}\left[u^{[5]}(W)\cdot W\right]}{\mathbb{E}\left[u^{[5]}(W)\right]},$$

and hence that

$$\mathbb{E}\left[u'(W)\cdot W\right]\mathbb{E}\left[u^{[5]}(W)\right] > \mathbb{E}\left[u'(W)\right]\mathbb{E}\left[u^{[5]}(W)\cdot W\right]$$

and

$$\mathbb{E}\left[u'''(W)\cdot W\right]\mathbb{E}\left[u^{[5]}(W)\right] > \mathbb{E}\left[u'''(W)\right]\mathbb{E}\left[u^{[5]}(W)\cdot W\right].$$

Aggregating,

$$\left\{\mathbb{E}\left[u'(W) + \tfrac{1}{2}\cdot u'''(W)\cdot\sigma^2\right]W\right\}\cdot\mathbb{E}\left[u^{[5]}(W)\right] > \mathbb{E}\left[u'(W) + \tfrac{1}{2}\cdot u'''(W)\cdot\sigma^2\right]\cdot\mathbb{E}\left[u^{[5]}(W)\cdot W\right],$$

which is equivalent to Eq. (??).

For the second claim, note again that when $u(w) = -e^{-\alpha w}$, with $\alpha > 0$,

$$\frac{\mathbb{E}\left[u'(W)\cdot W\right]}{\mathbb{E}\left[u'(W)\right]} = \frac{\mathbb{E}(e^{-\alpha W}\cdot W)}{\mathbb{E}(e^{-\alpha W})} = \frac{\mathbb{E}\left[u'''(W)\cdot W\right]}{\mathbb{E}\left[u'''(W)\right]} = \frac{\mathbb{E}\left[u^{[5]}(W)\cdot W\right]}{\mathbb{E}\left[u^{[5]}(W)\right]},$$

so it follows from Theorem ?? that $\check{p} = \hat{p} = \bar{p}$, regardless of the variance and the kurtosis of the idiosyncratic shocks. □

Proof of Theorem 6: Obviously,

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}\theta} = \frac{\partial\hat{p}}{\partial q}\cdot\frac{\mathrm{d}q}{\mathrm{d}\theta} + \frac{\partial\hat{p}}{\partial\theta}.$$

For the first result, note that the first summand on the right-hand side of the last expression is negative, so a necessary condition for the sum to be positive is that the second summand be positive. For the second result, under the assumptions the first summand is non-negative, so the sum is positive if so is the second summand. In both cases, all one needs to observe is that $\partial\hat{p}/\partial\theta > 0$. The proof that this inequality is equivalent to Eq. (16) resembles to the arguments for Theorems 4 and ??, so we omit it. □

Proof of Theorem 7: We can write Eq. (15) as

$$\hat{p} = \frac{f(q) + \sigma^2\cdot g(q)}{\varphi(q) + \sigma^2\cdot\gamma(q)},$$

where

$$f(q) = \mathbb{E}[u'(W+q)]\cdot[E(W)+q],$$

$$g(q) = \frac{1}{2}\mathbb{E}[u'''(W+q)\cdot V]\cdot[E(W)+q],$$
$$\varphi(q) = \mathbb{E}[u'(W+q)\cdot(W+q)]$$

and

$$\gamma(q) = \frac{1}{2}\mathbb{E}[u'''(W+q)\cdot V\cdot(W+q)].$$

With this formulation, $\hat{p}$ is decreasing in q if, and only if,

$$[f'(q)+\sigma^2\cdot g'(q)]\cdot[\varphi(q)+\sigma^2\cdot\gamma(q)] < [\varphi'(q)+\sigma^2\cdot\gamma'(q)]\cdot[f(q)+\sigma^2\cdot g(q)],$$

which holds true if

$$f'(q)\cdot\varphi(q) < \varphi'(q)\cdot f(q) \tag{22}$$
$$f'(q)\cdot\gamma(q) \leq \varphi'(q)\cdot g(q) \tag{23}$$
$$g'(q)\cdot\varphi(q) \leq \gamma'(q)\cdot f(q) \tag{24}$$
$$g'(q)\cdot\gamma(q) \leq \gamma'(q)\cdot g(q). \tag{25}$$

Upon substitution, Eq. (22) is equivalent to

$$\{\mathbb{E}(U'')[\mathbb{E}(W)+q]+\mathbb{E}(U')\}\cdot\mathbb{E}[U'\cdot(W+q)] < \{\mathbb{E}[U''\cdot(W+q)]+\mathbb{E}(U')\}\cdot E(U')\cdot[\mathbb{E}(W)+q],$$

which is, by direct computation,

$$\{\mathbb{E}(U'')\cdot\mathbb{E}[U'\cdot(W+q)]-\mathbb{E}(U')\cdot\mathbb{E}[U''\cdot(W+q)]\}\cdot[\mathbb{E}(W)+q]+\mathbb{E}(U')\cdot\mathbb{C}\text{ov}(U',W) < 0. \quad (*)$$

Since $u' > 0$ and $u'' < 0$, we have that $\mathbb{E}(U') > 0$ and $\mathbb{C}\text{ov}(U',W) < 0$, so it suffices that

$$\mathbb{E}(U'')\cdot\mathbb{E}[U'\cdot(W+q)] \leq \mathbb{E}(U')\cdot\mathbb{E}[U''\cdot(W+q)],$$

for inequality $(*)$ to hold, as $\mathbb{E}(W)+q > 0$. As in the proof of Theorem 4, this is equivalent to

$$\frac{\mathbb{C}\text{ov}(U',W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\text{ov}(U'',W)}{\mathbb{E}(U'')},$$

which is one of the inequalities that are part of Eq. (18).

Similarly, Eq. (23) is equivalent to the requirement that the sum of

$$\{\mathbb{E}(U'')\cdot\mathbb{E}[U'''\cdot V\cdot(W+q)]-\mathbb{E}[U'''\cdot V]\cdot\mathbb{E}[U''\cdot(W+q)]\}\cdot[\mathbb{E}(W)+q] \quad (**)$$

and

$$\mathbb{E}(U')\cdot\mathbb{C}\text{ov}[U'''\cdot V,W] \quad (***)$$

be non-positive.

Since $u' > 0$ and $\mathbb{C}\text{ov}[U'''\cdot V,W] \leq 0$, we have that the expression in $(***)$ is non-positive. On the other hand, since $\mathbb{E}(W)+q > 0$, for inequality $(**)$ to hold it suffices that

$$\mathbb{E}(U'')\cdot\mathbb{E}[U'''\cdot V\cdot(W+q)] \leq \mathbb{E}[U'''\cdot V]\cdot\mathbb{E}[U''\cdot(W+q)],$$

which is equivalent to

$$\frac{\mathbb{C}\text{ov}[U'''\cdot V,W]}{\mathbb{E}[U'''\cdot V]} \geq \frac{\mathbb{C}\text{ov}(U'',W)}{\mathbb{E}(U'')}.$$

For Eqs. (22) and (23) to hold true, it thus suffices that

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')},\frac{\mathbb{C}\mathrm{ov}[U'''\cdot V,W]}{\mathbb{E}[U'''\cdot V]}\right\}\geq\frac{\mathbb{C}\mathrm{ov}(U'',W)}{\mathbb{E}(U'')}.$$

By a virtually identical analysis, using that $u'''>0$ and $u^{[4]}<0$, one can prove that

$$\min\left\{\frac{\mathbb{C}\mathrm{ov}(U',W)}{\mathbb{E}(U')},\frac{\mathbb{C}\mathrm{ov}[U'''\cdot V,W]}{\mathbb{E}[U'''\cdot V]}\right\}\geq\frac{\mathbb{C}\mathrm{ov}[U^{[4]}\cdot V,W]}{\mathbb{E}[U^{[4]}\cdot V]}$$

suffices for Eqs. (24) and (25) □

Proof of Theorem 8: Let $u(w)=-e^{-\alpha w}$, for some $\alpha>0$. Then,

$$u^{[n]}(w)=(-1)^{n+1}\alpha^n e^{-\alpha w}=(-\alpha)^n u(w),$$

which implies that $u^{[4]}<0$. Also,

$$\mathbb{C}\mathrm{ov}[u^{[n]}(W+q),w]=(-\alpha)^n\mathbb{C}\mathrm{ov}[u(W+q),w]$$

and

$$\mathbb{E}[u^{[n]}(W+q)]=(-\alpha)^n\mathbb{E}[u(W+q)],$$

which imply that

$$\frac{\mathbb{C}\mathrm{ov}[u^{[n]}(W+q),w]}{\mathbb{E}[u^{[n]}(W+q)]}=\frac{\mathbb{C}\mathrm{ov}[u(W+q),w]}{\mathbb{E}[u(W+q)]}$$

for all orders of differentiation. Corollary 4 implies that, with σ fixed, $\hat{p}$ is decreasing in q. If we can argue that the price of the asset is increasing in σ, then Corollary 3 implies that condition (16), which does not hold true, is necessary for $\hat{p}$ to be non-decreasing in Σ.

To see that, indeed, $q'>0$, note that

$$q=\mathbb{E}\left[u'(W+q+S)\cdot(W+q)\right]$$

since $v'=1$. This expression re-writes, in the case of exponential preferences, as

$$q=e^{-\alpha q}\cdot\{\mathbb{E}[u'(W+S)\cdot W]+\mathbb{E}[u'(W+S)]\cdot q\},$$

so

$$e^{\alpha q}=\frac{\mathbb{E}[u'(W+S)\cdot W]}{q}+\mathbb{E}[u'(W+S)].$$

This expression is transcendental, so we can only obtain q' by implicit differentiation:

$$\left\{e^{\alpha q}+\frac{\mathbb{E}[u'(W+S)\cdot W]}{q^2}\right\}\cdot q'=\frac{\partial}{\partial\sigma}\left\{\frac{\mathbb{E}[u'(W+S)\cdot W]}{q}+\mathbb{E}[u'(W+S)]\right\}.$$

Since exponential preferences are strictly increasing and strictly prudent, we know that

$$\mathbb{E}[u'(W+S)\cdot W]=\mathbb{E}\left\{\mathbb{E}[u'(W+S)\mid W]\cdot W\right\}$$

and

$$\mathbb{E}[u'(W+S)]=\mathbb{E}\left\{\mathbb{E}[u'(W+S)\mid W]\right\}$$

are both increasing in σ, which implies that $q'>0$, as needed. □

Proof of Theorem 9: The strategy for the proof is the same as in Theorem 8: we will argue that, under the assumption of the theorem, all the conditions that make Eq. (16) necessary for $\hat{p}$ to be non-decreasing in σ are satisfied, but not Eq. (16) itself. Using the functional forms $u'(w) = u'(1)w^{-\rho}$ and $\mathbb{V}(S \mid W) = \sigma W^2$, we need to prove the following:

(a) that q is increasing in σ, namely that implicitly differentiating

$$q = u'(1) \cdot \mathbb{E}[(W+q+S)^{-\rho} \cdot (W+q)] \tag{26}$$

with respect to σ yields $q' > 0$;

(b) Equation (17);

(c) that

$$\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')} \quad \text{and} \quad \frac{\mathbb{C}\mathrm{ov}[U''' \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U''' \cdot \mathbb{V}(S \mid W)]} \geq \frac{\mathbb{C}\mathrm{ov}[U^{[4]} \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U^{[4]} \cdot \mathbb{V}(S \mid W)]},$$

two of the inequalities in Eq. (18), which in the case are

$$\frac{\mathbb{E}[(W+q)^{-\rho} \cdot W]}{\mathbb{E}[(W+q)^{-\rho}]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+1)} \cdot W]}{\mathbb{E}[(W+q)^{-(\rho+1)}]} \tag{27}$$

and

$$\frac{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^2]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^2]}; \tag{28}$$

(d) that

$$\frac{\mathbb{C}\mathrm{ov}[U''' \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U''' \cdot \mathbb{V}(S \mid W)]} \geq \frac{\mathbb{C}\mathrm{ov}(U'', W)}{\mathbb{E}(U'')},$$

another one of the inequalities in Eq. (18), which is

$$\frac{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^2]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+1)} \cdot W]}{\mathbb{E}[(W+q)^{-(\rho+1)}]}; \tag{29}$$

(e) that

$$\frac{\mathbb{C}\mathrm{ov}(U', W)}{\mathbb{E}(U')} \geq \frac{\mathbb{C}\mathrm{ov}[U^{[4]} \cdot \mathbb{V}(S \mid W), W]}{\mathbb{E}[U^{[4]} \cdot \mathbb{V}(S \mid W)]}$$

the final inequality in Eq. (18), which is

$$\frac{\mathbb{E}[(W+q)^{-\rho} \cdot W]}{\mathbb{E}[(W+q)^{-\rho}]} \geq \frac{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+3)} \cdot W^2]} \tag{30}$$

under the functional forms; and

(f) that Eq. (16) fails, namely that

$$\frac{\mathbb{E}[(W+q)^{-\rho} \cdot W]}{\mathbb{E}[(W+q)^{-\rho}]} \leq \frac{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^3]}{\mathbb{E}[(W+q)^{-(\rho+2)} \cdot W^2]}. \tag{31}$$

Of course, we proceed one by one:

(a) From Eq. (26), by the implicit function theorem q' equals the product of

$$\frac{u'(1)}{\mathbb{E}\left[1+\rho(W+q+S)^{-(\rho+1)}\cdot(W+q)-(W+q+S)^{-\rho}\right]} \tag{$*$}$$

and

$$\mathbb{E}\left\{\frac{\partial}{\partial\sigma}\mathbb{E}[(W+q+S)^{-\rho}\mid W]\cdot(W+q)\right\}, \tag{$**$}$$

so long as the denominator on the former expression is non-zero. We actually want that denominator to be strictly positive, which is the case, since $\rho>0$, $W+q>0$ with probability one by assumption, and

$$\rho(w+q)>0 \Leftrightarrow 1+\frac{\rho(w+q)}{(w+q+s)^{\rho+1}}>\frac{1}{(w+q+s)^{\rho}}.$$

Since $u'(1)>0$, it follows that the term in Eq. ($*$) is strictly positive.

That the term in Eq. ($**$) is also positive is immediate, since $(w+q+s)^{-\rho}$ is strictly convex in s, and an increase in σ is a mean-preserving spread of S given W.

(b) For Eq. (17), it suffices for us to argue that $u'''(W+q)\cdot\mathbb{V}(S\mid W)$ and W are anti-comonotone with probability one. Letting the function

$$\eta(w)=u'''(w+q)\cdot\mathbb{V}(S\mid W=w)=\sigma\rho(\rho+1)u'(1)w^2,$$

we have that $\eta'(w)\leq 0$ so long as $w\geq 2q/\rho$. Since $q\leq \rho/2\inf\mathcal{W}$, by assumption, we have that this inequality holds with probability 1, as needed.

(c) Define now the function

$$h(n)=\frac{\mathbb{E}[(W+q)^{-n}\cdot W^{m}]}{\mathbb{E}[(W+q)^{-n}\cdot W^{m-1}]}$$

over $n>0$, given any $m\geq 0$. For Eqs. (27) and (28), it suffices to observe that q is non-increasing in n.

By direct computation, $h'(n)\leq 0$ if, and only if,

$$\mathbb{E}[(W+q)^{-n}\cdot W^{m}\cdot\ln(W+q)]\cdot\mathbb{E}[(W+q)^{-n}\cdot W^{m-1}]$$

is at least as large as

$$\mathbb{E}[(W+q)^{-n}\cdot W^{m-1}\cdot\ln(W+q)]\cdot\mathbb{E}[(W+q)^{-n}\cdot W^{m}].$$

Letting random variable V be i.i.d. with W, this is the requirement that

$$\mathbb{E}\left\{(W-V)\cdot(VW)^{m-1}\cdot[(V+q)(W+q)]^{-n}\cdot\ln(W+q)\right\}\geq 0.$$

This expectation is proportional, by a factor of $\Pr(V\neq W)/2$, to the sum of

$$\mathbb{E}\left\{(W-V)\cdot(VW)^{m-1}\cdot[(V+q)(W+q)]^{-n}\cdot\ln(W+q)\mid V>W\right\}$$

and

$$\mathbb{E}\Big\{(W-V)\cdot(VW)^{m-1}\cdot[(V+q)(W+q)]^{-n}\cdot\ln(W+q)\mid V<W\Big\}.$$

Since V and W follow the same distribution, the latter is

$$\mathbb{E}\Big\{(V-W)\cdot(VW)^{m-1}\cdot[(V+q)(W+q)]^{-n}\cdot\ln(V+q)\mid V>W\Big\},$$

so the sum equals

$$\mathbb{E}\Big\{(W-V)\cdot(VW)^{m-1}\cdot[(V+q)(W+q)]^{-n}\cdot[\ln(W+q)-\ln(V+q)]\mid V>W\Big\},$$

which is, indeed, non-negative.

(d) Note that Eq. (29) is equivalent to the requirement that

$$\mathbb{E}[(W+q)^{-(\rho+2)}W^3]\cdot\mathbb{E}[(W+q)^{-(\rho+1)}]\geq\mathbb{E}[(W+q)^{-(\rho+2)}W^2]\cdot\mathbb{E}[(W+q)^{-(\rho+1)}W].$$

With V defined as above, this is

$$\mathbb{E}\left\{V\cdot[(V+q)(W+q)]^{-(\rho+1)}\cdot\left(\frac{V^2}{V+q}-\frac{W^2}{W+q}\right)\right\}\geq 0,$$

or

$$\mathbb{E}\left\{(V-W)\cdot[(V+q)(W+q)]^{-(\rho+1)}\cdot\left(\frac{V^2}{V+q}-\frac{W^2}{W+q}\right)\;\middle|\;V>W\right\}\geq 0.$$

In order to guarantee this, we need to argue that

$$v>w\Rightarrow\frac{v^2}{v+q}\geq\frac{w^2}{w+q},$$

or, equivalently, that the ratio $w^2/(w+q)$ is non-decreasing for $w\in\mathcal{W}$. By direct computation, this is true since $\mathcal{W}\subseteq\mathbb{R}_{++}$ and $q\geq 0$.

(e) Using the same technique, Eq. (30) is equivalent to the requirement that

$$\mathbb{E}\Big[(V-W)\cdot W^2\cdot(V+q)^{-\rho}\cdot(W+q)^{-(\rho+3)}\Big]\geq 0,$$

or, equivalently, that

$$\mathbb{E}\left\{(V-W)\cdot[(V+q)(W+q)]^{-\rho}\cdot\left(\frac{W^2}{(W+q)^3}-\frac{V^2}{(V+q)^3}\right)\;\middle|\;V>W\right\}\geq 0.$$

For this, it suffices that the ratio $w^2/(w+q)^3$ be non-increasing at all $w\in\mathcal{W}$. This is guaranteed, indeed, by the assumption that $q\leq 1/2\inf\mathcal{W}$.

(f) Finally, note again that Eq. (31) is equivalent to

$$\mathbb{E}[(V-W)\cdot V^2\cdot(V+q)^{-(\rho+2)}(W+q)^{-\rho}]\geq 0,$$

or

$$\mathbb{E}\left\{(V-W)\cdot[(V+q)(W+q)]^{-\rho}\cdot\left(\frac{V^2}{(V+q)^2}-\frac{W^2}{(W+q)^2}\right)\;\middle|\;V>W\right\}\geq 0.$$

For this inequality to hold true, it suffices that $w/(w+q)$ be non-increasing at all $w\in\mathcal{W}$, which is true since $\mathcal{W}\subseteq\mathbb{R}_{++}$ and $q\geq 0$. □