Stochastic Discount Factors in Incomplete Markets

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Abstract

When the markets are incomplete, the law of one price no longer guarantees the uniqueness of the stochastic discount factor (SDF), resulting in a set of admissible SDFs, which complicates the study of financial market equilibrium, portfolio optimization, and derivative securities. This paper first proposes a discrete-time framework for estimating this set of SDFs, where there are extra states that cannot be hedged by the marketed assets. Without specifying the cause of incompleteness, we show that the constructed incomplete market SDF set has a unique boundary point, and shrinks to this point only when the market completes. This property allows us to develop a novel measure for market incompleteness based upon the Wasserstein metric, which estimates the least distance between the probability distributions of the complete and incomplete market SDFs. To facilitate the parametrization of market incompleteness for implementation, we then consider in detail a continuous-time framework, in which incompleteness arises from stochastic jumps in asset prices, and we demonstrate that the theoretical results developed under the discrete-time setting still hold true. We further apply our results to study the evolution of market incompleteness in four world's largest stock markets. Our findings indicate that an increase in market incompleteness is usually associated with financial crises or policy changes that raise the likelihood of unanticipated risks.

Keywords: Stochastic discount factor; incomplete market; degree of market incompleteness

1 Introduction

Stochastic discount factor (henceforth SDF) forms the basis for all asset pricing and provides a summary of investor preferences for payoffs over different states of the world. Under the law of one price (henceforth LOOP), the asset pricing equation established by Harrison and Kreps (1979), Harrison and Pliska (1981) and Hansen and Jagannathan (1991) implies that asset prices today are a function of their expected future payoffs discounted by the SDF. When markets are complete, the asset pricing equation leads to a unique SDF, whereas there is a multiplicity of admissible SDFs that satisfy the equation in the absence of complete markets (Hansen and Jagannathan, 1991; Boyle et al., 2008; Kaido and White, 2009), thus complicating the study of financial market equilibrium, portfolio optimization, and derivative securities (Skiadas, 2007; Staum, 2007; Boyle et al., 2008). It is therefore essential to establish a framework for characterizing the incomplete market SDF set, and assess the extent of market incompleteness.

Markets are incomplete when perfect risk transfer is impossible, and this incompleteness can be caused by a variety of factors, including but not limited to market frictions, such as trading costs and portfolio constraints, and an insufficient number of marketed assets relative to the class of risks to be hedged, such as jumps or volatility in underlying asset prices (Jackwerth, 2004; Staum, 2007; Willems and Morbee, 2008; Bondarenko and Longarela, 2009; Mnif, 2012; Marroquı et al., 2013; Kwak et al., 2014; Cheridito et al., 2016; Bouzianis and Hughston, 2020). To model the SDF under incomplete markets in a general setup, we first consider a discrete-time setting, while the cause of incompleteness is not specified. Particularly, we regard markets as incomplete when there are extra states relative to the traded assets, either as a result of frictions or idiosyncratic risk that cannot be diversified by trading the spanning assets in the market. We demonstrate that the constructed incomplete market SDF set has a unique boundary point, and only shrinks to this point when the market completes. This nice property allows us to examine features of the incomplete market SDF set, and enables us to determine the degree of market incompleteness.

To facilitate the empirical implementation of our results, we parameterize the market incompleteness in a continuous-time setting and propose that the unhedgeable risk is caused by a specific, but practically realistic source of incompleteness – stochastic jumps, where prices exhibit positive probabilities of unexpected changes, regardless of the interval between successive observations. Jump diffusion processes have been frequently used to model asset pricing, and their empirical performance in fitting the time-series properties of the asset price has been extensively evidenced by a number of studies (Dritschel and Protter, 1999; Svishchuk et al., 2000; Bellamy, 2001; Andersen et al., 2002; Carr et al., 2002; Geman, 2002; Willems and Morbee, 2008; Bouzianis and Hughston, 2020; Aït-Sahalia et al., 2021). In most cases, jumps cause incompleteness, except in very simple or unusual models, whence the market offers sufficient trading opportunities (Dritschel and Protter, 1999; Staum, 2007). As such, inspired by Merton (1976)'s work, whereby the total change in price should be a combination of the normal and abnormal price vibrations, our continuous-time framework considers complete markets as those in which asset prices are subject only to normal fluctuations, and incomplete markets as those with a positive likelihood of experiencing unanticipated changes in price. We demonstrate that the theoretical results developed in the discrete-time counterpart are still valid in the continuous-time setting, and we further use those results to establish the degree of market incompleteness.

In the literature, one popular measure for the degree of market incompleteness is through the correlation between the derivative price and its basis asset values (Cass and Citanna, 1998; Marin and Rahi, 2000; Dávila et al., 2017; Chen et al., 2021), where a lower correlation indicates a greater degree of incompleteness, and the market is complete only when the correlation reaches 100%. Another measure employs the root-mean-squared error between the payoff function of the derivative and the value of the optimal-replication portfolio constructed by the underlying securities (Bertsimas et al., 2001). The degree of incompleteness is thus determined by the extent to which the replicated portfolio is able to correctly price the derivative of the underlying assets.

Our approach is distinct from the previous ones in that instead of focusing on the linkage between the prices of derivative securities and their underlying securities, we only concern the prices of the primitive assets. In particular, considering that SDFs summarize investor preferences for payoffs across different states of the world, it is natural to define the degree of market incompleteness as how much the investor's risk preference under incomplete markets diverges from that under complete markets. The empirical implementation of this measure is summarized as follows, and we will elaborate in Section 2 and 3 with discrete- and continuous-time examples. After constructing the incomplete market SDF set and determining its corresponding complete SDF boundary point using the asset prices, we employ the distance between their probability distributions to estimate the degree of market incompleteness. As the complete market SDF is the boundary point of the incomplete market SDF set, this distance vanishes only if the extra-state probability is zero, that is, when the incomplete market SDF set degenerates into a unique complete market SDF. It can be challenging to gauge this distance, since the complete and incomplete SDFs have probability distributions of different dimensions, i.e., there are extra states with positive probabilities under incomplete markets compared with complete markets. A natural solution to this problem is the Wasserstein metric, a widely adopted measure in estimating the distance between distributions whose support differs, and its value reflects the least cost required to transform from one distribution to another (Mallows, 1972; Del Barrio et al., 1999; Villani, 2009; Nguyen, 2011).

The remainder of this paper proceeds as follows. Sections 2 and 3 sketch the discrete- and continuous-time frameworks to model SDF under incomplete markets, and show the applicability of our model in assessing the evolution of market incompleteness. Section 4 provides the empirical analysis and investigates the evolution of incompleteness in four of the world's largest stock markets. Section 5 concludes with a summary and a discussion of directions for future research.

2 Discrete-time Setting

In this section, we model the SDF set and the market incompleteness under three discrete-time setups, where each case is denoted as one risk-free bond$-A$ asset(s)$-T$ periods$-S$ states with $A \in \mathbb{N}$, and $T \geq 2 \in \mathbb{N}$. The number of traded assets is assumed to be less than the number of states at the end of each period, i.e., $A + 1 < S$, so that the markets are incomplete, while the cause of this incompleteness is not imposed. Then, there is a set of SDFs identified by the distribution of observed asset prices (Boyle et al., 2008; Kaido and White, 2009).

To motivate our study, we begin with a two-period market that has only one additional state relative to the number of traded assets in Sections 2.1 and 2.2. We formalize the setup as follows.

Assumption 2.1. *Suppose that there are one risk-free bond and $A \in \mathbb{N}$ risky assets. We consider a two-period market, $t \in \{0, 1\}$, with trading occurring on dates $t = 0, 1$. The outcome of the second period, $t = 1$, is uncertain, and represented by a finite set $\Omega = \{\omega^s\}_{s=1,2,\ldots,S}$ comprising $S = A + 2$ states of nature. Let $\mathcal{F}$ be the set of events with all subsets of Ω and P be the physical probability measure such that $P : \mathcal{F} \to \mathbb{R}$. There exists a set $\mathcal{P}$ of complete probability measures on $(\Omega, \mathcal{F})$ such that $P \in \mathcal{P}$. Letting $P(\omega^s) = \pi^s$ be the probability of state ω^s, π^s are strictly positive scalars for all $s = 1, 2, \ldots, S$ in incomplete markets, while $\pi^S = 0$ when the markets are complete.*

Assumption 2.1 has three implications. First, there are only two periods in the economy, and thus, we do not index the states by time in the subsequent two sections. We will extend our setup to multiperiods, where t takes the value from a finite sequence of real numbers in $[0, 1]$ that are equally-spaced, and continuous time, where t is generalized to take any value in $[0, 1]$. Second, without loss of generality, the last state is assumed to be the extra one, which is caused by an unknown source of market incompleteness, resulting in an idiosyncratic risk that cannot be hedged by the existing marketed assets. Third, our basic design requires the markets to be either complete, with the same number of marketed assets and states, or incomplete, with only one extra state. In Section 2.4, this restriction is relaxed to a finite number of extra states, and π^S is extended to a vector such that[1] $[\boldsymbol{\pi}^{\bar{s}}]_{\bar{s}=A+2,A+3,\dots,S} \in \mathbb{R}^{S-1-A}_{++}$. Then, the market completes only when[2] $[\boldsymbol{\pi}^{\bar{s}}]_{\bar{s}=A+2,A+3,\dots,S} = \mathbf{0}_{S-1-A}$.

2.1 One risk-free bond, one risky asset, two periods, three states (1-1-2-3)

Suppose that we have one risk-free bond and one risky primitive asset in the economy, and there are three states at period $t = 1$ such that $\Omega = (\omega^1, \omega^2, \omega^3)$, correspondingly there exists a set of physical probabilities

$$\boldsymbol{\Pi} = \left\{ \boldsymbol{\pi} = \left[\pi^1, \pi^2, \pi^3\right]^\top \in \mathbb{R}^3_{++} : \sum_{s=1}^{3} \pi^s = 1 \right\}. \tag{2.1}$$

As we have two assets, the gross rate of returns realized at the second period are of length two[3], i.e., $\boldsymbol{r}(\omega^s) = \left[r^{1,s}, 1\right]^\top$, where $r^{a,s}$ denotes the return for risky asset a in state s and, for simplicity, the return of the bond is 1, suggesting a zero risk-free rate. Let $\boldsymbol{r} = [\boldsymbol{r}(\omega^1), \boldsymbol{r}(\omega^2), \boldsymbol{r}(\omega^3)]$, we assume that the second-moment matrix of $\boldsymbol{r}$, $\mathbb{E}[\boldsymbol{r}\boldsymbol{r}^\top]$, is nonsingular, so that the cases where the entries of $\boldsymbol{r}$ is linearly dependent are ruled out. This restriction also guarantees that LOOP holds trivially for linear combinations of $\boldsymbol{r}$ (Hansen and Jagannathan, 1991). We can treat $\boldsymbol{r}$ as payoffs for assets with price one, and the asset pricing equation is expressed in the form[4]

$$\mathbb{E}_{\boldsymbol{\pi}}[\boldsymbol{r}\boldsymbol{M}] = \sum_{s=1}^{3} \boldsymbol{r}(\omega^s)\, \boldsymbol{M}(\omega^s)\, \pi^s = \mathbf{1}_2, \tag{2.2}$$

where the subscript of $\mathbb{E}$ is used to specify which probability measure is being used to compute the expectation. As discussed in Kaido and White (2009), the SDF $\boldsymbol{M}$ is a non-zero $\mathcal{F}$-measurable random variable such that $\boldsymbol{M} : \Omega \to \mathbb{M}_{\boldsymbol{\pi}}$, where $\mathbb{M}_{\boldsymbol{\pi}}$ is the set of SDFs under $P \in \mathcal{P}$ that satisfies Equation 2.2:

$$\mathbb{M}_{\boldsymbol{\pi}} := \{\boldsymbol{M} : \mathbb{E}_{\boldsymbol{\pi}}[\boldsymbol{r}\boldsymbol{M}] = \mathbf{1}_2\}. \tag{2.3}$$

[1] We write $\boldsymbol{v} \in \mathbb{R}^n_{++}$ for a vector that is strictly positive in all its coordinates.
[2] $\mathbf{0}_n$ denotes a zero vector of size n.
[3] For vectors and matrices, we shall use the superscript '$\top$' to denote transpose.
[4] $\mathbf{1}_n$ is a vector of ones in $\mathbb{R}^n$.

Let $M^s \equiv \boldsymbol{M}(\omega^s)$ and $\alpha = \pi^3 M^3 \in \mathbb{R}_*$ be the free variable[5,6], for any $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, we can think of $\boldsymbol{M}$ as a vector in $\mathbb{R}^3_*$, where the three coordinates give the values of $\boldsymbol{M}$ on the three possible outcomes. Thus, Equation 2.3 can be rewritten as[7]

$$\mathbb{M}_{\boldsymbol{\pi}} = \left\{ \boldsymbol{M} \in \mathbb{R}^3_* : \begin{bmatrix} M^1 \\ M^2 \\ M^3 \end{bmatrix} = \begin{bmatrix} \frac{1-r^{1,2}}{r^{1,1}-r^{1,2}} \\ \frac{r^{1,1}-1}{r^{1,1}-r^{1,2}} \\ 0 \end{bmatrix} \boldsymbol{\pi}^{-1} + \alpha \begin{bmatrix} \frac{r^{1,2}-r^{1,3}}{r^{1,1}-r^{1,2}} \\ -\frac{r^{1,1}-r^{1,3}}{r^{1,1}-r^{1,2}} \\ 1 \end{bmatrix} \boldsymbol{\pi}^{-1}, \alpha \in \mathbb{R}_* \right\}.$$

Lastly, we write the combined set $\boldsymbol{C}$ of $\mathbb{M}_{\boldsymbol{\pi}}$'s for all $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ as $\boldsymbol{C} := \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \boldsymbol{\Pi}\}$.

2.1.1 Set properties of $\boldsymbol{C}$

The following proposition provides the limit and boundary points of the set of probability measures in Equation 2.1, which will later be used to explore the boundary point of the constructed incomplete market SDF set $\boldsymbol{C}$. The proof is in Appendix A.1.

Proposition 2.1. *Consider the metric space* $(\bar{\boldsymbol{\Pi}}, d)$ *such that*

$$\bar{\boldsymbol{\Pi}} = \left\{ \boldsymbol{\pi} = \left[\pi^1, \pi^2, \pi^3\right]^\top : \sum_{s=1}^{3} \pi^s = 1, \pi^1, \pi^2 > 0, \pi^3 \geq 0 \right\}$$

and d *is the Euclidean distance metric. Then, the set of limit points of* $\boldsymbol{\Pi}$ *in* $(\bar{\boldsymbol{\Pi}}, d)$ *is*

$$L(\boldsymbol{\Pi}) = \left\{ \boldsymbol{\pi} = \left[\pi^1, \pi^2, \pi^3\right]^\top : \sum_{s=1}^{3} \pi^s = 1, \pi^1, \pi^2 > 0, \pi^3 \geq 0 \right\}$$

and the set of boundary points of $\boldsymbol{\Pi}$ *in* $\bar{\boldsymbol{\Pi}}$ *is*

$$\partial \boldsymbol{\Pi} = \left\{ \boldsymbol{\pi} = \left[\pi^1, \pi^2, \pi^3\right]^\top : \sum_{s=1}^{3} \pi^s = 1, \pi^1, \pi^2 > 0, \pi^3 = 0 \right\}.$$

The probability set $\bar{\boldsymbol{\Pi}}$ has the first two states being strictly positive and the last state being nonnegative, which therefore, covers all complete and incomplete market scenarios described in As-

[5] $\boldsymbol{M}$ can be also thought as the discounted Radon–Nikodym derivative, where the Radon–Nikodym derivative $\mathcal{D}$ is defined as a $\mathcal{F}$-measurable random variable such that for any $A \in \mathcal{F}, Q(A) = \int_A \mathcal{D} dP$ (Kaido and White, 2009) with Q be the risk-neutral probability measure. In our setup, assuming zero risk-free interest rate, $\boldsymbol{M} = \mathbb{E}[dQ/dP|\mathcal{F}]$. Since Q and P are equivalent in measure, they agree on which events have zero probability, and hence, $\boldsymbol{M}$ is non-zero.

[6] We write $\boldsymbol{v} \in \mathbb{R}^n_*$ for a vector that is non-zero in all its coordinates.

[7] To simplify our notation, for two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same dimensions, $\boldsymbol{AB}$ is their element-wise product with the same dimension as $\boldsymbol{A}$ and its element expressed as $(\boldsymbol{AB})_i = \boldsymbol{A}_i \times \boldsymbol{B}_i$. Similarly, for a vector $\boldsymbol{A}$, the element-wise power of a real number x on it is $\boldsymbol{A}^x$, i.e., $(\boldsymbol{A}^x)_i = (\boldsymbol{A}_i)^x$; for a real number x, the element-wise power of a vector $\boldsymbol{A}$ on it is $x^{\boldsymbol{A}}$, i.e., $(x^{\boldsymbol{A}})_i = x^{\boldsymbol{A}_i}$.

sumption 2.1. Proposition 2.1 implies that there is a unique boundary point for $\mathbf{\Pi}$ in $(\bar{\mathbf{\Pi}}, d)$ when $\lim \pi^3 \to 0$, which is compatible with Assumption 2.1 such that the incompleteness is introduced through a non-tradable risk with positive likelihood of occurrence.

In accordance to Proposition 2.1, the next result presents that the combined incomplete market SDF set $\boldsymbol{C}$ has the complete market SDF on its boundary, and its proof can be found in Appendix A.2.

Theorem 2.1. *Consider a metric space* $\left(\bar{\boldsymbol{C}}, d_1\right)$ *with* $\bar{\boldsymbol{C}} := \left\{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \bar{\mathbf{\Pi}}\right\}$ *and* d_1 *being the Wasserstein distance such that for* $\boldsymbol{x}, \boldsymbol{y} \in \bar{\mathbf{\Pi}}$,

$$d_1\left(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}\right) = \inf_{\boldsymbol{w}} \left\{ \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2\left(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}\right) : \boldsymbol{w} \in W(\boldsymbol{x}, \boldsymbol{y}) \right\}, \tag{2.4}$$

where[8] $W(\boldsymbol{x}, \boldsymbol{y}) := \left\{\boldsymbol{w} \in \mathbb{R}_+^{S_{\boldsymbol{y}} \times S_{\boldsymbol{x}}} : \boldsymbol{w}^\top \mathbf{1}_{S_{\boldsymbol{y}}} = \boldsymbol{x}, \boldsymbol{w}\boldsymbol{x} = \boldsymbol{y}\right\}$ *is the set of transport plans between* $\boldsymbol{x}$ *and* $\boldsymbol{y}$. $S_{\boldsymbol{z}}$ *is the number of states with non-zero probabilities and the superscript* $s_{\boldsymbol{z}}$ *is the index of the elements in the vector under the physical probability* $\boldsymbol{z} \in \bar{\mathbf{\Pi}}$. *For all* $s_{\boldsymbol{x}} = 1, 2, \ldots, S_{\boldsymbol{x}}$ *and* $s_{\boldsymbol{y}} = 1, 2, \ldots, S_{\boldsymbol{y}}$,

$$d_2\left(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}\right) = \left|v^{s_{\boldsymbol{x}}} - v^{s_{\boldsymbol{y}}}\right| + \left|u^{s_{\boldsymbol{x}}} - u^{s_{\boldsymbol{y}}}\right|, \tag{2.5}$$

where

$$\begin{cases} \boldsymbol{v}(\boldsymbol{x}) = \begin{bmatrix} \frac{1-r^{1,2}}{r^{1,1}-r^{1,2}} \\ \frac{r^{1,1}-1}{r^{1,1}-r^{1,2}} \\ 0 \end{bmatrix} \boldsymbol{x}^{-1} \text{ and } \boldsymbol{u}(\boldsymbol{x}) = \begin{bmatrix} \frac{r^{1,2}-r^{1,3}}{r^{1,1}-r^{1,2}} \\ -\frac{r^{1,1}-r^{1,3}}{r^{1,1}-r^{1,2}} \\ 1 \end{bmatrix} \boldsymbol{x}^{-1}, & \boldsymbol{x} \in \mathbf{\Pi}; \\ \boldsymbol{v}(\boldsymbol{x}) = \begin{bmatrix} \frac{1-r^{1,2}}{r^{1,1}-r^{1,2}} \\ \frac{r^{1,1}-1}{r^{1,1}-r^{1,2}} \end{bmatrix} \boldsymbol{x}^{-1} \text{ and } \boldsymbol{u}(\boldsymbol{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \boldsymbol{x}^{-1}, & \boldsymbol{x} \in \partial\mathbf{\Pi}. \end{cases}$$

Then, the set of limit points of $\boldsymbol{C}$ *in* $(\bar{\boldsymbol{C}}, d_1)$ *is*

$$L(\boldsymbol{C}) = \left\{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \mathbf{\Pi}\right\} \cup \left\{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \partial\mathbf{\Pi}\right\},$$

where, for any $\boldsymbol{\pi} \in \partial\mathbf{\Pi}$,

$$\mathbb{M}_{\boldsymbol{\pi}} = \left\{ \begin{bmatrix} M^1 \\ M^2 \end{bmatrix} = \begin{bmatrix} \frac{1-r^{1,2}}{r^{1,1}-r^{1,2}} \\ \frac{r^{1,1}-1}{r^{1,1}-r^{1,2}} \end{bmatrix} \boldsymbol{\pi}^{-1} \right\},$$

and the set of boundary points of $\boldsymbol{C}$ *is denoted as* $\partial\boldsymbol{C} = \left\{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \partial\mathbf{\Pi}\right\}$.

[8] We write $A \in \mathbb{R}_+^{M \times N}$ for matrix of dimension $M \times N$ that is non-negative in all its elements.

Theorem 2.1 utilizes the Wasserstein metric as the distance measure, which is a natural way to compare two probability distributions with different supports, and thus, suitable to quantify the divergence of the incomplete market SDFs from the complete market one. Based on Theorem 2.1, we have the following Lemma 2.1.1, suggesting that, for every probability measure $\boldsymbol{x} \in \boldsymbol{\Pi}$, there is a probability measure $\boldsymbol{y}^* \in \partial\boldsymbol{\Pi}$ that minimizes the distance between $\mathbb{M}_{\boldsymbol{x}} \in \boldsymbol{C}$ and $\mathbb{M}_{\boldsymbol{y}} \in \partial\boldsymbol{C}$. The proof of this lemma is presented in Appendix A.3, and we will further utilize it in the discussion of set properties of $\boldsymbol{C}$ and the measure for market incompleteness.

Lemma 2.1.1. *For every $\mathbb{M}_{\boldsymbol{x}} \in \boldsymbol{C}$, there exists $\mathbb{M}_{\boldsymbol{y}^*}$ such that*

$$\mathbb{M}_{\boldsymbol{y}^*} = \underset{\mathbb{M}_{\boldsymbol{y}} \in \partial\boldsymbol{C}}{\arg\min}\ d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}).$$

The next result develops an overview of the incomplete market SDF set, where $\boldsymbol{C}$ is *convex*, *open*, *bounded*, and *not compact*. The proof is provided in Appendix A.4.

Theorem 2.2. *Let $\boldsymbol{\Pi}$ be the set of physical probabilities satisfying Equation 2.1.*

Let $\mathbb{M}_{\boldsymbol{\pi}}$ be the identified SDF set in the 1-1-2-3 case satisfying Equation 2.3 under $\boldsymbol{\pi} \in \boldsymbol{\Pi}$.

Let $\boldsymbol{C}$ be the combined SDF set such that $\boldsymbol{C} = \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \boldsymbol{\Pi}\}$. Then, $\boldsymbol{C}$ is a convex set.

Let $\left(\bar{\boldsymbol{C}}, d_1\right)$ be the metric space such that $\bar{\boldsymbol{C}} = \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \bar{\boldsymbol{\Pi}}\}$ and for $\boldsymbol{x}, \boldsymbol{y} \in \bar{\boldsymbol{\Pi}}$, d_1 is as defined in Equation 2.4. Then, $\boldsymbol{C}$ is open, bounded and not compact under $\left(\bar{\boldsymbol{C}}, d_1\right)$.

2.1.2 Measure for market incompleteness

As discussed in Theorems 2.1 and 2.2, we can naturally employ the metric d_1 defined in Equation 2.4 to measure for market incompleteness. Given that at $t = 0$, the complete and incomplete market SDFs are assumed to be 1, and the distance between them is 0 following the metric d_1, for every $\boldsymbol{x} \in \bar{\boldsymbol{\Pi}}$, the degree of market incompleteness measured at $t = 1$ is defined as in Equation 2.6, which is the least transport cost from the SDF set $\mathbb{M}_{\boldsymbol{x}} \in \bar{\boldsymbol{C}}$ to the complete market SDF set $\mathbb{M}_{\boldsymbol{y}} \in \partial\boldsymbol{C}$:

$$\begin{aligned} MI(\boldsymbol{x}) &= \min_{\mathbb{M}_{\boldsymbol{y}} \in \partial\boldsymbol{C}} d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) \\ &= \min_{\boldsymbol{y} \in \partial\boldsymbol{\Pi}} \inf_{\boldsymbol{w}} \left\{ \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2\left(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}\right) : \boldsymbol{w} \in W(\boldsymbol{x}, \boldsymbol{y}) \right\}. \end{aligned} \tag{2.6}$$

Let $\boldsymbol{y}^* = \arg\min_{\boldsymbol{y} \in \partial\boldsymbol{\Pi}} d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})$, since d_1 is a valid metric and the complete market SDF is the boundary point of the incomplete market SDF set, the degree of market incompleteness equals zero only when markets are complete, i.e., $\lim_{\mathbb{M}_{\boldsymbol{x}} \to \mathbb{M}_{\boldsymbol{y}^*}} MI(\boldsymbol{x}) = 0$. As this degree increases (decreases), the cost to transport the incomplete market SDF to the complete market SDF increases (decreases), which implies more (less) divergence of the current market from completeness.

2.2 One Risk-free Bond, A Risky Assets, Two Periods, $A+2$ States (1-A-2-($A+2$))

This section extends the previous economy by having $A \geq 2$ risky primitive assets and $A+2$ states at $t = 1$ such that $\Omega = (\omega^s)_{s=1,2,\ldots,A+2}$. Correspondingly, for $P \in \mathcal{P}$, there is a set of physical probabilities

$$\mathbf{\Pi} = \left\{ \left[\pi^1, \pi^2, \ldots, \pi^{A+2}\right]^\top \in \mathbb{R}_{++}^{A+2} : \sum_{s=1}^{A+2} \pi^s = 1 \right\}. \tag{2.7}$$

In each state s, assuming a zero risk-free interest rate, the gross rate of return vector realized is of length $A+1$ and denoted as $\boldsymbol{r}(\omega^s) = \left[r^{1,s}, r^{2,s}, \ldots, r^{A,s}, 1\right]^\top$. Let $\boldsymbol{r} = [\boldsymbol{r}(\omega^1), \boldsymbol{r}(\omega^2), \ldots, \boldsymbol{r}(\omega^{A+2})]$, the second-moment matrix of $\boldsymbol{r}$ is again nonsingular. Recall that based on the asset pricing equation

$$\mathbb{E}_{\boldsymbol{\pi}}[\boldsymbol{r}\boldsymbol{M}] = \sum_{s=1}^{A+2} \boldsymbol{r}\left(\omega^s\right) \boldsymbol{M}\left(\omega^s\right) \pi^s = \mathbf{1}_{A+1}, \tag{2.8}$$

the SDF is a non-zero $\mathcal{F}$-measurable random variable such that $\boldsymbol{M} : \Omega \to \mathbb{M}_{\boldsymbol{\pi}}$, where $\mathbb{M}_{\boldsymbol{\pi}}$ is the set of SDFs under $P \in \mathcal{P}$ that satisfies Equation 2.8:

$$\mathbb{M}_{\boldsymbol{\pi}} = \left\{\boldsymbol{M} : \mathbb{E}_{\boldsymbol{\pi}}[\boldsymbol{r}\boldsymbol{M}] = \mathbf{1}_{A+1}\right\}. \tag{2.9}$$

Let $M^s \equiv \boldsymbol{M}(\omega^s)$ and $\alpha = \pi^{A+2} M^{A+2} \in \mathbb{R}_*$ be the free variable, for any $\boldsymbol{\pi} \in \mathbf{\Pi}$, we can think of $\boldsymbol{M}$ as a vector in $\mathbb{R}_*^{A+2}$, where each coordinate gives the value of $\boldsymbol{M}$ on the corresponding outcome. Thus, Equation 2.9 implies that[9]

$$\boldsymbol{r}' \boldsymbol{M}_{(1:A+1)*} \boldsymbol{\pi}_{(1:A+1)*} = \mathbf{1}_{A+1} + \alpha\left(-\boldsymbol{r}''\right),$$

where $\boldsymbol{r}' = (\boldsymbol{r})_{*(1:A+1)}$ and $\boldsymbol{r}'' = (\boldsymbol{r})_{*(A+2)}$, and we can be rewrite it as

$$\mathbb{M}_{\boldsymbol{\pi}} = \left\{\boldsymbol{M} \in \mathbb{R}_*^{A+2} : \boldsymbol{M} = \boldsymbol{v}(\boldsymbol{\pi}) + \alpha \boldsymbol{u}(\boldsymbol{\pi}), \alpha \in \mathbb{R}_*\right\},$$

where

$$\boldsymbol{v}(\boldsymbol{\pi}) = \begin{bmatrix} (\boldsymbol{r}')^{-1} \mathbf{1}_{A+1} \\ 0 \end{bmatrix} \boldsymbol{\pi}^{-1} \text{ and } \boldsymbol{u}(\boldsymbol{\pi}) = \begin{bmatrix} -(\boldsymbol{r}')^{-1}(\boldsymbol{r}'') \\ 1 \end{bmatrix} \boldsymbol{\pi}^{-1}.$$

[9] For a matrix $\boldsymbol{A}$, the i^{th} row of the matrix is denoted as $\boldsymbol{A}_{i*}$ and the j^{th} column of the matrix is denoted as $\boldsymbol{A}_{*j}$.

Lastly, the combined set $\boldsymbol{C}$ of $\mathbb{M}_{\boldsymbol{\pi}}$'s for all $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ is defined as $\boldsymbol{C} := \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \boldsymbol{\Pi}\}$.

2.2.1 Set properties of $\boldsymbol{C}$

As in the 1-1-2-3 case, we start by showing that the complete market SDF is indeed the boundary point of the incomplete market SDF set. The following proposition demonstrates that the probability distribution under complete markets is the boundary point of the set of probabilities under incomplete markets, and its proof is discussed in Appendix A.5.

Proposition 2.2. *Consider the metric space* $(\bar{\boldsymbol{\Pi}}, d)$ *such that*

$$\bar{\boldsymbol{\Pi}} = \left\{ \boldsymbol{\pi} = \left[\pi^1, \pi^2, \ldots, \pi^{A+2}\right]^\top : \sum_{s=1}^{A+2} \pi^s = 1, \pi^s > 0 \text{ for } s = 1, 2, \ldots, A+1, \pi^{A+2} \geq 0 \right\}$$

and d *is the Euclidean distance metric. Then, the set of limit points of* $\boldsymbol{\Pi}$ *in* $(\bar{\boldsymbol{\Pi}}, d)$ *is*

$$L(\boldsymbol{\Pi}) = \left\{ \boldsymbol{\pi} = \left[\pi^1, \pi^2, \ldots, \pi^{A+2}\right]^\top : \sum_{s=1}^{3} \pi^s = 1, \pi^s > 0 \text{ for } s = 1, 2, \ldots, A+1, \pi^{A+2} \geq 0 \right\}$$

and the set of boundary points of $\boldsymbol{\Pi}$ *in* $\bar{\boldsymbol{\Pi}}$ *is*

$$\partial \boldsymbol{\Pi} = \left\{ \boldsymbol{\pi} = \left[\pi^1, \pi^2, \ldots, \pi^{A+2}\right]^\top : \sum_{s=1}^{A+2} \pi^s = 1, \pi^s > 0 \text{ for } s = 1, 2, \ldots, A+1, \pi^{A+2} = 0 \right\}.$$

Then, the following result corroborates with Theorem 2.1 that the constructed incomplete SDF set indeed has its boundary point to be the complete market SDF under the defined metric space. The proof of this result is presented in Appendix A.6

Theorem 2.3. *Consider the metric space* $\left(\bar{\boldsymbol{C}}, d_1\right)$ *with* $\bar{\boldsymbol{C}} = \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \bar{\boldsymbol{\Pi}}\}$ *and* d_1 *being the Wasserstein distance such that for* $\boldsymbol{x}, \boldsymbol{y} \in \bar{\boldsymbol{\Pi}}$,

$$d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) = \inf_{\boldsymbol{w}} \left\{ \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2\left(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}\right) : \boldsymbol{w} \in W(\boldsymbol{x}, \boldsymbol{y}) \right\}, \tag{2.10}$$

where $W(\boldsymbol{x}, \boldsymbol{y}) := \{\boldsymbol{w} \in \mathbb{R}_+^{S_{\boldsymbol{y}} \times S_{\boldsymbol{x}}} : \boldsymbol{w}^\top \mathbf{1}_{S_{\boldsymbol{y}}} = \boldsymbol{x}, \boldsymbol{w}\boldsymbol{x} = \boldsymbol{y}\}$ *is the set of transport plans between* $\boldsymbol{x}$ *and* $\boldsymbol{y}$. $S_{\boldsymbol{z}}$ *is the number of states with non-zero probabilities and the subscript* $s_{\boldsymbol{z}}$ *is the index of the elements in the vector under the physical probability* $\boldsymbol{z} \in \bar{\boldsymbol{\Pi}}$. *For all* $s_{\boldsymbol{x}} = 1, 2, \ldots, S_{\boldsymbol{x}}$ *and* $s_{\boldsymbol{y}} = 1, 2, \ldots, S_{\boldsymbol{y}}$,

$$d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = \left|v^{s_{\boldsymbol{x}}} - v^{s_{\boldsymbol{y}}}\right| + \left|u^{s_{\boldsymbol{x}}} - u^{s_{\boldsymbol{y}}}\right|,$$

where

$$\begin{cases} \boldsymbol{v}(\boldsymbol{x}) = \begin{bmatrix} (\boldsymbol{r}')^{-1}\mathbf{1}_{A+1} \\ 0 \end{bmatrix}\boldsymbol{x}^{-1} \text{ and } \boldsymbol{u}(\boldsymbol{x}) = \begin{bmatrix} -(\boldsymbol{r}')^{-1}(\boldsymbol{r}'') \\ 1 \end{bmatrix}\boldsymbol{x}^{-1}, & \boldsymbol{x} \in \boldsymbol{\Pi} \\ \boldsymbol{v}(\boldsymbol{x}) = (\boldsymbol{r}')^{-1}\mathbf{1}_{A+1}\boldsymbol{x}^{-1} \text{ and } \boldsymbol{u}(\boldsymbol{x}) = \mathbf{0}_{A+1}\boldsymbol{x}^{-1}, & \boldsymbol{x} \in \partial\boldsymbol{\Pi} \end{cases}$$

Then, the set of limit points of $\boldsymbol{C}$ *in* $(\bar{\boldsymbol{C}}, d_1)$ *can be denoted as*

$$L(\boldsymbol{C}) = \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \boldsymbol{\Pi}\} \cup \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \partial\boldsymbol{\Pi}\},$$

where, for any $\boldsymbol{\pi} \in \partial\boldsymbol{\Pi}$,

$$\mathbb{M}_{\boldsymbol{\pi}} = \left\{\boldsymbol{M} = \left[(\boldsymbol{r}')^{-1}\mathbf{1}_{A+1}\boldsymbol{\pi}^{-1}_{1:(A+1)}\right]\right\},$$

and the set of boundary points of $\boldsymbol{C}$ *is then* $\partial\boldsymbol{C} = \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \partial\boldsymbol{\Pi}\}$.

Based on Theorem 2.3, we can then derive the following lemma, which will be employed further in the discussion of set properties and the degree of market incompleteness. Its proof is shown in Appendix A.7

Lemma 2.2.1. *For every* $\mathbb{M}_{\boldsymbol{x}} \in \boldsymbol{C}$, *there exists* $\mathbb{M}_{\boldsymbol{y}^*}$ *such that*

$$\mathbb{M}_{\boldsymbol{y}^*} = \underset{\mathbb{M}_{\boldsymbol{y}} \in \partial\boldsymbol{C}}{\arg\min}\; d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}).$$

The next result establishes the *convexity*, *openness*, *boundedness*, and *non-compactness*, for $\boldsymbol{C}$ in the 1-A-2-(A+2) case with its proof discussed in Appendix A.8

Theorem 2.4. *Let* $\boldsymbol{\Pi}$ *be the set of all the probability density measures under* P *satisfying Equation 2.7.*

Let $\mathbb{M}_{\boldsymbol{\pi}}$ *be the identified SDF set in the 1-A-2-(A+2) case satisfying Equation 2.9 given* $\boldsymbol{\pi}$ *in* $\boldsymbol{\Pi}$.

Let $\boldsymbol{C}$ *be the combined SDF set such that* $\boldsymbol{C} = \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \boldsymbol{\Pi}\}$. *Then,* $\boldsymbol{C}$ *is a convex set.*

Let $(\bar{\boldsymbol{C}}, d_1)$ *be the metric space such that* $\bar{\boldsymbol{C}} = \{\mathbb{M}_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \bar{\boldsymbol{\Pi}}\}$ *and for* $\boldsymbol{x}, \boldsymbol{y} \in \bar{\boldsymbol{\Pi}}$, d_1 *is as defined in Equation 2.10. Then,* $\boldsymbol{C}$ *is open, bounded and not compact under* $(\bar{\boldsymbol{C}}, d_1)$.

2.2.2 Measure for market incompleteness

Similar to the 1-1-2-3 case, based upon Theorems 2.3 and 2.4, we adopt d_1 defined in Equation 2.10 as the measure for market incompleteness. Given that at $t = 0$, the complete and incomplete market SDFs are assumed to be 1, and the distance between them is 0 following the metric d_1, for every

$\boldsymbol{x} \in \bar{\boldsymbol{\Pi}}$, the degree of market incompleteness is defined as in Equation 2.11, which is the least transport cost from $\mathbb{M}_{\boldsymbol{x}} \in \bar{\boldsymbol{C}}$ to $\mathbb{M}_{\boldsymbol{y}} \in \partial \boldsymbol{C}$:

$$MI(\boldsymbol{x}) = \min_{\mathbb{M}_{\boldsymbol{y}} \in \partial \boldsymbol{C}} d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}). \tag{2.11}$$

Let $\boldsymbol{y}^* = \arg\min_{\boldsymbol{y} \in \partial \boldsymbol{\Pi}} d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})$, since d_1 is a valid metric and the complete market SDF is the boundary point of the incomplete market SDF set, the degree of market incompleteness equals zero only when the markets become complete, i.e., $\lim_{\mathbb{M}_{\boldsymbol{x}} \to \mathbb{M}_{\boldsymbol{y}^*}} MI(\boldsymbol{x}) = 0$. A higher (lower) degree suggests more (less) cost required to transport between the complete and incomplete market SDF sets, implying that the market diverges more (less) from the complete market.

2.3 One risk-free bond, One Risky Asset, Three Periods, Three States (1-1-3-3)

We now extend our layout to a three-period financial market. Consider a time interval $[0,1]$, let the complete filtered probability space characterized by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,1]}, P)$. The filtration $\{\mathcal{F}_t\} = \{\mathcal{F}_t\}_{t \in [0,1]}$ is assumed to satisfy the usual properties (Protter, 2005). There are 2 equally-spaced subperiods in $[0,1]$, and let $h = 1/2$ be the time window. Suppose we have two long-lived assets, one risk-free bond and one risky primitive asset, available for trading at time points $\{0, 1/2, 1\}$, and three states at each $t = \{kh\}_{k=1,2}$ such that $\Omega_t = (\omega_t^1, \omega_t^2, \omega_t^3)$. Letting $P(\omega_t^s) = \pi_t^s$ be the physical probability of state ω_t^s, the corresponding set of physical probabilities under P is

$$\boldsymbol{\Pi}_t = \left\{ \boldsymbol{\pi}_t = \left[\pi_t^1, \pi_t^2, \pi_t^3\right]^\top \in \mathbb{R}_{++}^3 : \sum_{s=1}^{3} \pi_t^s = 1 \right\}. \tag{2.12}$$

Assuming a zero risk-free interest rate, the gross rate of asset returns realized at $t = kh$ in states s is of length two and denoted as $\boldsymbol{r}_t(\omega_t^s) = [r_t^{1,s}, 1]^\top$, where $r_t^{a,s}$ is the return of the risky asset a in state s at time t. Let $\boldsymbol{r}_t = [\boldsymbol{r}_t(\omega_t^1), \boldsymbol{r}_t(\omega_t^2), \boldsymbol{r}_t(\omega_t^3)]$, we assume as in the previous sections that the second-moment matrix of $\boldsymbol{r}_t$ is nonsingular. Then, each subperiod $[(k-1)h, kh]$ can be viewed as a two-period model as in the 1-1-2-3 case, and we have the random variable $\boldsymbol{m}_t : \Omega_t \to \mathbb{m}_{\boldsymbol{\pi}_t}$, where $\mathbb{m}_{\boldsymbol{\pi}_t}$ is the set of subperiod SDFs under $P_t \in \mathcal{P}_t$ that satisfies the asset pricing equation:

$$\mathbb{m}_{\boldsymbol{\pi}_t} := \{\boldsymbol{m}_t : \mathbb{E}_{\boldsymbol{\pi}_t}[\boldsymbol{r}_t \boldsymbol{m}_t] = \mathbf{1}_2\}.$$

Here, we denote the above SDF set as the one-period SDF set with $\boldsymbol{m}_t$ being the SDF that discount the asset payoff at time kh to its price at time $(k-1)h$ for $k = 1, 2$, and the multiperiod SDF at t is defined as $\boldsymbol{M}_t = \Pi_{k=1}^{t/h}\boldsymbol{m}_{kh}$ (Cochrane, 2009), which prices a k-period payoff and satisfies the following equation

$$\mathbb{E}_{\boldsymbol{\pi}_{kh}}[\boldsymbol{r}_{kh}\boldsymbol{M}_{kh}] = \boldsymbol{M}_{(k-1)h}.$$

Then, the multiperiod SDF set at t can be written in the form $\mathbb{M}_{\boldsymbol{\pi}_t} = \Pi_{k=1}^{t/h}\mathbb{m}_{\boldsymbol{\pi}_{kh}}$.

To examine the evolution of market incompleteness over time, we focus on the single-period SDF set at time t such that for every $\boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t$

$$\mathbb{m}_{\boldsymbol{\pi}_t} = \left\{ \boldsymbol{m}_t \in \mathbb{R}^3_* : \begin{bmatrix} m_t^1 \\ m_t^2 \\ m_t^3 \end{bmatrix} = \begin{bmatrix} \frac{1-r_t^{1,2}}{r_t^{1,1}-r_t^{1,2}} \\ \frac{r_t^{1,1}-1}{r_t^{1,1}-r_t^{1,2}} \\ 0 \end{bmatrix} \boldsymbol{\pi}_t^{-1} + \alpha \begin{bmatrix} \frac{r_t^{1,2}-r_t^{1,3}}{r_t^{1,1}-r_t^{1,2}} \\ -\frac{r_t^{1,1}-r_t^{1,3}}{r_t^{1,1}-r_t^{1,2}} \\ 1 \end{bmatrix} \boldsymbol{\pi}_t^{-1}, \alpha \in \mathbb{R}_* \right\},$$

where $m_t^s \equiv \boldsymbol{m}_t(\omega_t^s)$ and $\alpha = (1 - \pi_t^1 - \pi_t^2)m_t^3$, and the set that contains all single-period SDFs in each period t is $\boldsymbol{c}_t := \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t\}$.

Further, analogous to the results proved in the 1-1-2-3 case, the following proposition and theorems hold.

Proposition 2.3. *For $t = kh$, consider the metric space $(\bar{\boldsymbol{\Pi}}_t, d)$ such that*

$$\bar{\boldsymbol{\Pi}}_t = \left\{ \boldsymbol{\pi}_t = \left[\pi_t^1, \pi_t^2, \pi_t^3\right]^\top : \sum_{s=1}^{3} \pi_t^s = 1, \pi_t^1, \pi_t^2 > 0, \pi_t^3 \ge 0 \right\}$$

and d is the Euclidean distance metric. Then, the set of limit points of $\boldsymbol{\Pi}_t$ in $(\bar{\boldsymbol{\Pi}}_t, d)$ is

$$L(\boldsymbol{\Pi}_t) = \left\{ \boldsymbol{\pi}_t = \left[\pi_t^1, \pi_t^2, \pi_t^3\right]^\top : \sum_{s=1}^{3} \pi_t^s = 1, \pi_t^1, \pi_t^2 > 0, \pi_t^3 \ge 0 \right\},$$

and the set of boundary points of $\boldsymbol{\Pi}_t$ in $\bar{\boldsymbol{\Pi}}_t$ is

$$\partial\boldsymbol{\Pi}_t = \left\{ \boldsymbol{\pi}_t = \left[\pi_t^1, \pi_t^2, \pi_t^3\right]^\top : \sum_{s=1}^{3} \pi_t^s = 1, \pi_t^1, \pi_t^2 > 0, \pi_t^3 = 0 \right\}.$$

Theorem 2.5. *Consider the metric space $(\bar{\boldsymbol{c}}_t, d_1)$ such that $\bar{\boldsymbol{c}}_t = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \bar{\boldsymbol{\Pi}}_t\}$ and for $\boldsymbol{x}_t, \boldsymbol{y}_t \in \bar{\boldsymbol{\Pi}}_t$,*

$$d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t}) = \inf_{\boldsymbol{w}_t} \left\{ \sum_{s_{\boldsymbol{x}_t}=1}^{S_{\boldsymbol{x}_t}} \sum_{s_{\boldsymbol{y}_t}=1}^{S_{\boldsymbol{y}_t}} w_t^{s_{\boldsymbol{x}_t} s_{\boldsymbol{y}_t}} x_t^{s_{\boldsymbol{x}_t}} d_2\left(m_t^{s_{\boldsymbol{x}_t}}, m_t^{s_{\boldsymbol{y}_t}}\right) : \boldsymbol{w}_t \in W(\boldsymbol{x}_t, \boldsymbol{y}_t) \right\}, \tag{2.13}$$

where $W(\boldsymbol{x}_t, \boldsymbol{y}_t) := \{\boldsymbol{w}_t \in \mathbb{R}_+^{S_{\boldsymbol{y}_t} \times S_{\boldsymbol{x}_t}} : \boldsymbol{w}_t^\top \mathbf{1}_{S_{\boldsymbol{y}_t}} = \boldsymbol{x}_t, \boldsymbol{w}_t\boldsymbol{x}_t = \boldsymbol{y}_t\}$ is the set of transport plans between $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$. $S_{\boldsymbol{z}_t}$ is the number of states with non-zero probabilities and the subscript $s_{\boldsymbol{z}_t}$ is the index *of the elements in the vector under the physical probability* $\boldsymbol{z}_t \in \bar{\boldsymbol{\Pi}}_t$*. For all* $s_{\boldsymbol{x}_t} = 1, 2, \ldots, S_{\boldsymbol{x}_t}$ *and* $s_{\boldsymbol{y}_t} = 1, 2, \ldots, S_{\boldsymbol{y}_t}$,

$$d_2\left(m_t^{s_{\boldsymbol{x}_t}}, m_t^{s_{\boldsymbol{y}_t}}\right) = \left|v^{s_{\boldsymbol{x}_t}} - v^{s_{\boldsymbol{y}_t}}\right| + \left|u^{s_{\boldsymbol{x}_t}} - u^{s_{\boldsymbol{y}_t}}\right|,$$

where

$$\begin{cases} \boldsymbol{v}(\boldsymbol{x}_t) = \begin{bmatrix} \frac{1-r_t^{1,2}}{r_t^{1,1}-r_t^{1,2}} \\ \frac{r_t^{1,1}-1}{r_t^{1,1}-r_t^{1,2}} \\ 0 \end{bmatrix} \boldsymbol{x}_t^{-1} \text{ and } \boldsymbol{u}(\boldsymbol{x}_t) = \begin{bmatrix} \frac{r_t^{1,2}-r_t^{1,3}}{r_t^{1,1}-r_t^{1,2}} \\ -\frac{r_t^{1,1}-r_t^{1,3}}{r_t^{1,1}-r_t^{1,2}} \\ 1 \end{bmatrix} \boldsymbol{x}_t^{-1}, & \boldsymbol{x}_t \in \boldsymbol{\Pi}_t; \\ \boldsymbol{v}(\boldsymbol{x}_t) = \begin{bmatrix} \frac{1-r_t^{1,2}}{r_t^{1,1}-r_t^{1,2}} \\ \frac{r_t^{1,1}-1}{r_t^{1,1}-r_t^{1,2}} \end{bmatrix} \boldsymbol{x}_t^{-1} \text{ and } \boldsymbol{u}(\boldsymbol{x}_t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \boldsymbol{x}_t^{-1}, & \boldsymbol{x}_t \in \partial\boldsymbol{\Pi}_t. \end{cases}$$

Then, the set of limit points of $\boldsymbol{c}_t$ *in* $\bar{\boldsymbol{c}}_t$ *can be denoted as*

$$L(\boldsymbol{c}_t) = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t\} \cup \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \partial\boldsymbol{\Pi}_t\},$$

where, for any $\boldsymbol{\pi}_t \in \partial\boldsymbol{\Pi}_t$,

$$\mathbb{m}_{\boldsymbol{\pi}_t} = \left\{ \begin{bmatrix} m_t^1 \\ m_t^2 \end{bmatrix} = \begin{bmatrix} \frac{1-r_t^{1,2}}{r_t^{1,1}-r_t^{1,2}} \\ \frac{r_t^{1,1}-1}{r_t^{1,1}-r_t^{1,2}} \end{bmatrix} \boldsymbol{\pi}_t^{-1} \right\},$$

and the set of boundary points of $\boldsymbol{c}_t$ *is then* $\partial\boldsymbol{c}_t = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \partial\boldsymbol{\Pi}_t\}$.

Theorem 2.6. *Let* $\boldsymbol{\Pi}_t$ *be the set of physical probability measures satisfying Equation 2.12. Let* $\mathbb{m}_{\boldsymbol{\pi}_t}$ *be the identified SDF set in the 1-1-3-3 case given the probability measure* $\boldsymbol{\pi}_t$ *in* $\boldsymbol{\Pi}_t$*, and let* $\boldsymbol{c}_t$ *be the combined SDF set such that* $\boldsymbol{c}_t = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t\}$*. Then,* $\boldsymbol{c}_t$ *is a convex set.*

Let $(\bar{\boldsymbol{c}}_t, d_1)$ *be the metric space such that* $\bar{\boldsymbol{c}}_t = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \bar{\boldsymbol{\Pi}}_t\}$ *and for* $\boldsymbol{x}_t, \boldsymbol{y}_t \in \bar{\boldsymbol{\Pi}}_t$*,* d_1 *is as defined in Equation 2.13. Then,* $\boldsymbol{c}_t$ *is open, bounded and not compact under* $(\bar{\boldsymbol{c}}_t, d_1)$*.*

Last, we derive the degree of market incompleteness based on the set properties in Theorems 2.5 and 2.6. Again, since at time 0, the complete and incomplete market SDFs are assumed to be 1, the distance between them is 0 following the metric d_1. Thus, given $\{\boldsymbol{x}_{kh} \in \bar{\boldsymbol{\Pi}}_{kh}\}_{k=1}^{t/h}$, the degree of market incompleteness at t is defined as in Equation 2.14, which is the average of the least transport costs from $\mathbb{m}_{\boldsymbol{x}_{kh}} \in \bar{\boldsymbol{c}}_{kh}$ to $\mathbb{m}_{\boldsymbol{y}_{kh}} \in \partial\boldsymbol{c}_{kh}$ from time h to t, indicating that we weigh the degree of market incompleteness equally across subperiods.

$$MI\left(\{\boldsymbol{x}_{kh}\}_{k=0}^{t/h}\right) = \frac{h}{t} \sum_{k=0}^{t/h} \min_{\mathbb{m}_{\boldsymbol{y}_{kh}} \in \partial\boldsymbol{c}_{kh}} d_1\left(\mathbb{m}_{\boldsymbol{x}_{kh}}, \mathbb{m}_{\boldsymbol{y}_{kh}}\right). \tag{2.14}$$

As the subperiod degrees of market incompleteness are functions of their subperiod asset returns, which are uncorrelated, we take the average of them so that $MI\left(\{\boldsymbol{x}_{kh}\}_{k=0}^{t/h}\right)$ is not monotonic in t. Moreover, since d_1 is a valid metric and the complete market SDF is the boundary point of the incomplete market SDF set in each subperiod, the degree of market incompleteness equals zero only when markets are dynamically complete, i.e., when markets are complete at every subperiods. Hence, the estimated $\hat{M}I\left(\{\boldsymbol{x}_{kh}\}_{k=0}^{1}\right)$ and $\hat{M}I\left(\{\boldsymbol{x}_{kh}\}_{k=0}^{2}\right)$ depict the evolution of market incompleteness over the time interval $[0,1]$.

2.4 Generalization of the Discrete-time Setting

Finally, we consider a generalized discrete-time setting in which there are a finite number of additional states rather than only one, while still allowing any type of incompleteness in the market. The setup is formalized as follows.

Assumption 2.2. *Suppose that there are one risk-free bond and $A \in \mathbb{N}$ risky assets. Let $S \geq A+2$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, P)$ be the complete filtered probability space, where Ω, $\mathcal{F}$ and P are the same as defined in Assumption 2.1, and the filtration $\{\mathcal{F}_t\} = \{\mathcal{F}_t\}_{t\in[0,1]}$ is assumed to satisfy the usual properties (Protter, 2005). Suppose that there are $K \geq 1$ equally-spaced subperiods in $[0,1]$, and let $h = 1/K$ be the time window. All assets are long-lived and available for trading at time points $\{0, h, \ldots, (K-1)h, 1\}$, and there are S states at each $t = \{kh\}_{k=1,2,\ldots,K}$ such that $\Omega_t = (\omega_t^s)_{s=1,2,\ldots,S}$. Let $P_t(\omega_t^s) = \pi_t^s$ be the physical probability of state ω_t^s, where π_t^s are strictly positive scalars for all s in incomplete markets, while $[\boldsymbol{\pi}_t^{\bar{s}}]_{\bar{s}=A+2,A+3,\ldots,S} = \mathbf{0}_{S-A-1}$ when the markets are complete. There exists a set $\mathcal{P}_t$ of complete probability measures on $(\Omega_t, \mathcal{F}_t)$ at each $t = \{kh\}_{k=1,\ldots,K}$ such that $P_t \in \mathcal{P}_t$.*

Assumption 2.2 is a generalization of Assumption 2.1, where the number of jumps at time t is no longer restricted to be one, and we allow for multiple extra states in each subperiod. Correspondingly, the set of physical probabilities at t when the markets are incomplete is[10]

$$\boldsymbol{\Pi}_t = \left\{ \left[\pi_t^1, \pi_t^2, \ldots, \pi_t^S\right] \in \mathbb{R}_{++}^S : \sum_{s=1}^{S} \pi_t^s = 1 \right\}. \tag{2.15}$$

At time $t = kh$, the gross rate of return vector of length $A+1$ realized at state s is $\boldsymbol{r}_t(\omega_t^s) = \left[r_t^{1,s}, r_t^{2,s}, \ldots, r_t^{A,s}, r_t^0\right]^\top$, where $r_t^{a,s}$ denotes the return of the risky asset a in state s, and r_t^0 denotes the risk-free rate. Let $\boldsymbol{r}_t = [\boldsymbol{r}_t(\omega_t^s)]_{s=1,2,\ldots,S}$, when the assumption that the second-moment of $\boldsymbol{r}_t$ is

[10] Here, we restrict all π_t^s's to be strictly positive under incomplete markets, instead of letting $\pi_t^s \geq 0$ for $s \geq A+2$, because the latter can be simply reduced to a lower-dimensional case. For instance, if $\pi_t^S = 0$ under both complete and incomplete markets, then, our setup can be reduced to an $(S-1)$-dimensional case.

nonsingular holds, at the end of each subperiod t, we have the random variable $\boldsymbol{m}_t : \Omega_t \to \mathbb{m}_{\boldsymbol{\pi}_t}$, where $\mathbb{m}_{\boldsymbol{\pi}_t}$ is the set of SDFs under $P_t \in \mathcal{P}_t$ that satisfies the asset pricing equation:

$$\mathbb{m}_{\boldsymbol{\pi}_t} := \{\boldsymbol{m}_t : \mathbb{E}_{\boldsymbol{\pi}_t}[\boldsymbol{r}_t \boldsymbol{m}_t] = \mathbf{1}_{A+1}.\}\,. \tag{2.16}$$

Subsequently, let $\boldsymbol{\alpha}_t = [\pi_t^{\bar{s}} m_t^{\bar{s}}]_{\bar{s}=A+2,A+3,\ldots,S} \in \mathbb{R}_*^{S-A-1}$ be the vector of free variables, we can derive the single-period SDF set at time t in the form such that for every $\boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t$

$$\mathbb{m}_{\boldsymbol{\pi}_t} = \left\{\boldsymbol{m}_t \in \mathbb{R}_*^S : \boldsymbol{m}_t = \boldsymbol{v}_t(\boldsymbol{\pi}_t) + \boldsymbol{u}_t(\boldsymbol{\pi}_t)\boldsymbol{\alpha}_t, \boldsymbol{\alpha}_t \in \mathbb{R}_*^{S-A-1}\right\},$$

where

$$\boldsymbol{v}_t(\boldsymbol{\pi}_t) = \begin{bmatrix} (\boldsymbol{r}_t')^{-1}\,\mathbf{1}_{A+1} \\ \mathbf{0}_{S-A-1} \end{bmatrix} \boldsymbol{\pi}_t^{-1} \text{ and } \boldsymbol{u}_t(\boldsymbol{\pi}_t) = \begin{bmatrix} -(\boldsymbol{r}_t')^{-1}(\boldsymbol{r}_t'') \\ \mathbf{1}_{S-A-1} \end{bmatrix} \boldsymbol{\pi}_t^{-1}$$

with $\boldsymbol{r}_t' = (\boldsymbol{r}_t)_{*(1:A+1)}$ and $\boldsymbol{r}_t'' = (\boldsymbol{r}_t)_{*(A+2:S)}$. Lastly, the combined set $\boldsymbol{c}_t$ of $\mathbb{m}_{\boldsymbol{\pi}_t}$'s for all $\boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t$ is defined as $\boldsymbol{c}_t := \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t\}$.

Now, we demonstrate that the results in previous special cases hold in the generalized setting. The following proposition indicates that the probability distribution under complete markets is the boundary point of the set of probabilities under incomplete markets, and its proof is discussed in Appendix A.9.

Proposition 2.4. *Consider the metric space $(\bar{\boldsymbol{\Pi}}_t, d)$ such that*

$$\bar{\boldsymbol{\Pi}}_t = \boldsymbol{\Pi}_t \cup \left\{\boldsymbol{\pi}_t = \left[\pi_t^1, \pi_t^2, \ldots, \pi_t^S\right]^\top : \sum_{s=1}^{S} \pi_t^s = 1, \pi_t^s > 0 \text{ for } s = 1, 2, \ldots, A+1, \right.$$
$$\left. \pi_t^s = 0 \text{ for } s = A+2, A+3\ldots, S\right\}$$

and d is the Euclidean distance metric.

Then, the set of limit points of $\boldsymbol{\Pi}_t$ in $(\bar{\boldsymbol{\Pi}}_t, d)$ is $L(\boldsymbol{\Pi}_t) = \bar{\boldsymbol{\Pi}}_t$, and the set of boundary points of $\boldsymbol{\Pi}_t$ in $\bar{\boldsymbol{\Pi}}_t$ is

$$\partial\boldsymbol{\Pi}_t = \left\{\boldsymbol{\pi}_t = \left[\pi_t^1, \pi_t^2, \ldots, \pi_t^S\right]^\top : \sum_{s=1}^{S} \pi_t^s = 1, \pi_t^s > 0 \text{ for } s = 1, 2, \ldots, A+1, \right.$$
$$\left. \pi_t^s = 0 \text{ for } s = A+2, A+3\ldots, S\right\}.$$

Then, we can establish the following result such that the constructed incomplete set SDF has its boundary point to be the complete market SDF under the defined metric space. The proof of this result is presented in Appendix A.10

Theorem 2.7. *Consider the metric space* $(\bar{\boldsymbol{c}}_t, d_1)$ *with* $\bar{\boldsymbol{c}}_t = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi_t} \in \bar{\boldsymbol{\Pi}}_t\}$ *and* d_1 *is as defined in Equation 2.13. Then, the set of limit points of* $\boldsymbol{c}_t$ *in* $(\bar{\boldsymbol{c}}_t, d_1)$ *is*

$$L(\boldsymbol{c}_t) = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t\} \cup \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \partial\boldsymbol{\Pi}_t\},$$

where, for any $\boldsymbol{\pi}_t \in \partial\boldsymbol{\Pi}_t$,

$$\mathbb{m}_{\boldsymbol{\pi}_t} = \left\{\boldsymbol{m}_t = \left[(\boldsymbol{r}_t')^{-1} \mathbf{1}_{A+1} (\boldsymbol{\pi}_t)^{-1}_{1:(A+1)}\right]\right\},$$

and the set of boundary points of $\boldsymbol{c}_t$ *is then* $\partial\boldsymbol{c}_t = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \partial\boldsymbol{\Pi}_t\}$.

The following lemma, derived from Theorem 2.7, implies that for every probability measure $\boldsymbol{x}_t \in \boldsymbol{\Pi}_t$, there is a probability measure $\boldsymbol{y}_t^* \in \partial\boldsymbol{\Pi}_t$ that minimizes the distance between $\mathbb{m}_{\boldsymbol{x}_t} \in \boldsymbol{c}_t$ and $\mathbb{m}_{\boldsymbol{y}_t} \in \partial\boldsymbol{c}_t$. This lemma enables us to further explore the set properties of the incomplete market SDF as well as the degree of market incompleteness. Its proof is presented in Appendix A.11.

Lemma 2.4.1. *For every* $\mathbb{m}_{\boldsymbol{x}_t} \in \boldsymbol{c}_t$, *there exists* $\mathbb{m}_{\boldsymbol{y}_t^*}$ *such that*

$$\mathbb{m}_{\boldsymbol{y}_t^*} = \underset{\mathbb{m}_{\boldsymbol{y}_t} \in \boldsymbol{c}_t}{\arg\min} \{d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t})\}.$$

The next result establishes the *convexity*, *openness*, *boundedness*, and *non-compactness*, for $\boldsymbol{c}_t$ in the generalized case with its proof discussed in Appendix A.12.

Theorem 2.8. *Let* $\boldsymbol{\Pi}_t$ *be the set of all the probability density measures under* P_t *satisfying Equation 2.15.*

Let $\mathbb{m}_{\boldsymbol{\pi}_t}$ *be the identified SDF set in the generalized case satisfying Equation 2.16 given* $\boldsymbol{\pi}_t$ *in* $\boldsymbol{\Pi}_t$.

Let $\boldsymbol{c}_t$ *be the combined SDF set such that* $\boldsymbol{c}_t = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \boldsymbol{\Pi}_t\}$. *Then,* $\boldsymbol{c}_t$ *is a convex set.*

Let $(\bar{\boldsymbol{c}}_t, d_1)$ *be the metric space such that* $\bar{\boldsymbol{c}}_t = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \bar{\boldsymbol{\Pi}}_t\}$ *and for* $\boldsymbol{x}_t, \boldsymbol{y}_t \in \bar{\boldsymbol{\Pi}}_t$, d_1 *is as defined in Equation 2.13. Then,* $\boldsymbol{c}_t$ *is open, bounded and not compact under* $(\bar{\boldsymbol{c}}_t, d_1)$.

Last, based on the set properties in Theorems 2.7 and 2.8, given $\left\{\boldsymbol{x}_{kh} \in \bar{\boldsymbol{\Pi}}_{kh}\right\}_{k=1}^{t/h}$, the degree of market incompleteness at t is defined as in Equation 2.17, which is the mean of the least transport costs from $\mathbb{m}_{\boldsymbol{x}_{kh}} \in \bar{\boldsymbol{c}}_{kh}$ to $\mathbb{m}_{\boldsymbol{y}_{kh}} \in \partial \boldsymbol{c}_{kh}$ from time 0 up to t[11]:

$$MI\left(\{\boldsymbol{x}_{kh}\}_{k=1}^{t/h}\right) = \frac{h}{t}\sum_{k=0}^{t/h} \min_{\mathbb{m}_{\boldsymbol{y}_{kh}} \in \partial \boldsymbol{c}_{kh}} d_1\left(\mathbb{m}_{\boldsymbol{x}_{kh}}, \mathbb{m}_{\boldsymbol{y}_{kh}}\right). \tag{2.17}$$

The subperiod degrees of market incompleteness are functions of their subperiod asset returns, which are uncorrelated, then by taking the average of these subperiod transport costs, we get the degree of market incompleteness at t, $MI\left(\{\boldsymbol{x}_{kh}\}_{k=0}^{t/h}\right)$, which is not monotonic in t. Moreover, as d_1 is a valid metric and the complete market SDF is the boundary point of the incomplete market SDF set in each subperiod, the degree of market incompleteness equals zero only when the markets are dynamically complete, i.e., when the markets are complete at every subperiods. Hence, the estimated $\hat{M}I\left(\{\boldsymbol{x}_{kh}\}_{k=0}^{t/h}\right)$ depict the evolution of market incompleteness over the time interval $[0, 1]$.

3 Continuous-time Setting

The modelling of the SDF set and the degree of market incompleteness in the continuous-time setting is similar to that used in its discrete-time counterpart, but there are differences. Particularly, in order to implement our approach in empirical works, we further parameterize the market incompleteness by specifying that the asset prices are generated by the jump diffusion processes, which constitute an important class of incomplete market models and are realistic in practice (Kaido and White, 2009).

Same as Assumption 2.2, we have the time interval $[0, 1]$ and the complete filtered probability space characterized by $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, P)$. There exists a set $\mathcal{P}$ of complete probability measures on $(\Omega, \mathcal{F})$ such that $P \in \mathcal{P}$. Suppose that there are $A \in \mathbb{N}$ risky assets and that, in incomplete markets, $\mathbb{R}^A$ −valued asset price process $\{\boldsymbol{S}_t\}$ solves the stochastic differential equation (SDE)

$$\frac{d\boldsymbol{S}_t}{\boldsymbol{S}_{t-}} = \boldsymbol{\mu}_t^B dt + \boldsymbol{\sigma}_t^B d\boldsymbol{B}_t + \boldsymbol{J}_t d\tilde{\boldsymbol{N}}_t, \tag{3.1}$$

where $\{\boldsymbol{\mu}_t^B\}$ is an $\mathbb{R}^A$-valued adapted drift process, $\{\boldsymbol{\sigma}_t^B\}$ is an $\mathbb{R}^{A\times A}$-valued adapted diffusion coefficient process. $\boldsymbol{J}_t$ is a random jump amplitude, which is predictable and $\boldsymbol{J}_t > -1$, implying that all elements in $\boldsymbol{S}_t$ remain positive, consistent with the limited liability provision (Aït-Sahalia et al., 2009). Then, it is convenient to have $\boldsymbol{J}_t = \exp(\boldsymbol{Q}_t) - \boldsymbol{1}_A$ as in Hanson and Westman (2002), where $\boldsymbol{Q}_t$ follows a normal distribution with mean $\boldsymbol{\mu}_t^J$ and standard deviation $\boldsymbol{\sigma}_t^J$. $\{\boldsymbol{B}_t\}$ is a vector of A independent Brownian motions under P and $\tilde{\boldsymbol{N}}_t = \boldsymbol{N}_t - \boldsymbol{v}_t(dx)t$ is the compensated martingales of

[11] Again, we assume that the complete and incomplete market SDFs are 1 at time 0, and the distance between them is 0 following the metric d_1.

Poisson process $\boldsymbol{N}_t$ with mean measure $\boldsymbol{v}_t(dx)t$, where $\boldsymbol{v}_t(dx) \geq 0$ is taken to be the Lévy measure associated with an A-dimensional pure-jump Lévy process. Thus, $\boldsymbol{v}_t(dx)$ is a σ-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$[12] such that $\boldsymbol{v}_t(\{0\}) = 0$, suggesting that $\boldsymbol{v}$ does not have mass on 0, and

$$\int_{\mathbb{R}} \min\left(1, |x|^2\right) \boldsymbol{v}_t(dx) < \infty, \tag{3.2}$$

so the jumps have finite variation. $\{\boldsymbol{B}_t\}$ and $\tilde{\boldsymbol{N}}_t$ are independent under P and are adapted to $\{\mathcal{F}_t\}$. We require that

$$\mathbb{P}\left[\int_0^t \left(\left|\boldsymbol{\mu}_s^B\right| + \boldsymbol{\sigma}_s^{B2} + \boldsymbol{J}_s^2 \boldsymbol{v}_s(dx)\right) ds < \infty\right] = 1 \tag{3.3}$$

for $t \geq 0$, which is a sufficient restriction to ensure that the integral with respect to the compensated Poisson random measure exists for both small and large jumps. We assume that the market is built with a risk-free bond with a known rate of return of r_t.

Given the $\mathbb{R}^A$-valued adapted processes $\{\boldsymbol{\psi}_t\}_{t\geq 0}$ and $\{\boldsymbol{\gamma}_t\}_{t\geq 0}$, the Girsanov transformation defines the new adapted processes $\{\bar{\boldsymbol{B}}_t\}$ and $\{\bar{\boldsymbol{N}}_t\}$ by adjusting the original Brownian motion and the compensated martingales of Poisson process:

$$\bar{\boldsymbol{B}}_t = \boldsymbol{B}_t + \int_0^t \boldsymbol{\psi}_s ds \text{ and } \bar{\boldsymbol{N}}_t = \tilde{\boldsymbol{N}}_t + \int_0^t \boldsymbol{v}_s(dx)\boldsymbol{\gamma}_s ds.$$

Then, the asset return process under the risk-neutral probability measure is

$$\frac{d\boldsymbol{S}_t}{\boldsymbol{S}_{t-}} = r_t \mathbf{1}_A^B dt + \boldsymbol{\sigma}_t^B d\bar{\boldsymbol{B}}_t + \boldsymbol{J}_t d\bar{\boldsymbol{N}}_t$$

and the existence of the SDF holds only for $(\boldsymbol{\psi}_t, \boldsymbol{\gamma}_t)$ such that

$$\boldsymbol{\mu}_t^B - r_t \mathbf{1}_A - \boldsymbol{\sigma}_t^B \boldsymbol{\psi}_t - \boldsymbol{J}_t \boldsymbol{\gamma}_t \boldsymbol{v}_t(dx) = 0, \ a.s. - P.$$

Such vectors are called the market prices of risk, where $\{\boldsymbol{\psi}_t\}_{t\geq 0}$ is the adapted Brownian market price of risk and $\{\boldsymbol{\gamma}_t\}_{t\geq 0}$ is the predictable jump market price of risk, and $\boldsymbol{\gamma}_t < 1$ for $t \geq 0$.

Let $\boldsymbol{\phi}_t = \left(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)\right) \in \boldsymbol{\Phi}_t$, where

$$\boldsymbol{\Phi}_t = \left\{\left(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)\right) : \boldsymbol{v}_t(dx) > \mathbf{0}_A\right\}$$

is an admissible parameter space under $P \in \mathcal{P}$. When markets are incomplete, the market prices of

[12] We use $\mathcal{B}(\mathbb{R})$ to denote the Borel σ-algebra.

risk form a set

$$\boldsymbol{\Gamma}(\boldsymbol{\phi}_t) = \left\{(\boldsymbol{\psi}_t, \boldsymbol{\gamma}_t) : \boldsymbol{\mu}_t^B - r_t \mathbf{1}_A - \boldsymbol{\sigma}_t^B \boldsymbol{\psi}_t - \boldsymbol{J}_t \boldsymbol{\gamma}_t \boldsymbol{v}_t(dx) = 0\right\}. \tag{3.4}$$

Let $\boldsymbol{\alpha}_t = \ln\left[(\mathbf{1}_A - \boldsymbol{\gamma}_t)^{-1}\right]$, Equation 3.4 can be written in the form

$$\boldsymbol{\Gamma}(\boldsymbol{\phi}_t) = \Big\{ (\boldsymbol{\psi}_t, \boldsymbol{\gamma}_t) : \boldsymbol{\psi}_t = (\boldsymbol{\sigma}_t^B)^{-1} \left(\boldsymbol{\mu}_t^B - r_t \mathbf{1}_A\right) - \left(\boldsymbol{\sigma}_t^B\right)^{-1} \left(\boldsymbol{J}_t \left(\mathbf{1}_A - e^{-\boldsymbol{\alpha}_t}\right) \boldsymbol{v}_t(dx)\right), \\ \boldsymbol{\gamma}_t = \mathbf{1}_A - e^{-\boldsymbol{\alpha}_t}, \boldsymbol{\alpha}_t \in \mathbb{R}^A \Big\}. \tag{3.5}$$

Correspondingly, for $(\boldsymbol{\psi}_t, \boldsymbol{\gamma}_t) \in \boldsymbol{\Gamma}(\boldsymbol{\phi}_t)$, the SDF process $\{\boldsymbol{M}(\boldsymbol{\phi}_t)\}_{t\geq 0}$ follows the dynamic form

$$\frac{d\boldsymbol{M}(\boldsymbol{\phi}_t)}{\boldsymbol{M}(\boldsymbol{\phi}_{t-})} = -\left[r_t \mathbf{1}_A dt + \boldsymbol{\psi}_t d\boldsymbol{B}_t + \boldsymbol{\gamma}_t d\tilde{\boldsymbol{N}}_t\right]$$

with the solution

$$\boldsymbol{M}(\boldsymbol{\phi}_t) = \exp\left(-\int_0^t r_s \mathbf{1}_A ds - \int_0^t \boldsymbol{\psi}_s d\boldsymbol{B}_s - \frac{1}{2}\int_0^t \boldsymbol{\psi}_s^2 ds\right) \\ \times \exp\left(-\int_0^t \boldsymbol{\alpha}_s d\tilde{\boldsymbol{N}}_s - \int_0^t \left(e^{-\boldsymbol{\alpha}_s} - \mathbf{1}_A + \boldsymbol{\kappa}_s\right) \boldsymbol{v}_t(dx) ds\right).$$

We shall restrict $\boldsymbol{M}(\boldsymbol{\phi}_t)$ to be a P-square integrable martingale over the time interval $[0,1]$, i.e., $\sup_{t\in[0,1]} \mathbb{E}\left[\boldsymbol{M}^2(\boldsymbol{\phi}_t)\right] < \infty$. Then, the SDF set is

$$\mathbb{M}(\boldsymbol{\phi}_t) = \Bigg\{\boldsymbol{M}(\boldsymbol{\phi}_t) = \exp\left(-\int_0^t r_s \mathbf{1}_A ds - \int_0^t \boldsymbol{\psi}_s d\boldsymbol{B}_s - \frac{1}{2}\int_0^t \boldsymbol{\psi}_s^2 ds\right) \\ \times \exp\left(-\int_0^t \boldsymbol{\alpha}_s d\tilde{\boldsymbol{N}}_s - \int_0^t \left(e^{-\boldsymbol{\alpha}_s} - \mathbf{1}_A + \boldsymbol{\alpha}_s\right) \boldsymbol{v}_t(dx) ds\right), \\ \boldsymbol{\alpha}_t = \ln\left[(\mathbf{1}_A - \boldsymbol{\gamma}_t)^{-1}\right], (\boldsymbol{\psi}_t, \boldsymbol{\gamma}_t) \in \boldsymbol{\Gamma}(\boldsymbol{\phi}_t)\Bigg\},$$

and the set that contains all SDFs under $P \in \mathcal{P}$ is defined as $\boldsymbol{C}_t := \{\mathbb{M}_t(\boldsymbol{\phi}_t), \boldsymbol{\phi}_t \in \boldsymbol{\Phi}_t\}$. Analogous to the discrete case, in order to analyze the evolution of the degree of market incompleteness, we frame the following discussion in terms of the SDF set including all possible SDFs that price payoff over a small time interval dt:

$$\begin{aligned}\mathbb{m}(\boldsymbol{\phi}_t) &= \{\boldsymbol{m}(\boldsymbol{\phi}_t) = \boldsymbol{M}(\boldsymbol{\phi}_t)/\boldsymbol{M}(\boldsymbol{\phi}_{t-}), \boldsymbol{\phi}_t \in \boldsymbol{\Phi}_t\} \\ &= \Bigg\{\boldsymbol{m}(\boldsymbol{\phi}_t) = \exp\left(-r_t \mathbf{1}_A dt - \boldsymbol{\psi}_t d\boldsymbol{B}_t - \frac{1}{2}\psi_t^2 dt\right) \times \exp\left(-\boldsymbol{\alpha}_t d\tilde{\boldsymbol{N}}_t - (e^{-\boldsymbol{\alpha}_t} - \mathbf{1}_A + \boldsymbol{\alpha}_t)\, \boldsymbol{v}_t(dx) dt\right), \\ &\qquad \boldsymbol{\alpha}_t = \ln\left[(\mathbf{1}_A - \boldsymbol{\gamma}_t)^{-1}\right], (\boldsymbol{\psi}_t, \boldsymbol{\gamma}_t) \in \boldsymbol{\Gamma}(\boldsymbol{\phi}_t)\Bigg\}, \end{aligned} \quad (3.6)$$

and this SDF discounts the payoff at t to its price at $t - dt$. Accordingly, the set that contains all $\mathbb{m}(\boldsymbol{\phi}_t)$ for $P \in \mathcal{P}$ is $\boldsymbol{c}_t := \{\mathbb{m}(\boldsymbol{\phi}_t), \boldsymbol{\phi}_t \in \boldsymbol{\Phi}_t\}$.

3.1 Set properties of $\boldsymbol{C}_t$

Similar to the discrete setups, we first verify that the boundary point of the proposed SDF set is indeed the one under the complete market. Let $\bar{\boldsymbol{\Phi}}_t := \{(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)) : \boldsymbol{v}_t(dx) \geq \mathbf{0}_A\}$ be the admissible parameter space, the following proposition establishes the limit and boundary points of $\boldsymbol{\Phi}_t$ under $\bar{\boldsymbol{\Phi}}_t$. The proof is shown in Appendix A.13.

Proposition 3.1. *Consider the metric space* $(\bar{\boldsymbol{\Phi}}_t, d)$ *such that*

$$\bar{\boldsymbol{\Phi}}_t := \left\{\left(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)\right) : \boldsymbol{v}_t(dx) \geq \mathbf{0}_A\right\}$$

and d is the Euclidean norm. Then, the set of limit points of $\boldsymbol{\Phi}_t$ *in* $(\bar{\boldsymbol{\Phi}}_t, d)$ *is*

$$L(\boldsymbol{\Phi}_t) = \left\{\left(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)\right) : \boldsymbol{v}_t(dx) \geq \mathbf{0}_A\right\},$$

and the set of boundary points of $\boldsymbol{\Phi}_t$ *in* $(\bar{\boldsymbol{\Phi}}_t, d)$ *is*

$$\partial\boldsymbol{\Phi}_t = \left\{\left(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)\right) : \boldsymbol{v}_t(dx) = \mathbf{0}_A\right\}.$$

The next result indicates that with the continuous-time setup, the complete market SDF is indeed the boundary point of the incomplete market SDF set. The proof is presented in Appendix A.14.

Theorem 3.1. *Consider the metric space* $(\bar{\boldsymbol{c}}_t, d_3)$ *such that* $\bar{\boldsymbol{c}}_t = \{\mathbb{m}(\boldsymbol{\phi}_t), \boldsymbol{\phi}_t \in \bar{\boldsymbol{\Phi}}_t\}$, *and for* $\boldsymbol{\phi}_t, \boldsymbol{\phi}_t' \in \bar{\boldsymbol{\Phi}}_t$ *satisfies Equation 3.3, let* $P(\boldsymbol{\phi}_t)$ *and* $P(\boldsymbol{\phi}_t')$ *denote the physical probability measures in* $\mathcal{P}$,

$$d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}_t')) = \inf_{w_t}\left\{\int d_4(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}_t')) dw_t : w_t \in W(P(\boldsymbol{\phi}_t), P(\boldsymbol{\phi}_t'))\right\}, \quad (3.7)$$

where $W(P(\phi_t), P(\phi_t')) := \{w_t : \int w_t dP(\phi_t') = P(\phi_t), \int w_t dP(\phi_t) = P(\phi_t')\}$ *is the set of transport plans between* $P(\phi_t)$ *and* $P(\phi_t')$*, and*

$$d_4\left(\mathbb{m}(\phi_t), \mathbb{m}(\phi_t')\right) = \left|f(\phi_t) - f(\phi_t')\right|$$

with

$$f(\phi_t) = \exp\left(-r_t\mathbf{1}_A dt - g(\phi_t)d\boldsymbol{B}_t - \frac{1}{2}g(\phi_t)^2 dt\right) \times \exp\left(-d\tilde{\boldsymbol{N}}_t - e^{-1}\boldsymbol{v}_t(dx)dt\right),$$

$$g(\phi_t) = \left(\boldsymbol{\sigma}_t^B\right)^{-1}\left(\boldsymbol{\mu}_t^B - r_t\mathbf{1}_A\right) - \left(\boldsymbol{\sigma}_t^B\right)^{-1}\left(\boldsymbol{J}_t\left(\mathbf{1}_A - e^{-\mathbf{1}_A}\right)\boldsymbol{v}_t(dx)\right),$$

and $\boldsymbol{J}_t = \exp(\boldsymbol{Q}_t) - \mathbf{1}_A, \boldsymbol{Q}_t \sim N(\boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^{J2})$.

Then, the set of limit points of $\boldsymbol{c}_t$ *in* $\bar{\boldsymbol{c}}_t$ *can be denoted as* $L(\boldsymbol{c}_t) = \{\mathbb{m}(\phi_t), \phi_t \in \bar{\boldsymbol{\Phi}}_t\}$ *and the set of boundary points of* $\boldsymbol{c}_t$ *is then* $\partial\boldsymbol{c}_t = \{\mathbb{m}(\phi_t), \phi_t \in \partial\boldsymbol{\Phi}_t\}$*, where for any* $\phi_t \in \partial\boldsymbol{\Phi}_t$*,*

$$\mathbb{m}(\phi_t) = \left\{\boldsymbol{m}(\phi_t) = \exp\left(-r_t\mathbf{1}_A dt - \boldsymbol{\psi}_t d\boldsymbol{B}_t - \frac{1}{2}\boldsymbol{\psi}_t^2 dt\right), \boldsymbol{\mu}_t^B - r_t\mathbf{1}_A - \boldsymbol{\sigma}_t^B\boldsymbol{\psi}_t = 0\right\}. \tag{3.8}$$

Based on Theorem 3.1, we derive the following lemma, which will later be incorporated in the discussion of set properties and the degree of market incompleteness. The proof of Lemma 3.1.1 is presented in Appendix A.15.

Lemma 3.1.1. *For every* $\mathbb{m}(\phi_t) \in \boldsymbol{c}_t$*, there exists* $\mathbb{m}(\phi_t^*)$ *such that*

$$\mathbb{m}(\phi_t^*) = \underset{\mathbb{m}(\phi_t') \in \partial\boldsymbol{c}_t}{\arg\min}\left\{d_3\left(\mathbb{m}(\phi_t), \mathbb{m}(\phi_t')\right)\right\}.$$

The next theorem establishes the properties of the incomplete market SDF set, and the proof is provided in Appendix A.16.

Theorem 3.2. *Let* $\mathbb{m}(\phi_t)$ *be the identified SDF set given that* $\phi_t \in \boldsymbol{\Phi}_t$*, and let* $\boldsymbol{c}_t$ *be the combined SDF set such that* $\boldsymbol{c}_t = \{\mathbb{m}(\phi_t), \phi_t \in \boldsymbol{\Phi}_t\}$*. Then,* $\boldsymbol{c}_t$ *is a convex set.*

Let $(\bar{\boldsymbol{c}}_t, d_3)$ *be the metric space such that* $\bar{\boldsymbol{c}}_t = \{\mathbb{m}(\phi_t), \phi_t \in \bar{\boldsymbol{\Phi}}_t\}$*, and* d_3 *is as defined in Equation 3.7. Then,* $\boldsymbol{c}_t$ *is open, bounded and not compact under* $(\bar{\boldsymbol{c}}_t, d_3)$*.*

3.2 Measure for market incompleteness

Based upon Theorems 3.1 and 3.2, given $\left\{\phi_i \in \bar{\boldsymbol{\Phi}}_i\right\}_{i\in[0,t]}$, the degree of market incompleteness at t is defined as in Equation 3.9, which is the mean of the least transport cost process from $\mathbb{m}(\phi_i) \in \bar{\boldsymbol{c}}_i$ to

$\mathbb{m}(\boldsymbol{\phi}_i') \in \partial \boldsymbol{c}_i$ over $[0,t]$[13]:

$$MI_t\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right) = \mathbb{E}_t\left[\min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right)\right]. \tag{3.9}$$

Since d_3 is a valid metric and the complete market SDF is the boundary point of the incomplete market SDF set, the degree of market incompleteness equals zero only when the market is dynamically complete, i.e., the distance between complete and incomplete market SDF sets measured by d_3 reduces to zero at every i over the time period $[0,t]$.

The following pointwise properties of $MI(\cdot)$ indicate that $MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right)$ is continuous and not monotone in t, which enable us to implement our theoretical results in empirical studies and examine the evolution of market incompleteness over time. The proof is presented in Appendix A.17.

Theorem 3.3. *The degree of market incompleteness $MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right)$ is continuous on the time interval $[0,1]$ and is not monotone in t.*

4 Application

This section illustrates the degree of market incompleteness estimation with four countries' major stock market index composites. We first present the layout of a simple but important special case of our continuous-time setup, which will be used for the demonstration of our market incompleteness measure. Then, we describe the data in Section 4.1 and the parameter estimations in Section 4.2.

Throughout this section, we consider a running example as follows.

Assumption 4.1. *Let $\boldsymbol{R}_t := \ln \boldsymbol{S}_t - \ln \boldsymbol{S}_{t-}$ be a vector of $A \in \mathbb{N}$ log-returns observed at $t \in [0,1]$. Suppose the degree of market incompleteness is evaluated at K equally-spaced time points $\{kh\}_{k=1,\ldots,K}$ with the time window $h = 1/K$. To simplify the notation, we use the subscript k to denote the parameter that characterizes the return in the time period $[(k-1)h, kh]$.*
When markets are incomplete, let $\{\boldsymbol{B}_t\}$ be a vector of $A \in \mathbb{N}$ independent standard Brownian motions under P and $\boldsymbol{N}_t$ be the Poisson process with mean measure $\boldsymbol{v}_k(dx)t$, where $\boldsymbol{v}_k(dx) \geq 0$ is taken to be the Lévy measure associated with an A-dimensional pure-jump Lévy process. $\{\boldsymbol{B}_t\}$ and $\{\boldsymbol{N}_t\}$ are

[13] Same as in the discrete setting, we assume that the complete and incomplete market SDFs are 1 at time 0, and the distance between them is 0 following the metric d_3.

independent and adapted to the filtration $\{\mathcal{F}_t\}$. $\boldsymbol{R}_t$ *solves the SDE*[14]

$$\boldsymbol{R}_t = \left(\boldsymbol{\mu}_k^B - \boldsymbol{\sigma}_k^{B^2}/2 - \boldsymbol{v}_k(dx)\boldsymbol{\mu}_k^J\right)dt + \boldsymbol{\sigma}_k^B d\boldsymbol{B}_t + \boldsymbol{Q}_k d\boldsymbol{N}_t,$$

where $\boldsymbol{\mu}_k^B \in \mathbb{R}^A$, $\boldsymbol{\sigma}_k^B \in \mathbb{R}^{A\times A}$, $\boldsymbol{Q}_k$ *follows a normal distribution with mean* $\boldsymbol{\mu}_k^J \in \mathbb{R}^A$ *and standard deviation* $\boldsymbol{\sigma}_k^J \in \mathbb{R}^{A\times A}$, *and* dt *is estimated by the observational interval. Moreover, both* $\boldsymbol{\sigma}_k^B$ *and* $\boldsymbol{\sigma}_k^J$ *are diagonal matrices, and the price of the risk-free bond has a known constant rate of return* r_k.
When the markets are complete, let $\{\boldsymbol{B}_t\}$ *be a vector of* $A \in \mathbb{N}$ *independent standard Brownian motions under* P. $\boldsymbol{R}_t$ *solves the SDE*

$$\boldsymbol{R}_t = \left(\boldsymbol{\mu}_k^C - \boldsymbol{\sigma}_k^{C^2}/2\right)dt + \boldsymbol{\sigma}_k^C d\boldsymbol{B}_t,$$

where $\boldsymbol{\mu}_k^C \in \mathbb{R}^A$, $\boldsymbol{\sigma}_k^C \in \mathbb{R}^{A\times A}$.

Assumption 4.1 ensures that the market prices of risk always lie in a nonrandom time-invariant set over a given time period $[(k-1)h, kh]$. Specifically, for $t \in [(k-1)h, kh]$, $k = 1, \ldots, K$, and $\boldsymbol{\alpha}_k = \ln\left[(\mathbf{1}_A - \boldsymbol{\gamma}_k)^{-1}\right]$, Equation 3.5 becomes

$$\Gamma(\boldsymbol{\phi}_k) = \Big\{(\boldsymbol{\psi}, \boldsymbol{\gamma}) : \boldsymbol{\psi}_k = \left(\boldsymbol{\sigma}_k^B\right)^{-1}\left(\boldsymbol{\mu}_k^B - r_k\mathbf{1}_A\right) - \left(\boldsymbol{\sigma}_k^B\right)^{-1}\left(\boldsymbol{J}_k\left(\mathbf{1}_A - e^{-\boldsymbol{\alpha}_k}\right)\boldsymbol{v}_k(dx)\right),$$
$$\boldsymbol{\gamma}_k = \mathbf{1}_A - e^{-\boldsymbol{\alpha}_k}, \boldsymbol{\alpha}_k \in \mathbb{R}^A\Big\}.$$

Then, under incomplete markets, the SDF set in Equation 3.6 can be written as

$$\mathbb{m}(\boldsymbol{\phi}_k) = \Big\{\boldsymbol{m}(\boldsymbol{\phi}_k) = \exp\left(-r_k\mathbf{1}_A dt - \boldsymbol{\psi}_k d\boldsymbol{B}_t - \frac{1}{2}\boldsymbol{\psi}_k^2 dt\right)$$
$$\times \exp\left(-\boldsymbol{\alpha}_k d\tilde{\boldsymbol{N}}_t - \left(e^{-\boldsymbol{\alpha}_k} - \mathbf{1}_A + \boldsymbol{\alpha}_k\right)\boldsymbol{v}_k(dx)dt\right), (\boldsymbol{\psi}_k, \boldsymbol{\gamma}_k) \in \boldsymbol{\Gamma}(\boldsymbol{\phi}_k)\Big\}.$$

Under complete markets, the SDF set in Equation 3.8 can be written as

$$\mathbb{m}\left(\boldsymbol{\phi}_k^C\right) = \left\{\boldsymbol{m}\left(\boldsymbol{\phi}_k^C\right) = \exp\left(-r_k\mathbf{1}_A dt - \boldsymbol{\psi}_k^C d\boldsymbol{B}_t - \frac{1}{2}\boldsymbol{\psi}_k^{C^2} dt\right), \boldsymbol{\mu}_k^C - r_k\mathbf{1}_A - \boldsymbol{\sigma}_k^C\boldsymbol{\psi}_k = 0\right\}.$$

Hence, given $\{\boldsymbol{\phi}_i \in \boldsymbol{\Phi}_i\}_{i=1}^k$, the degree of market incompleteness at kh is

$$MI_{kh}\left(\{\boldsymbol{\phi}_i\}_{i=1}^k\right) = \frac{1}{k}\sum_{i=1}^k \min_{\boldsymbol{\phi}_i^C \in \partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i^C)\right),$$

[14]Given that the value of interest is usually the log-return on asset, we transform Equation 3.1 using the stochastic chain rule for Markov processes in continuous time, the detailed derivation can be found in Kushner (1967) and Gihman and Skorohod (2012).

where

$$d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i^C)\right) = \inf_{w_i}\left\{\int d_4\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i^C)\right) dw_i : w_i \in W\left(P(\boldsymbol{\phi}_i), P(\boldsymbol{\phi}_i^C)\right)\right\}.$$

$W\left(P(\boldsymbol{\phi}_i), P(\boldsymbol{\phi}_i^C)\right) := \left\{w_i : \int w_i dP(\boldsymbol{\phi}_i^C) = P(\boldsymbol{\phi}_i), \int w_i dP(\boldsymbol{\phi}_i) = P(\boldsymbol{\phi}_i^C)\right\}$ is the set of transport plans between $P(\boldsymbol{\phi}_i)$ and $P(\boldsymbol{\phi}_i^C)$, and $d_4\left(\mathbb{M}(\boldsymbol{\phi}_i), \mathbb{M}(\boldsymbol{\phi}_i^C)\right) = \left|f(\boldsymbol{\phi}_i) - f(\boldsymbol{\phi}_i^C)\right|$ with

$$f(\boldsymbol{\phi}_i) = \exp\left(-r_i\mathbf{1}_A dt - g(\boldsymbol{\phi}_i)d\boldsymbol{B}_t - \frac{1}{2}g(\boldsymbol{\phi}_i)^2 dt\right) \times \exp\left(-d\tilde{\boldsymbol{N}}_t - e^{-1}\boldsymbol{v}_i(dx)dt\right),$$

$$g(\boldsymbol{\phi}_i) = (\boldsymbol{\sigma}_i^B)^{-1}(\boldsymbol{\mu}_i^B - r_i\mathbf{1}_A) - (\boldsymbol{\sigma}_i^B)^{-1}\left(\boldsymbol{J}_i(\mathbf{1}_A - e^{-1})\boldsymbol{v}_i(dx)\right),$$

and $\boldsymbol{J}_i = \exp(\boldsymbol{Q}_i) - 1, \boldsymbol{Q}_i \sim N(\boldsymbol{\mu}_i^J, {\boldsymbol{\sigma}_i^J}^2)$;

$$f\left(\boldsymbol{\phi}_i^C\right) = \exp\left(-r_i\mathbf{1}_A dt - g\left(\boldsymbol{\phi}_i^C\right)d\boldsymbol{B}_t - \frac{1}{2}g^2\left(\boldsymbol{\phi}_i^C\right)dt\right)$$

and

$$g\left(\boldsymbol{\phi}_i^C\right) = \left(\boldsymbol{\sigma}_i^C\right)^{-1}\left(\boldsymbol{\mu}_i^C - r_i\mathbf{1}_A\right).$$

4.1 Data Description

Our empirical study analyzes the financial markets of China, Japan, the United Kingdom (UK), and the United States (US) using publicly available data from Yahoo Finance. Due to the availability of data, the Chinese and the US samples begin in 1994, the UK sample begins in 1995, whereas the Japanese sample begins in 1999, and all samples end in 2021. We use the stock data from CSI 300 index for China, Nikkei 225 index for Japan, and FTSE 350 index for the UK and S&P 500 for the US[15].

The stock data is collected on a daily basis, and to examine the evolution of market incompleteness, we divided the full sample into yearly blocks, i.e., for the US market, there are 27 sub-samples, then $K = 27$ and $h = 1/27$. The daily log return (hereinafter, the return) is calculated, and assuming 252 trading days per year, dt is estimated by $\Delta = 1/252$. Further, stocks with less than one-month of data are excluded from each subsample in order to eliminate outliers and ensure the reliability of

[15] The CSI 300 is a capitalization-weighted index that replicates the performance of the top 300 stocks traded on the Shanghai Stock Exchange and the Shenzhen Stock Exchange. The Nikkei 225 index measures the performance of 225 large, publicly owned companies in Japan that span a wide range of industry sectors. The FTSE 350 is a capitalization-weighted index composed of the 350 largest companies listed on the London Stock Exchange. The S&P 500 index is a capitalization-weighted index that represents around 80% of the market capitalization of the New York Stock Exchange.

the estimates.

4.2 Estimation Algorithm

At each subperiod $[(k-1)h, kh]$, we first estimate the parameters $\boldsymbol{\phi}_k$ under incomplete market assumption using the maximum likelihood estimation (MLE) method, and the parameters $\boldsymbol{\phi}_k^C$ under the complete market assumption using the analytical closed-form expression. To the best of our knowledge, there are not yet an analytic expression of the optimal parameter values for jump diffusion models, and thus, we employ the MATLAB function $fminsearchbnd$, which is developed based upon $fminsearch$ to find the minimum value of the constrained multivariable function using derivative-free method for our estimation. As a prerequisite to applying the $fminsearchbnd$ method, we must first establish an initial estimation of the parameters based on the empirical data. Consistent with Merton (1976)'s definition, in this study, we say that there is a jump in the process when the absolute value of return exceeds some threshold $\epsilon > 0$, which is determined as the minimum absolute value of the 5% and 95% quantiles of returns[16], and then, we divide the empirical return data into two groups $\mathcal{B}$ and $\mathcal{J}$, which include returns with absolute values less than or equal to ϵ and those with absolute values larger than ϵ, respectively.

Here, the initial estimation of the intensity parameter, $\hat{\boldsymbol{v}}_k(dx)$, is measured as the number of jumps in period $[(k-1)h, kh]$, and for simplicity, we estimate the initial parameters $\boldsymbol{\phi}_k$ assuming that there is only one jump for a return process that belongs to group $\mathcal{J}$. Then, as discussed in Hanson and Westman (2002), the expectation and variance of the process for $t \in [(k-1)h, kh]$ are

$$\mathbb{E}(\boldsymbol{R}_t^J) = \mathbb{E}[\boldsymbol{R}_t | \boldsymbol{N}_t = 1] = \left(\boldsymbol{\mu}_k^B - \boldsymbol{\sigma}_k^{B^2}/2 - \boldsymbol{v}_k(dx)\boldsymbol{\mu}_k^J\right)\Delta + \boldsymbol{\mu}_k^J$$

and

$$\mathrm{Var}(\boldsymbol{R}_t^J) = \mathrm{Var}[\boldsymbol{R}_t | \boldsymbol{N}_t = 1] = \boldsymbol{\sigma}_k^{B^2}\Delta + \boldsymbol{\sigma}_k^{J^2}.$$

Hence, $\hat{\boldsymbol{\mu}}_k^J$ and $\hat{\boldsymbol{\sigma}}_k^J$ are estimated from the above equations such that

$$\begin{cases} \hat{\boldsymbol{\mu}}_k^J = \left(\hat{\mathbb{E}}\left(\boldsymbol{R}_t^J\right) - \left(\hat{\boldsymbol{\mu}}_k^B - \hat{\boldsymbol{\sigma}}_k^{B^2}/2\right)\Delta\right)\left(\mathbf{1}_A - \hat{\boldsymbol{v}}_k(dx)\Delta\right)^{-1} \\ \left(\hat{\boldsymbol{\sigma}}_k^J\right)^2 = \hat{\mathrm{Var}}\left(\boldsymbol{R}_t^J\right) - \hat{\boldsymbol{\sigma}}_k^{B^2}\Delta, \end{cases}$$

[16] Other quantiles can be adopted to determine ϵ, while as discussed in Tang (2018), in this case MLE is not strongly depending on the value of ϵ.

where $\hat{\mathbb{E}}(\boldsymbol{R}_t^J)$ and $\hat{\mathrm{Var}}(\boldsymbol{R}_t^J)$ are the sample mean and variance of the empirical returns in group $\mathcal{J}$.

When there are no jumps, the expectation and variance of the return of the process for $t \in [(k-1)h, kh]$ are

$$\mathbb{E}(\boldsymbol{R}_t^B) = \mathbb{E}[\boldsymbol{R}_t | \boldsymbol{N}_t = 0] = \left(\boldsymbol{\mu}_k^B - {\boldsymbol{\sigma}_k^B}^2/2\right)\Delta$$

and

$$\mathrm{Var}(\boldsymbol{R}_t^B) = \mathrm{Var}[\boldsymbol{R}_t | \boldsymbol{N}_t = 0] = {\boldsymbol{\sigma}_k^B}^2 \Delta.$$

The parameters $\hat{\boldsymbol{\mu}}_k^B$ and $\hat{\boldsymbol{\sigma}}_k^B$ can be estimated from the above formulas such that

$$\begin{cases} \hat{\boldsymbol{\mu}}_k^B = \left(2\hat{\mathbb{E}}\left(\boldsymbol{R}_t^B\right) + \hat{\mathrm{Var}}\left(\boldsymbol{R}_t^B\right)\Delta\right)(2\Delta)^{-1} \\ (\hat{\boldsymbol{\sigma}}_k^B)^2 = \hat{\mathrm{Var}}\left(\boldsymbol{R}_t^B\right)/\Delta, \end{cases} \tag{4.1}$$

where $\hat{\mathbb{E}}(\boldsymbol{R}_t^B)$ and $\hat{\mathrm{Var}}(\boldsymbol{R}_t^B)$ are the sample mean and variance of the empirical returns in group $\mathcal{B}$.

Let $\boldsymbol{R}_{\Delta t} \coloneqq \ln \boldsymbol{S}_t - \ln \boldsymbol{S}_{t-\Delta}$ denote the log-return observed at $t \in [(k-1)h, kh]$, the initial estimates are then used to numerically optimize the likelihood function, given that the probability density function of returns at Δt is:

$$\varphi_{\boldsymbol{R}_{\Delta t}}(\boldsymbol{x}; \boldsymbol{\phi}_k) = \sum_{z=0}^{\infty} p_z(\boldsymbol{v}_k(dx)\Delta)\varphi_n\left(\boldsymbol{x} \middle| \left(\boldsymbol{\mu}_k^B - {\boldsymbol{\sigma}_k^B}^2/2 - \boldsymbol{v}_k(dx)\boldsymbol{\mu}_k^J\right)\Delta + \boldsymbol{\mu}_k^J z, {\boldsymbol{\sigma}_k^B}^2\Delta + {\boldsymbol{\sigma}_k^J}^2 z^2\right),$$

where $p_z(\boldsymbol{v}_k(dx)dt) = p(d\boldsymbol{N}_t = z) = \exp(-\boldsymbol{v}_k(dx)dt)(\boldsymbol{v}_k(dx)dt)^z/z!$ for $z = 0, 1, \ldots$ and φ_n is the normal density function (Hanson and Westman, 2002). In a multivariate economy defined in Assumption 4.1, returns are independent over time, so that the objective function of the MLE method is

$$L(\boldsymbol{\phi}_k) = \Pi_{i=1}^I \varphi_{\boldsymbol{R}_{\Delta t}}(\boldsymbol{x}_i; \boldsymbol{\phi}_k),$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_I)$ is the empirical log-return data. To estimate the five parameters, we then minimize the minus log-likelihood function:

$$-\ln L(\boldsymbol{\phi}_k) = -\sum_{i=1}^{I} \ln \varphi_{\boldsymbol{R}_{\Delta t}}(\boldsymbol{x}_i; \boldsymbol{\phi}_k).$$

Next, we numerically estimate the degree of market incompleteness at kh given $\left\{\hat{\boldsymbol{\phi}}_i\right\}_{i=1}^k$ and

$\{\hat{\phi}_i^C\}_{i=1}^k$ as follows.

(i). For each asset $a = 1, 2, \ldots, A$, at time point ih for $i = 1, \ldots, k$, generate 1000 replications of $dB_{n,i}^a \sim N(0, \Delta)$, $d\tilde{N}_{n,i}^a = dN_{n,i}^a - \hat{v}_i^a(dx)\Delta$ with $dN_{n,i}^a \sim Poisson(\hat{v}_i^a(dx)\Delta)$ and the observation window $\Delta = 1/252$ being the approximation for dt, and $\hat{J}_{n,i}^a = \exp(\hat{Q}_{n,i}^a) - 1$ with $\hat{Q}_{n,i}^a \sim N(\hat{\mu}_i^{J,a}, \hat{\sigma}_i^{J,a^2})$.

(ii). For each $n = 1, 2, \ldots, 1000$ replication, calculate

$$f_n(\hat{\phi}_i^a) = \exp\left(-\hat{r}_i\Delta - g_n\left(\hat{\phi}_i^a\right) dB_{n,i}^a - \frac{1}{2} g_n^2\left(\hat{\phi}_i^a\right)\Delta\right) \times \exp\left(-d\tilde{N}_{n,i}^a - e^{-1}\hat{v}_i^a(dx)\Delta\right)$$

where

$$g_n\left(\hat{\phi}_i^a\right) = \left(\hat{\sigma}_i^{B,a}\right)^{-1}\left(\hat{\mu}_i^{B,a} - \hat{r}_i\right) - \left(\hat{\sigma}_i^{B,a}\right)^{-1}\left(\hat{J}_{n,i}^a(1 - e^{-1})\hat{v}_i^a(dx)\right)$$

under incomplete markets, and

$$f_n\left(\hat{\phi}_i^{C,a}\right) = \exp\left(-\hat{r}_i\Delta - g_n\left(\hat{\phi}_i^{C,a}\right) dB_{n,i}^a - \frac{1}{2} g_n^2\left(\hat{\phi}_i^{C,a}\right)\Delta\right)$$

where $g_n\left(\hat{\phi}_i^{C,a}\right) = \left(\hat{\sigma}_i^{C,a}\right)^{-1}\left(\hat{\mu}_i^{C,a} - \hat{r}_i\right)$ with $\hat{\mu}_i^{C,a}$ and $\hat{\sigma}_i^{C,a}$ estimated following Equation 4.1 under complete markets.

(iii). Using the 1000 observations of $f_n\left(\hat{\phi}_i^a\right)$ and $f_n\left(\hat{\phi}_i^{C,a}\right)$, we find the empirical cumulative distributions $F\left(x; \hat{\phi}_i^a\right)$ and $F\left(x; \hat{\phi}_i^{C,a}\right)$ for the probability measures $P(\hat{\phi}_i^a)$ and $P(\hat{\phi}_i^{C,a})$ respectively.

(iv). Derive the distance metric for each stock a at time i (Frohmader and Volkmer, 2021)

$$d_3^a\left(\mathbb{m}\left(\hat{\phi}_i^a\right), \mathbb{m}\left(\hat{\phi}_i^{C,i}\right)\right) = \int_{\mathbb{R}} \left|F\left(x; \hat{\phi}_i^a\right) - F\left(x; \hat{\phi}_i^{C,a}\right)\right| dx.$$

(v). Compute the degree of market incompleteness at kh for $k = 1, \ldots, K$,

$$\hat{MI}\left(\{\hat{\phi}_i\}_{i=1}^k\right) = \frac{1}{k}\sum_{i=1}^k \frac{1}{A}\sum_{a=1}^A d_3^a\left(\mathbb{m}\left(\hat{\phi}_i\right), \mathbb{m}\left(\hat{\phi}_i^C\right)\right).$$

4.3 Estimation Results

Figure 1 displays the evolution of the degree of market incompleteness for the four stock markets. The market often sees an increase in MI when there is a rising level of panic. Namely, all three developed markets experienced peaks in MI during the period 2007-2009 due to the global financial crisis, in which asset prices experienced unexpected jumps due the presence of significant unhedgeable risks in the market. In a similar manner, the value of MI spiked both during the mini-crash in the

UK stock market in 1997 (Hua et al., 2020) as well as during the collapse of the Chinese stock market in 2015 (Han et al., 2019). Government regulation policies toward the stock market can also influence its completeness. In 1995, the sharp decline in MI on the Chinese market was attributed to a policy change, which adjusted settlement dates to the next business day ($T+1$) instead of the same day ($T+0$)[17] (Xu, 2000). In the Japanese market, MI rose in 2000 due to deregulation policies, such as decontrolling brokerage commissions and reducing securities transaction taxes (Takaishi, 2022). We also observe that the Chinese stock market has a significantly higher degree of market incompleteness, implying that the market is susceptible to more risks that cannot be diversified away by the spanning of traded assets, which accords with the literature that emerging markets are inherently riskier (Sharkasi et al., 2006; Saranya and Prasanna, 2014).

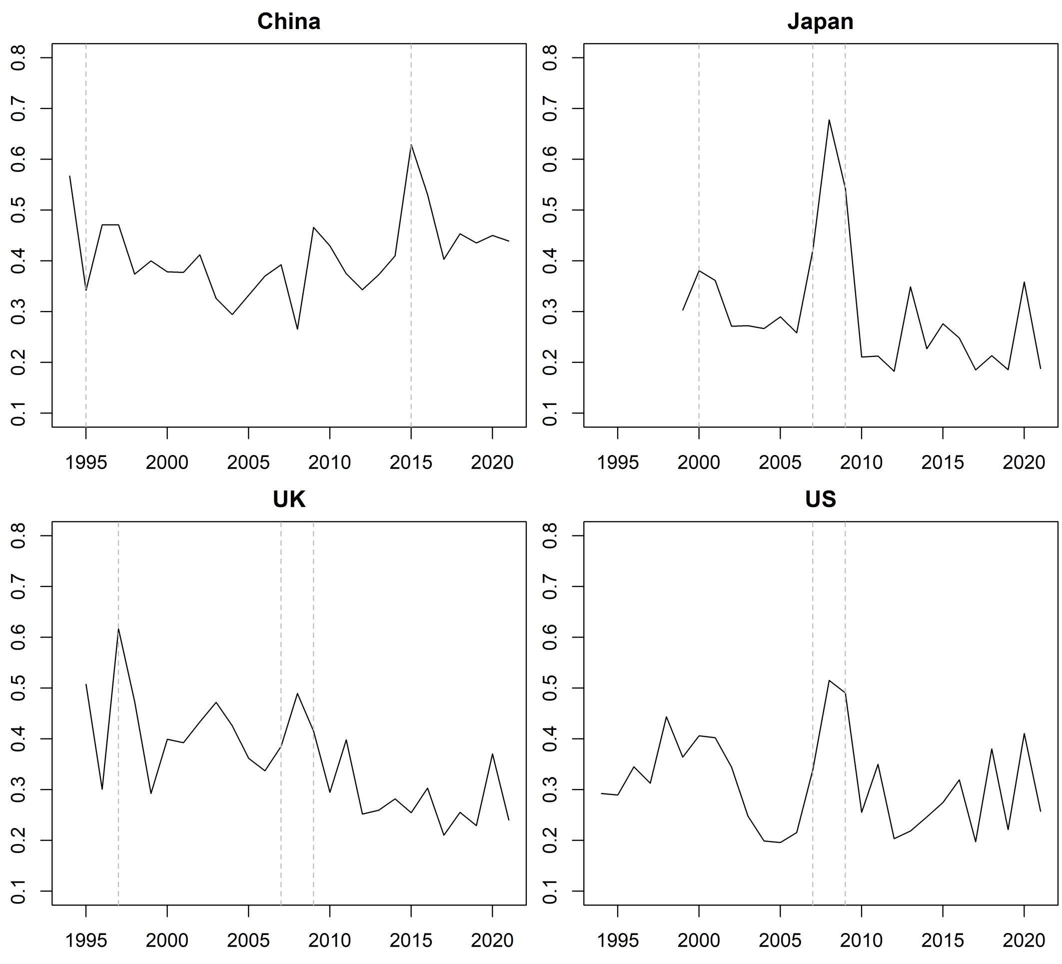

Figure 1: Evolution of the degree of market incompleteness

[17] $T+1$ came into effect on January 1, 1995, replacing $T+0$.

5 Conclusion

This paper studies an econometric framework useful for estimating the set of SDFs in the absence of complete markets. The investigation of set properties reveals that the complete market SDF is the unique boundary point of the incomplete market SDF set, which only degenerates to its complete counterpart when the likelihood of unanticipated risks vanish. This feature allows us to introduce a novel measure for market incompleteness, which is the distance between the probability distributions of the complete and incomplete market SDFs. We use the Wasserstein metric to construct our measure since it naturally deals with distributions with different supports.

A possible implementation of this measure is presented in which we examine the evolution of market incompleteness in the four largest stock markets worldwide, including both emerging and developed markets. The results are consistent with our construction of incomplete markets, whereby the increase (decrease) in market incompleteness correlates to financial crises or policy changes that raise (lower) the likelihood of unhedgeable risks.

To maintain a sharp focus on our results, we have considered in detail a specific but practically realistic type of incomplete market resulting from stochastic jumps in the continuous-time setting, and applied the results in the empirical study. Nevertheless, as shown in the discrete-time setting, our framework applies more broadly, and the extension to asset prices generated by other stochastic processes is another interesting possibility worth exploring in future work. Methods of estimation and inference for more general asset-price generating processes will then refine the measurement for market incompleteness as well as the assessment of misspecification caused by imposing complete market assumptions in financial market equilibrium, portfolio strategy, and risk pricing studies.

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Appendix: Proofs of Propositions and Theorems

Before proceeding to the proofs, we first recall the following definitions regarding limit and boundary points, set's convexity, openness, boundedness, and compactness properties.

Definition 5.1 (**Limit Point**). *Let $(\mathcal{S}, d)$ be the metric space and $\mathcal{C} \subseteq \mathcal{S}$. $x \in \mathcal{S}$ is a **limit point** of $\mathcal{C}$ if $\forall \epsilon > 0$, there is a point $y \in \mathcal{C} \backslash \{x\}$ with $d(x, y) < \epsilon$*

Definition 5.2 (**Boundary point**). *Let $(\mathcal{S}, d)$ be the metric space, if $\mathcal{C}$ is a subset of $\mathcal{S}$, a point $x \in \mathcal{S}$ is a **boundary point** of $\mathcal{C}$ if every neighbourhood of x contains at least one point in $\mathcal{C}$ and at least one point not in $\mathcal{C}$.*

Definition 5.3 (**Convex set**). *Let $\mathcal{S}$ be an affine space over some ordered field. A subset $\mathcal{C}$ of $\mathcal{S}$ is **convex** if, for all x and y in $\mathcal{C}$, the line segment connecting x and y is included in $\mathcal{C}$. This means that the affine combination*

$$\rho x + (1 - \rho) y \in \mathcal{C},$$

for all $x, y \in \mathcal{C}$, and ρ in the interval $[0, 1]$.

Definition 5.4 (**Open set**). *A subset $\mathcal{C}$ of a metric space $(\mathcal{S}, d)$ is **open** if every element, x, in $\mathcal{C}$ has a neighbourhood centred at x with radius ϵ lying in the set (i.e., $\mathcal{B}(x, \epsilon) \subset \mathcal{C}$).*

Definition 5.5 (**Bounded set**). *A set $\mathcal{C}$ in a metric space $(\mathcal{S}, d)$ is **bounded** if it has a finite generalized diameter. In other words, there is an $R < \infty$ such that $d(x, y) \leq R$ for all $x, y \in \mathcal{C}$.*

Definition 5.6 (**Compact set**). *For any subset $\mathcal{C}$ in a metric space $(\mathcal{S}, d)$, an **open cover** is a collection of sets $\{\boldsymbol{G}_n\}$ which are open in $(\mathcal{S}, d)$, such that $\mathcal{C} \subset \bigcup_n \{\boldsymbol{G}_n\}$. $\mathcal{C}$ is **compact** if and only if every open cover of $\mathcal{C}$ has a finite subcover.*

A.1 Proof of Proposition 2.1

Proof. Let $\boldsymbol{x} = \left[x^1, x^2, x^3\right]^\top$ be any point in $L(\boldsymbol{\Pi})$ and $\epsilon > 0$, we prove that there is $\boldsymbol{y} = \left[y^1, y^2, y^3\right]^\top \in \boldsymbol{\Pi} \backslash \{\boldsymbol{x}\}$ such that $d(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$. Let $y^1 = x^1$, $y^2 = x^2 - \delta$, and $y^3 = x^3 + \delta$, where $\delta < \min\left\{x^2, 1 - x^3, \epsilon/\sqrt{2}\right\}$. Then, $\sum_{s=1}^3 y^s = 1$ and $y^s > 0$ for $s = 1, 2, 3$ imply that $\boldsymbol{y} \in \boldsymbol{\Pi} \backslash \{\boldsymbol{x}\}$. Since $d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{2\delta^2} < \sqrt{\epsilon^2} = \epsilon$, $\boldsymbol{x} \in L(\boldsymbol{\Pi})$.

Since $\partial \boldsymbol{\Pi} \subset L(\boldsymbol{\Pi})$, every $\boldsymbol{x} \in \partial \boldsymbol{\Pi}$ is an element in $L(\boldsymbol{\Pi})$. Therefore, for $\epsilon > 0$, there is at least one point in $\mathcal{B}(\boldsymbol{x}, \epsilon)$ that is also an element of $\boldsymbol{\Pi}$. Now, consider $\boldsymbol{y} = \left[y^1, y^2, y^3\right]^\top \in \mathcal{B}(\boldsymbol{x}, \epsilon)$ in that $y^1 = x^1 + \delta$, $y^2 = x^2 - \delta$, and $y^3 = x^3 = 0$, where $\delta < \min\left\{1 - x^1, x^2, \epsilon/\sqrt{2}\right\}$. Then, $\sum_{s=1}^3 y^s = 1$, $y^1, y^2 > 0$ and $y^3 = 0$ imply that $\boldsymbol{y} \notin \boldsymbol{\Pi}$. Hence, $\boldsymbol{x} \in \partial \boldsymbol{\Pi}$. □

A.2 Proof of Theorem 2.1

To prove that d_1 is a valid metric, we first show that, d_2 in Equation 2.5 is a valid metric satisfying the following conditions.

1. $d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = 0$ if and only if $M^{s_{\boldsymbol{x}}} = M^{s_{\boldsymbol{y}}}$.

   Proof. ($\Rightarrow$) If $d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = 0$, we must have $|v^{s_{\boldsymbol{x}}} - v^{s_{\boldsymbol{y}}}| + |u^{s_{\boldsymbol{x}}} - u^{s_{\boldsymbol{y}}}| = 0$ for $s_{\boldsymbol{x}} = 1, 2, \ldots, S_{\boldsymbol{x}}$ and $s_{\boldsymbol{y}} = 1, 2, \ldots, S_{\boldsymbol{y}}$. Since $M^{s_{\boldsymbol{x}}}$ and $M^{s_{\boldsymbol{y}}}$ are non-zero, we must have $v^{s_{\boldsymbol{x}}} = v^{s_{\boldsymbol{y}}}$ and $u^{s_{\boldsymbol{x}}} = u^{s_{\boldsymbol{y}}}$, and thus, $M^{s_{\boldsymbol{x}}} = M^{s_{\boldsymbol{y}}}$.

   ($\Leftarrow$) If $M^{s_{\boldsymbol{x}}} = M^{s_{\boldsymbol{y}}}$, we have $v^{s_{\boldsymbol{x}}} = v^{s_{\boldsymbol{y}}}$ and $u^{s_{\boldsymbol{x}}} = u^{s_{\boldsymbol{y}}}$, and thus, $d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = 0$. □

2. $d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = d_2(M^{s_{\boldsymbol{y}}}, M^{s_{\boldsymbol{x}}})$.

   Proof.
$$\begin{aligned} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) &= |v^{s_{\boldsymbol{x}}} - v^{s_{\boldsymbol{y}}}| + |u^{s_{\boldsymbol{x}}} - u^{s_{\boldsymbol{y}}}| \\ &= |v^{s_{\boldsymbol{y}}} - v^{s_{\boldsymbol{x}}}| + |u^{s_{\boldsymbol{y}}} - u^{s_{\boldsymbol{x}}}| \\ &= d_2(M^{s_{\boldsymbol{y}}}, M^{s_{\boldsymbol{x}}}). \end{aligned}$$
   □

3. $d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{z}}}) \le d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) + d_2(M^{s_{\boldsymbol{y}}}, M^{s_{\boldsymbol{z}}})$.

   Proof.
$$\begin{aligned} & d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) + d_2(M^{s_{\boldsymbol{y}}}, M^{s_{\boldsymbol{z}}}) \\ &= |v^{s_{\boldsymbol{x}}} - v^{s_{\boldsymbol{y}}}| + |v^{s_{\boldsymbol{y}}} - v^{s_{\boldsymbol{z}}}| + |u^{s_{\boldsymbol{x}}} - u^{s_{\boldsymbol{y}}}| + |u^{s_{\boldsymbol{y}}} - u^{s_{\boldsymbol{z}}}| \\ &\ge |v^{s_{\boldsymbol{x}}} - v^{s_{\boldsymbol{y}}} + v^{s_{\boldsymbol{y}}} - v^{s_{\boldsymbol{z}}}| + |u^{s_{\boldsymbol{x}}} - u^{s_{\boldsymbol{y}}} + u^{s_{\boldsymbol{y}}} - u^{s_{\boldsymbol{z}}}| \\ &= |v^{s_{\boldsymbol{x}}} - v^{s_{\boldsymbol{z}}}| + |u^{s_{\boldsymbol{x}}} - u^{s_{\boldsymbol{z}}}| \\ &= d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{z}}}) \end{aligned}$$

   Hence, d_2 is a valid metric. □

Then, let
$$\boldsymbol{w}^* = \arg\inf_{\boldsymbol{w}} \left\{ \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) : \boldsymbol{w} \in W(\boldsymbol{x}, \boldsymbol{y}) \right\}$$
where $W(\boldsymbol{x}, \boldsymbol{y}) \coloneqq \left\{ \boldsymbol{w} \in \mathbb{R}_+^{S_{\boldsymbol{y}} \times S_{\boldsymbol{x}}} : \boldsymbol{w}^\top \mathbf{1}_{S_{\boldsymbol{y}}} = \boldsymbol{x}, \boldsymbol{w}\boldsymbol{x} = \boldsymbol{y} \right\}$ is the set of transport plans between $\boldsymbol{x}$ and $\boldsymbol{y}$ and $\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}, \mathbb{M}_{\boldsymbol{z}} \in \bar{\boldsymbol{C}}$, we prove that d_1 is a valid metric that satisfies the following conditions.

1. $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) = 0$ if and only if $\mathbb{M}_{\boldsymbol{x}} = \mathbb{M}_{\boldsymbol{y}}$.

   Proof. ($\Rightarrow$) If $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) = 0$, then we have

$$\sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = 0,$$

   implying that $w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = 0$ for all pairs of $(s_{\boldsymbol{x}}, s_{\boldsymbol{y}})$. Since $w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \geq 0$ and d_2 is a valid metric, we either have $w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} = 0$ or $d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = 0$. Since $\sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} = 1$, we can have one and only one $s'_{\boldsymbol{y}} \leq S_{\boldsymbol{y}}$ such that $M^{s_{\boldsymbol{x}}} = M^{s'_{\boldsymbol{y}}}$, in which case, we have $w^*_{s'_{\boldsymbol{y}} s_{\boldsymbol{x}}} = 1$, and since $\sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} = y^{s_{\boldsymbol{y}}}$, the $s'_{\boldsymbol{y}}$ must be distinct for different $s_{\boldsymbol{x}}$. Hence, $x^{s_{\boldsymbol{x}}} = y^{s_{\boldsymbol{y}}}$ for the $s_{\boldsymbol{x}}$ such that $M^{s_{\boldsymbol{x}}} = M^{s_{\boldsymbol{y}}}$, which entails that $\mathbb{M}_{\boldsymbol{x}} = \mathbb{M}_{\boldsymbol{y}}$.

   ($\Leftarrow$) If $\mathbb{M}_{\boldsymbol{x}} = \mathbb{M}_{\boldsymbol{y}}$, we can have $d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) = 0$ and $w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} = 1$ for every $s_{\boldsymbol{x}} = s_{\boldsymbol{y}}$. Since $\sum_{j=1}^{S_{\boldsymbol{y}}} w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} = 1$ and $w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \geq 0$, $w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} = 0$ for all $s_{\boldsymbol{x}} \neq s_{\boldsymbol{y}}$. Hence, $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) = 0$. □

2. $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) = d_1(\mathbb{M}_{\boldsymbol{y}}, \mathbb{M}_{\boldsymbol{x}})$.

   Proof.

$$\begin{aligned}
d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) &= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) \\
&= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} \frac{w^{*\prime}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} y^{s_{\boldsymbol{y}}}}{w^{*\prime}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} y^{s_{\boldsymbol{y}}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) \\
&= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w^{*\prime}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} y^{s_{\boldsymbol{y}}} \frac{w^*_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}}}{w^{*\prime}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} y^{s_{\boldsymbol{y}}}} d_2(M^{s_{\boldsymbol{y}}}, M^{s_{\boldsymbol{x}}}) \\
&= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w^{*\prime}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} y^{s_{\boldsymbol{y}}} d_2(M^{s_{\boldsymbol{y}}}, M^{s_{\boldsymbol{x}}}) \\
&= d_1(\mathbb{M}_{\boldsymbol{y}}, \mathbb{M}_{\boldsymbol{x}}),
\end{aligned}$$

   where $w^* \in W(\boldsymbol{x}, \boldsymbol{y})$ and $w^{*\prime} \in W(\boldsymbol{y}, \boldsymbol{x})$. □

3. $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{z}}) \leq d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) + d_1(\mathbb{M}_{\boldsymbol{y}}, \mathbb{M}_{\boldsymbol{z}})$.

Proof.

$$
\begin{aligned}
d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{z}}) &= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{z}}=1}^{S_{\boldsymbol{z}}} w^{*\prime\prime}_{s_{\boldsymbol{x}} s_{\boldsymbol{z}}} z^{s_{\boldsymbol{z}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{z}}}) \\
&= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} \sum_{s_{\boldsymbol{z}}=1}^{S_{\boldsymbol{z}}} w^{*}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} w^{*\prime}_{s_{\boldsymbol{z}} s_{\boldsymbol{y}}} y^{s_{\boldsymbol{y}}} \frac{w^{*\prime\prime}_{s_{\boldsymbol{x}} s_{\boldsymbol{z}}} z^{s_{\boldsymbol{z}}}}{w^{*}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} w^{*\prime}_{s_{\boldsymbol{z}} s_{\boldsymbol{y}}} y^{s_{\boldsymbol{y}}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{z}}}) \\
&\leq \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} \sum_{s_{\boldsymbol{z}}=1}^{S_{\boldsymbol{z}}} w^{*}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} \frac{w^{*\prime\prime}_{s_{\boldsymbol{x}} s_{\boldsymbol{z}}} z^{s_{\boldsymbol{z}}}}{w^{*}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) \\
&+ \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} \sum_{s_{\boldsymbol{z}}=1}^{S_{\boldsymbol{z}}} w^{*\prime}_{s_{\boldsymbol{z}} s_{\boldsymbol{y}}} y^{s_{\boldsymbol{y}}} \frac{w^{*}_{s_{\boldsymbol{x}} s_{\boldsymbol{z}}} z^{s_{\boldsymbol{z}}}}{w^{*\prime}_{s_{\boldsymbol{z}} s_{\boldsymbol{y}}} y^{s_{\boldsymbol{y}}}} d_2(M^{s_{\boldsymbol{y}}}, M^{s_{\boldsymbol{z}}}) \\
&= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w^{*}_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) + \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} \sum_{s_{\boldsymbol{z}}=1}^{S_{\boldsymbol{z}}} w^{*\prime}_{s_{\boldsymbol{z}} s_{\boldsymbol{y}}} y^{s_{\boldsymbol{y}}} d_2(M^{s_{\boldsymbol{y}}}, M^{s_{\boldsymbol{z}}}) \\
&= d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) + d_1(\mathbb{M}_{\boldsymbol{y}}, \mathbb{M}_{\boldsymbol{z}}),
\end{aligned}
$$

where $w^* \in W(\boldsymbol{x}, \boldsymbol{y})$, $w^{*\prime} \in W(\boldsymbol{y}, \boldsymbol{z})$ and $w^{*\prime\prime} \in W(\boldsymbol{x}, \boldsymbol{z})$.

Hence, $\left(\bar{\boldsymbol{C}}, d_1\right)$ is a valid metric space. □

Now, we are ready to prove for Theorem 2.1.

Proof. Let $\mathbb{M}_{\boldsymbol{x}}$ be any point in $L(\boldsymbol{C})$, and thus, $\boldsymbol{x} = \left[x^1, x^2, x^3\right]^\top \in L(\boldsymbol{\Pi})$, and $\epsilon > 0$, we prove that there is $\mathbb{M}_{\boldsymbol{y}} \in \boldsymbol{C} \backslash \mathbb{M}_{\boldsymbol{x}}$ such that $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) < \epsilon$, where $\boldsymbol{y} = \left[y^1, y^2, y^3\right]^\top \in \boldsymbol{\Pi} \backslash \{\boldsymbol{x}\}$. Let $y^1 = x^1 - \delta$, $y^2 = x^2$, and $y^3 = x^3 + \delta$. Since $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) = 0$ if and only if $\mathbb{M}_{\boldsymbol{x}} = \mathbb{M}_{\boldsymbol{y}}$ and $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) \geq 0$, thus, we can choose δ satisfying the following conditions:

$$
d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) < \epsilon \text{ and } \delta < \min\left\{x^1, 1 - x^3\right\},
$$

so that we have $\sum_{s=1}^{3} y^s = 1$ and $y^s > 0$ for $s = 1, 2, 3$, implying that $\boldsymbol{y} \in \boldsymbol{\Pi} \backslash \{\boldsymbol{x}\}$. Therefore, $\mathbb{M}_{\boldsymbol{y}} \in \mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon)$ such that $\mathbb{M}_{\boldsymbol{y}} \in \boldsymbol{C} \backslash \mathbb{M}_{\boldsymbol{x}}$, and thus, $\mathbb{M}_{\boldsymbol{x}} \in L(\boldsymbol{C})$.

Since $\partial \boldsymbol{C} \subset L(\boldsymbol{C})$, every $\mathbb{M}_{\boldsymbol{x}} \in \partial \boldsymbol{C}$ is an element in $L(\boldsymbol{C})$. Therefore, for $\epsilon > 0$, there is at least one point in $\mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon)$ that is also an element of $\boldsymbol{C}$. Now, consider $\mathbb{M}_{\boldsymbol{y}} \in \mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon)$ in that $y^1 = x^1 + \delta$, $y^2 = x^2 - \delta$, and $y^3 = x^3 = 0$, where $\delta < \min\left\{1 - x^1, x^2\right\}$ and satisfies the following condition $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) < \epsilon$. Then, $y^s > 0$ for $s = 1, 2$, and $y^3 = 0$, implying that $\boldsymbol{y} \notin \boldsymbol{\Pi}$, and thus, $\mathbb{M}_{\boldsymbol{y}} \notin \boldsymbol{C}$. Hence, $\mathbb{M}_{\boldsymbol{x}} \in \partial \boldsymbol{C}$. □

A.3 Proof of Lemma 2.1.1

Proof. Suppose that $(M_n^1, M_n^2) \to (M^1, M^2) \in \partial\boldsymbol{C}$ with $\pi_n^1 + \pi_n^2 = 1$ and $1/\pi_n^1, 1/\pi_n^2 > 0$. If it were $1/\pi^1 = 0$, then $\pi_n^1 \to \infty$, but $\pi_n^1 \le \pi_n^1 + \pi_n^2 = 1$, so that's impossible. Similarly, we cannot have $1/\pi_n^2 = 0$. Then, $\lim_{n\to\infty} \pi_n^1 + \pi_n^2 = \pi^1 + \pi^2 = 1$ so that $(M_n^1, M_n^2) \in \partial\boldsymbol{C}$, and $\partial\boldsymbol{C}$ is closed. Since $\partial\boldsymbol{C}$ is a non-empty subspace of $\bar{\boldsymbol{C}}$, taking $\mathbb{M}_{\boldsymbol{x}} \in \boldsymbol{C}$, there exists a closed ball $\mathcal{B} = \mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon)$ such that $\mathcal{B} \cap \partial\boldsymbol{C}$ is a non-empty compact set. So the function $\mathbb{M}_{\boldsymbol{y}} \mapsto d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})$ defined on $\mathcal{B} \cap \partial\boldsymbol{C}$ must achieve a minimum. That is, there is some $\mathbb{M}_{\boldsymbol{y}} = \mathbb{M}_{\boldsymbol{y}}^* \in \mathcal{B} \cap \partial\boldsymbol{C}$, which minimizes $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})$. Further, for $\mathbb{M}_{\boldsymbol{y}} \in \partial\boldsymbol{C} \setminus \mathcal{B}$, we have $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) > \epsilon \ge d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}^*)$, so it minimizes the distance on the whole of $\partial\boldsymbol{C}$. Moreover, since $\mathbb{M}_{\boldsymbol{y}}^* \notin \boldsymbol{C}$, $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}^*) > 0$. Hence, for every $\mathbb{M}_{\boldsymbol{x}} \in \boldsymbol{C}$, there exists $\mathbb{M}_{\boldsymbol{y}}^*$ such that

$$\mathbb{M}_{\boldsymbol{y}}^* = \underset{\mathbb{M}_{\boldsymbol{y}} \in \partial\boldsymbol{C}}{\arg\min} \{d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})\}.$$

□

A.4 Proof of Theorem 2.2

Proof. Let $\rho \in [0,1]$ and $\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}} \in \boldsymbol{C}$. The affine combination of $\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}$ is

$$\begin{aligned}\rho\,\mathbb{M}_{\boldsymbol{x}} + (1-\rho)\,\mathbb{M}_{\boldsymbol{y}} &= \{\boldsymbol{M} : \rho\,\mathbb{E}_{\boldsymbol{x}}[\boldsymbol{rM}] + (1-\rho)\,\mathbb{E}_{\boldsymbol{y}}[\boldsymbol{rM}] = \rho\mathbf{1}_2 + (1-\rho)\mathbf{1}_2\} \\ &= \{\boldsymbol{M} : \rho\,\mathbb{E}_{\boldsymbol{x}}[\boldsymbol{rM}] + (1-\rho)\,\mathbb{E}_{\boldsymbol{y}}[\boldsymbol{rM}] = \mathbf{1}_2\} \in \boldsymbol{C}.\end{aligned}$$

Hence, $\boldsymbol{C}$ is *convex*.

Let $\mathbb{M}_{\boldsymbol{x}} \in \boldsymbol{C}$ with $\boldsymbol{x} = [x^1, x^2, x^3]^\intercal \in \boldsymbol{\Pi}$. There is $\eta > 0$ such that

$$\eta = \min_{\mathbb{M}_{\boldsymbol{y}} \in \partial\boldsymbol{C}} \{d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})\},$$

where $\boldsymbol{y} \in \partial\boldsymbol{\Pi}$. Then, since $\boldsymbol{C} = \bar{\boldsymbol{C}} \setminus \partial\boldsymbol{C}$, by choosing $\epsilon < \eta$, we have $\mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon) \subset \boldsymbol{C}$. Hence, $\boldsymbol{C}$ is *open* in $\left(\bar{\boldsymbol{C}}, d_1\right)$.

Next, we prove that $\boldsymbol{C}$ is bounded in $\left(\bar{\boldsymbol{C}}, d_1\right)$. First, let $\boldsymbol{\kappa_v}$ and $\boldsymbol{\kappa_u}$ denote the coefficient vectors of $\boldsymbol{v}(\boldsymbol{x})$ and $\boldsymbol{u}(\boldsymbol{x})$, respectively, for $\boldsymbol{x} \in \bar{\boldsymbol{\Pi}}$. Given that $\boldsymbol{w} \in W(\boldsymbol{x}, \boldsymbol{y})$, we have

$$\begin{aligned}
\mathcal{D} &= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) \\
&= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} \left(|\kappa_{\boldsymbol{v}}^{s_{\boldsymbol{x}}}/x^{s_{\boldsymbol{x}}} - \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{y}}}/y^{s_{\boldsymbol{y}}}| + |\kappa_{\boldsymbol{u}}^{s_{\boldsymbol{x}}}/x^{s_{\boldsymbol{x}}} - \kappa_{\boldsymbol{u}}^{s_{\boldsymbol{y}}}/y^{s_{\boldsymbol{y}}}|\right) \\
&= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} \left(|w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{x}}} - w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{y}}} x^{s_{\boldsymbol{x}}}/y^{s_{\boldsymbol{y}}}| + |w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{u}}^{s_{\boldsymbol{x}}} - w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{u}}^{s_{\boldsymbol{y}}} x^{s_{\boldsymbol{x}}}/y^{s_{\boldsymbol{y}}}|\right).
\end{aligned}$$

Since $\sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} = y^{s_{\boldsymbol{y}}}$, for every $s_{\boldsymbol{x}} = 1, 2, \dots, S_{\boldsymbol{x}}, s_{\boldsymbol{y}} = 1, 2, \dots, S_{\boldsymbol{y}}$, $w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} \in [0, y^{s_{\boldsymbol{y}}}]$, and thus, $w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{y}}} x^{s_{\boldsymbol{x}}}/y^{s_{\boldsymbol{y}}} \in [0, \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{y}}}]$. Therefore, $\mathcal{D}$ is bounded, implying that there is $0 < R < \infty$ such that $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) \le R$ for all $\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}} \in \boldsymbol{C}$. Hence, $\boldsymbol{C}$ is *bounded*.

Lastly, to show that $\boldsymbol{C}$ is not compact, we just need one example of an open cover that has no finite open subcovers. Let $\{\boldsymbol{G}_n\} = \{\mathbb{M}_{\boldsymbol{\pi}} | \boldsymbol{\pi} \in \boldsymbol{\Pi}_n, n \in \mathbb{N}\}$, where

$$\{\boldsymbol{\Pi}_n\} = \left\{ \left[\pi^1, \pi^2, 1 - \sum_{i=1}^{2} \pi^s\right]^\top \in \mathbb{R}_{++}^3 : \sum_{i=1}^{2} \pi^s \in \left(\frac{1}{n}, \frac{n-1}{n}\right), n \in \mathbb{N} \right\}.$$

Notice that, if this gives us an invalid segment such as $(1, 0)$, we treat it as an empty element. Here, for any $b = \sum_{i=1}^{2} \pi^s \in (0, 1)$, the Archimedean Property provides an $n \in \mathbb{N}$ such that $n > \max\left\{\frac{1}{b}, \frac{1}{1-b}\right\}$. Then,

$$\begin{aligned}
& nb > 1 \text{ and } n - nb > 1 \\
\Rightarrow\ & 1 < nb < n - 1 \\
\Rightarrow\ & b \in \left(\frac{1}{n}, \frac{n}{n-1}\right).
\end{aligned}$$

Thus, every element of $\boldsymbol{\Pi}$ is in $\{\boldsymbol{\Pi}_n\}$ for some $n \in \mathbb{N}$, and therefore, every element of $\boldsymbol{C}$ is in $\{\boldsymbol{G}_n\}$ for some $n \in \mathbb{N}$, suggesting that $\boldsymbol{C} \subset \bigcup_{n=1}^{\infty} \{\boldsymbol{G}_n\}$. Moreover, since for any $n \in \mathbb{N}$, $\boldsymbol{G}_n$ has a neighbourhood centred at $\boldsymbol{G}_n$ with radius $\epsilon > 0$ lying in the set, $\{\boldsymbol{G}_n\}$ is an open cover of $\boldsymbol{C}$. Let $k, l \in \mathbb{N}$ such that $k > l > 2$, we have

$$\frac{1}{k} < \frac{1}{l} < 1 - \frac{1}{l} < 1 - \frac{1}{k} \Rightarrow \{\boldsymbol{\Pi}_l\} \subset \{\boldsymbol{\Pi}_k\} \Rightarrow \{\boldsymbol{G}_l\} \subset \{\boldsymbol{G}_k\}.$$

Therefore, for any finite $m \in \mathbb{N}$, $\bigcup_{n=1}^{m} \{\boldsymbol{G}_n\} = \{\boldsymbol{G}_m\} = \{\mathbb{M}_{\boldsymbol{\pi}} | \boldsymbol{\pi} \in \boldsymbol{\Pi}_m, m \in \mathbb{N}\}$. However, for any $m \in \mathbb{N}$, there exists $\boldsymbol{\Pi}_{m+1} \notin \{\boldsymbol{\Pi}_m\}$, while $\boldsymbol{\Pi}_{m+1} \in \boldsymbol{\Pi}$. Thus, there exists $\{\mathbb{M}_{\boldsymbol{\pi}} | \boldsymbol{\pi} \in \boldsymbol{\Pi}_{m+1}\} \notin \{\boldsymbol{G}_m\}$, while $\{\mathbb{M}_{\boldsymbol{\pi}} | \boldsymbol{\pi} \in \boldsymbol{\Pi}_{m+1}\} \in \boldsymbol{C}$. Therefore, $\{\boldsymbol{G}_n\}$ is an open cover of $\boldsymbol{C}$ that does not have a finite subcover.

Hence, $\boldsymbol{C}$ is *not compact.* □

A.5 Proof of Proposition 2.2

Proof. Let $\boldsymbol{x} = \left[x^1, x^2, \ldots, x^{A+2}\right]^\top$ be any point in $L(\boldsymbol{\Pi})$ and $\epsilon > 0$, we prove that there is $\boldsymbol{y} = \left[y^1, y^2, \ldots, y^{A+2}\right]^\top \in \boldsymbol{\Pi} \backslash \{\boldsymbol{x}\}$ such that $d(\boldsymbol{x}, \boldsymbol{y}) < \epsilon$. Let $y^s = x^s$ for $s = 1, 2, \ldots, A$, $y^{A+1} = x^{A+1} - \delta$, and $y^{A+2} = x^{A+2} + \delta$, where $\delta < \min\left\{x^{A+1}, 1 - x^{A+2}, \epsilon/\sqrt{2}\right\}$. Then, $\sum_{i=1}^{A+2} y^s = 1$ and $y^s > 0$ for $s = 1, 2, \ldots, A+2$ imply that $\boldsymbol{y} \in \boldsymbol{\Pi} \backslash \{\boldsymbol{x}\}$. Since $d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{2\delta^2} < \sqrt{\epsilon^2} = \epsilon$, $\boldsymbol{x} \in L(\boldsymbol{\Pi})$.

Since $\partial \boldsymbol{\Pi} \subset L(\boldsymbol{\Pi})$, every $\boldsymbol{x} \in \partial \boldsymbol{\Pi}$ is an element in $L(\boldsymbol{\Pi})$. Therefore, for $\epsilon > 0$, there is at least one point in $\mathcal{B}(\boldsymbol{x}, \epsilon)$ that is also an element of $\boldsymbol{\Pi}$. Now, consider $\boldsymbol{y} = \left[y^1, y^2, \ldots, y^{A+2}\right]^\top \in \mathcal{B}(\boldsymbol{x}, \epsilon)$ in that $y^1 = x^1 + \delta$, $y^2 = x^2 - \delta$, and $y^s = x^s$ for $s = 3, 4, \ldots, A+2$, where $\delta < \min\left\{1 - x^1, x^2, \epsilon/\sqrt{2}\right\}$. Then, $\sum_{s=1}^{A+2} y^s = 1$, $y^s > 0$ for $s = 1, 2, \ldots, A+1$ and $y^{A+2} = 0$ imply that $\boldsymbol{y} \notin \boldsymbol{\Pi}$. Hence, $\boldsymbol{x} \in \partial \boldsymbol{\Pi}$. □

A.6 Proof of Theorem 2.3

Proof. First of all, as proved in Appendix A.2, d_1 is a valid metric. Let $\mathbb{M}_{\boldsymbol{x}}$ be any point in $L(\boldsymbol{C})$, and thus, $\boldsymbol{x} = \left[x^1, x^2, \ldots, x^{A+2}\right]^\top \in L(\boldsymbol{\Pi})$. Let $\epsilon > 0$, we prove that there is $\mathbb{M}_{\boldsymbol{y}} \in \boldsymbol{C} \backslash \mathbb{M}_{\boldsymbol{x}}$ (equivalently, $\boldsymbol{y} = \left[y^1, y^2, \ldots, y^{A+2}\right]^\top \in \boldsymbol{\Pi} \backslash \{\boldsymbol{x}\}$) such that $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) < \epsilon$. Let $y^s = x^s$ for $s = 1, 2, \ldots, A$, $y^{A+1} = x^{A+1} - \delta$, and $y^{A+2} = x^{A+2} + \delta$. By choosing δ satisfying the following conditions:

$$d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) < \epsilon \text{ and } \delta < \min\left\{x^{A+1}, 1 - x^{A+2}\right\},$$

we have $\sum_{i=1}^{A+2} y^s = 1$ and $y^s > 0$ for $s = 1, 2, \ldots, A+2$ imply that $\boldsymbol{y} \in \boldsymbol{\Pi} \backslash \{\boldsymbol{x}\}$. Therefore, we can find $\mathbb{M}_{\boldsymbol{y}} \in \mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon)$ such that $\mathbb{M}_{\boldsymbol{y}} \in \boldsymbol{C}$, and thus, $\mathbb{M}_{\boldsymbol{x}} \in L(\boldsymbol{C})$.

Since $\partial \boldsymbol{C} \subset L(\boldsymbol{C})$, every $\mathbb{M}_{\boldsymbol{x}} \in \partial \boldsymbol{C}$ is an element in $L(\boldsymbol{C})$. Therefore, for $\epsilon > 0$, there is at least one point in $\mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon)$ that is also an element of $\boldsymbol{C}$. Now, consider $\mathbb{M}_{\boldsymbol{y}} \in \mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon)$ in that $y^1 = x^1 + \delta$ $y^2 = x^2 - \delta$, and $y^s = x^s$ for $s = 3, 4, \ldots, A+2$, where $\delta < \min\left\{1 - x^1, x^2\right\}$ and satisfies $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) < \epsilon$. Then, $y^s > 0$ for $s = 1, 2, \ldots, A+1$, and $y^{A+2} = 0$, implying that $\boldsymbol{y} \notin \boldsymbol{\Pi}$, and thus, $\mathbb{M}_{\boldsymbol{y}} \notin \boldsymbol{C}$. Hence, $\mathbb{M}_{\boldsymbol{x}} \in \partial \boldsymbol{C}$. □

A.7 Proof of Lemma 2.2.1

Proof. Suppose that $(M_n^1, M_n^2, \ldots, M_n^{A+1}) \to (M^1, M^2, \ldots, M^{A+1}) \in \partial \boldsymbol{C}$ with $\sum_{s=1}^{A+1} \pi_n^s = 1$ and $1/\pi_n^s > 0$ for $s = 1, 2, \ldots, A+1$. If it were $1/\pi^1 = 0$, then $\pi_n^1 \to \infty$, but $\pi_n^1 \leq \sum_{s=1}^{A+1} \pi_n^s = 1$, so that's impossible. Similarly, we cannot have $1/\pi_n^s = 0$ for any $s = 2, 3, \ldots, A+1$. Then, $\lim_{n\to\infty} \sum_{s=1}^{A+1} \pi_n^s = \sum_{s=1}^{A+1} \pi^s = 1$

so that $(M_n^1, M_n^2, \ldots, M_n^{A+1}) \in \partial \boldsymbol{C}$, and $\partial \boldsymbol{C}$ is closed. Since $\partial \boldsymbol{C}$ is a non-empty subspace of $\bar{\boldsymbol{C}}$, taking $\mathbb{M}_{\boldsymbol{x}} \in \boldsymbol{C}$, there exists a closed ball $\mathcal{B} = \mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon)$ such that $\mathcal{B} \bigcap \partial \boldsymbol{C}$ is a non-empty compact set. So the function $\mathbb{M}_{\boldsymbol{y}} \mapsto d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})$ defined on $\mathcal{B} \bigcap \partial \boldsymbol{C}$ must achieve a minimum. That is, there is some $\mathbb{M}_{\boldsymbol{y}} = \mathbb{M}_{\boldsymbol{y}}^* \in \mathcal{B} \bigcap \partial \boldsymbol{C}$, which minimizes $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})$. Further, for $\mathbb{M}_{\boldsymbol{y}} \in \partial \boldsymbol{C} \backslash \mathcal{B}$, we have $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}) > \epsilon \geq d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}^*)$, so it minimizes the distance on the whole of $\partial \boldsymbol{C}$. Moreover, since $\mathbb{M}_{\boldsymbol{y}}^* \notin \boldsymbol{C}$, $d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}^*) > 0$. □

A.8 Proof of Theorem 2.4

Proof. Let $\rho \in [0,1]$ and $\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}} \in \boldsymbol{C}$. The affine combination of $\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}$ is

$$\begin{aligned}\rho\, \mathbb{M}_{\boldsymbol{x}} + (1-\rho)\, \mathbb{M}_{\boldsymbol{y}} &= \{\boldsymbol{M} : \rho\, \mathbb{E}_{\boldsymbol{x}}[\boldsymbol{r}\boldsymbol{M}] + (1-\rho)\, \mathbb{E}_{\boldsymbol{y}}[\boldsymbol{r}\boldsymbol{M}] = \rho \mathbf{1}_{A+1} + (1-\rho)\mathbf{1}_{A+1}\} \\ &= \{\boldsymbol{M} : \rho\, \mathbb{E}_{\boldsymbol{x}}[\boldsymbol{r}\boldsymbol{M}] + (1-\rho)\, \mathbb{E}_{\boldsymbol{y}}[\boldsymbol{r}\boldsymbol{M}] = \mathbf{1}_{A+1}\} \in \boldsymbol{C}.\end{aligned}$$

Hence, $\boldsymbol{C}$ is *convex*.

Let $\mathbb{M}_{\boldsymbol{x}} \in \boldsymbol{C}$ with $\boldsymbol{x} \in \boldsymbol{\Pi}$. There is $\eta > 0$ such that

$$\eta = \min_{\mathbb{M}_{\boldsymbol{y}} \in \partial \boldsymbol{C}} \{d_1(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}})\},$$

where $\boldsymbol{y} \in \partial \boldsymbol{\Pi}$. Then, since $\boldsymbol{C} = \bar{\boldsymbol{C}} \backslash \partial \boldsymbol{C}$, by choosing $\epsilon < \eta$, we have $\mathcal{B}(\mathbb{M}_{\boldsymbol{x}}, \epsilon) \subset \boldsymbol{C}$. Hence, $\boldsymbol{C}$ is *open* in $\left(\bar{\boldsymbol{C}}, d_1\right)$.

Next, we prove that $\boldsymbol{C}$ is bounded in $\left(\bar{\boldsymbol{C}}, d_1\right)$. First, let $\boldsymbol{\kappa}_{\boldsymbol{v}}$ and $\boldsymbol{\kappa}_{\boldsymbol{u}}$ denote the coefficient vectors of $\boldsymbol{v}(\boldsymbol{x})$ and $\boldsymbol{u}(\boldsymbol{x})$, respectively, for $\boldsymbol{x} \in \bar{\boldsymbol{\Pi}}$. Given that $\boldsymbol{w} \in W(\boldsymbol{x}, \boldsymbol{y})$, we have

$$\begin{aligned}\mathcal{D} &= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} d_2(M^{s_{\boldsymbol{x}}}, M^{s_{\boldsymbol{y}}}) \\ &= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} (|\kappa_{\boldsymbol{v}}^{s_{\boldsymbol{x}}}/x^{s_{\boldsymbol{x}}} - \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{y}}}/y^{s_{\boldsymbol{y}}}| + |\kappa_{\boldsymbol{u}}^{s_{\boldsymbol{x}}}/x^{s_{\boldsymbol{x}}} - \kappa_{\boldsymbol{u}}^{s_{\boldsymbol{y}}}/y^{s_{\boldsymbol{y}}}|) \\ &= \sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} \sum_{s_{\boldsymbol{y}}=1}^{S_{\boldsymbol{y}}} \left(|w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_i^{\boldsymbol{v}} - w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{y}}} x^{s_{\boldsymbol{x}}}/y^{s_{\boldsymbol{y}}}| + |w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{u}}^{s_{\boldsymbol{x}}} - w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{u}}^{s_{\boldsymbol{y}}} x^{s_{\boldsymbol{x}}}/y^{s_{\boldsymbol{y}}}|\right)\end{aligned}$$

Since $\sum_{s_{\boldsymbol{x}}=1}^{S_{\boldsymbol{x}}} w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} = y^{s_{\boldsymbol{y}}}$, for every $s_{\boldsymbol{x}} = 1, 2, \ldots, S_{\boldsymbol{x}}, s_{\boldsymbol{y}} = 1, 2, \ldots, S_{\boldsymbol{y}}$, $w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} x^{s_{\boldsymbol{x}}} \in [0, y^{s_{\boldsymbol{y}}}]$, and thus, $w_{s_{\boldsymbol{y}} s_{\boldsymbol{x}}} \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{y}}} x^{s_{\boldsymbol{x}}}/y^{s_{\boldsymbol{y}}} \in [0, \kappa_{\boldsymbol{v}}^{s_{\boldsymbol{y}}}]$. Therefore, $\mathcal{D}$ is bounded, implying that there is $0 < R < \infty$ such that $d_1\left(\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}}\right) \leq R$ for all $\mathbb{M}_{\boldsymbol{x}}, \mathbb{M}_{\boldsymbol{y}} \in \boldsymbol{C}$. Hence, $\boldsymbol{C}$ is *bounded*.

Last, to show that $\boldsymbol{C}$ is not compact, we just need one example of an open cover that has no finite open subcovers. Let $\{\boldsymbol{G}_n\} = \{\mathbb{M}_{\boldsymbol{\pi}} \,|\, \boldsymbol{\pi} \in \boldsymbol{\Pi}_n, n \in \mathbb{N}\}$, where

$$\{\boldsymbol{\Pi}_n\} = \left\{\left[\pi^1, \pi^2, ..., \pi^{A+1}, 1 - \sum_{s=1}^{A+1} \pi^s\right]^\top \in (\mathbb{R})_{++}^{A+2} : \sum_{s=1}^{A+1} \pi^s \in \left(\frac{1}{n}, \frac{n-1}{n}\right), n \in \mathbb{N}\right\}.$$

Notice that, if this gives us an invalid segment such as $(1,0)$, we treat it as an empty element. Here, for any $b = \sum_{s=1}^{A+1} \pi^s \in (0,1)$, the Archimedean Property provides an $n \in \mathbb{N}$ such that $n > \max\left\{\frac{1}{b}, \frac{1}{1-b}\right\}$. Then,

$$\begin{aligned} & nb > 1 \text{ and } n - nb > 1 \\ \Rightarrow\ & 1 < nb < n - 1 \\ \Rightarrow\ & b \in \left(\frac{1}{n}, \frac{n}{n-1}\right). \end{aligned}$$

Thus, every element of $\boldsymbol{\Pi}$ is in $\{\boldsymbol{\Pi}_n\}$ for some $n \in \mathbb{N}$, and therefore, every element of $\boldsymbol{C}$ is in $\{\boldsymbol{G}_n\}$ for some $n \in \mathbb{N}$, suggesting that $\boldsymbol{C} \subset \bigcup_{n=1}^{\infty}\{\boldsymbol{G}_n\}$. Moreover, since for any $n \in \mathbb{N}$, $\boldsymbol{G}_n$ has a neighbourhood centred at $\boldsymbol{G}_n$ with radius $\epsilon > 0$ lying in the set, $\{\boldsymbol{G}_n\}$ is an open cover of $\boldsymbol{C}$. Let $k, l \in \mathbb{N}$ such that $k > l > 2$, we have

$$\frac{1}{k} < \frac{1}{l} < 1 - \frac{1}{l} < 1 - \frac{1}{k} \Rightarrow \{\boldsymbol{\Pi}_l\} \subset \{\boldsymbol{\Pi}_k\} \Rightarrow \{\boldsymbol{G}_l\} \subset \{\boldsymbol{G}_k\}.$$

Therefore, for any finite $m \in \mathbb{N}$, $\bigcup_{n=1}^{m}\{\boldsymbol{G}_n\} = \{\boldsymbol{G}_m\} = \{\mathbb{M}_{\boldsymbol{\pi}} \,|\, \boldsymbol{\pi} \in \boldsymbol{\Pi}_m, m \in \mathbb{N}\}$. However, for any $m \in \mathbb{N}$, there exists $\boldsymbol{\Pi}_{m+1} \notin \{\boldsymbol{\Pi}_m\}$, while $\boldsymbol{\Pi}_{m+1} \in \boldsymbol{\Pi}$. Thus, there exists $\{\mathbb{M}_{\boldsymbol{\pi}} \,|\, \boldsymbol{\pi} \in \boldsymbol{\Pi}_{m+1}\} \notin \{\boldsymbol{G}_m\}$, while $\{\mathbb{M}_{\boldsymbol{\pi}} \,|\, \boldsymbol{\pi} \in \boldsymbol{\Pi}_{m+1}\} \in \boldsymbol{C}$. Therefore, $\{\boldsymbol{G}_n\}$ is an open cover of $\boldsymbol{C}$ that does not have a finite subcover. Hence, $\boldsymbol{C}$ is *not compact.* □

A.9 Proof of Proposition 2.4

Proof. We first prove by induction that, for all $n \in \mathbb{Z}_+$ and $S = A+1+n$, there is $\boldsymbol{x}_t = \left[x_t^1, x_t^2, \ldots, x_t^S\right]^\top \in L(\boldsymbol{\Pi}_t)$ and $\boldsymbol{y}_t = \left[y_t^1, y_t^2, \ldots, y_t^S\right]^\top \in \boldsymbol{\Pi}_t \backslash \{\boldsymbol{x}_t\}$ such that $d(\boldsymbol{x}_t, \boldsymbol{y}_t) < \epsilon$ for $\epsilon > 0$.

Base case: When $n = 1$, $S = A+2$, let $\boldsymbol{x}_t = \left[x_t^1, x_t^2, \ldots, x_t^{A+2}\right]^\top$ be any point in $L(\boldsymbol{\Pi}_t)$. Let $\epsilon > 0$ and $\boldsymbol{y}_t = \left[y_t^1, y_t^2, \ldots, y_t^S\right]^\top \in \boldsymbol{\Pi}_t \backslash \{\boldsymbol{x}_t\}$ such that $y_t^s = x_t^s$ for $s = 1, 2, \ldots, A$, $y_t^{A+1} = x_t^{A+1} - \delta$, and $y_t^{A+2} = x_t^{A+2} + \delta$ and $\delta < \min\left\{x_t^{A+1}, 1 - x_t^{A+2}, \epsilon/\sqrt{2}\right\}$. Then, $\sum_{i=1}^{A+2} y_t^s = 1$ and $y_t^s > 0$ for $a = 1, 2, \ldots, A+2$, implying that $\boldsymbol{y}_t \in \boldsymbol{\Pi}_t \backslash \{\boldsymbol{x}_t\}$. Since $d(\boldsymbol{x}_t, \boldsymbol{y}_t) = \sqrt{2\delta^2} < \sqrt{\epsilon^2} = \epsilon$, $\boldsymbol{x}_t \in L(\boldsymbol{\Pi}_t)$.

Induction step: Let $k \in \mathbb{Z}_+$ be given and suppose our statement is true for $n = k$. Then, if $x_t^{A+k+2} = \eta > 0$, holding all other elements in $\boldsymbol{x}_t(n = k)$ and $\boldsymbol{y}_t(n = k)$ fixed, for any $i < A + k + 2$ with

$x_t^i(n=k) > \eta$[18], given $x_t^i(n=k+1) = x_t^i(n=k) - \eta$, there is $y_t^i(n=k+1) = y_t^i(n=k) - \eta$ such that

$$\begin{aligned}\left(x_t^i(n=k+1) - y_t^i(n=k+1)\right)^2 &= \left(x_t^i(n=k) - \gamma - y_t^i(n=k) + \gamma\right)^2 \\ &= \left(x_t^i(n=k) - y_t^i(n=k)\right)^2\end{aligned}$$

and $\left(x_t^{A+k+2}(n=k+1) - y_t^{A+k+2}(n=k+1)\right)^2 = 0$. Therefore, $d(\boldsymbol{x}_t(n=k+1), \boldsymbol{y}_t(n=k+1)) = d(\boldsymbol{x}_t(n=k), \boldsymbol{y}_t(n=k)) < \epsilon$. If $x_t^{A+k+2} = 0$, for any $x_t^i(n=k) > y_t^i(n=k)$, holding all other elements in $\boldsymbol{y}_t(n=k)$ fixed, let $y_t^{A+k+2}(n=k+1) = \gamma < x_t^i(n=k) - y_t^i(n=k)$ and $y_t^i(n=k+1) = y_t^i(n=k) - \gamma$, then

$$\begin{aligned}&\left(x_t^i(n=k+1) - y_t^i(n=k+1)\right)^2 + \left(x_t^{A+k+2}(n=k+1) - y_t^{A+k+2}(n=k+1)\right)^2 \\ &= \left(x_t^i(n=k) - y_t^i(n=k) + \gamma\right)^2 + \gamma^2 \\ &= \left(x_t^i(n=k) - y_t^i(n=k)\right)^2 - 2\gamma\left(x_t^i(n=k) - y_t^i(n=k)\right) + 2\gamma^2 \\ &< \left(x_t^i(n=k) - y_t^i(n=k)\right)^2.\end{aligned}$$

Therefore, $d(\boldsymbol{x}_t(n=k+1), \boldsymbol{y}_t(n=k+1)) < d(\boldsymbol{x}_t(n=k), \boldsymbol{y}_t(n=k)) < \epsilon$.

Conclusion: By the principal of induction, for all $n \in \mathbb{Z}_+$ and $S = A+1+n$, there is $\boldsymbol{x}_t = \left[x_t^1, x_t^2, \ldots, x_t^S\right]^\top \in L(\boldsymbol{\Pi}_t)$ and $\boldsymbol{y}_t = \left[y_t^1, y_t^2, \ldots, y_t^S\right]^\top \in \boldsymbol{\Pi}_t \backslash \{\boldsymbol{x}_t\}$ such that $d(\boldsymbol{x}_t, \boldsymbol{y}_t) < \epsilon$ for $\epsilon > 0$.

Since $\partial\boldsymbol{\Pi}_t \subset L(\boldsymbol{\Pi}_t)$, every $\boldsymbol{x}_t \in \partial\boldsymbol{\Pi}_t$ is an element in $L(\boldsymbol{\Pi}_t)$. Therefore, for $\epsilon > 0$, there is at least one point in $\mathcal{B}(\boldsymbol{x}_t, \epsilon)$ that is also an element of $\boldsymbol{\Pi}_t$. Now, consider $\boldsymbol{y}_t = \left[y_t^1, y_t^2, \ldots, y_t^S\right]^\top \in \mathcal{B}(\boldsymbol{x}_t, \epsilon)$ in that $y_t^1 = x_t^1 + \delta$, $y_t^2 = x_t^2 - \delta$, and $y_t^s = x_t^s$ for $s = 3, 4, \ldots, S$, where $\delta < \min\left\{1 - x_t^1, x_t^2, \epsilon/\sqrt{2}\right\}$. Then, $\sum_{s=1}^S y_t^s = 1$, $y_t^s > 0$ for $s = 1, 2, \ldots, A+1$, and $y_t^s = 0$ for $s = A+2, A+3, \ldots, S$ imply that $\boldsymbol{y}_t \notin \boldsymbol{\Pi}_t$. Hence, $\boldsymbol{x}_t \in \partial\boldsymbol{\Pi}_t$. □

A.10 Proof of Theorem 2.7

First notice that, similar to the proof in the 1-1-2-3 case, d_1 is a valid metric. Then, we prove by induction that, for all $n \in \mathbb{Z}_+$ and $S = A+1+n$, let $\mathbb{m}_{\boldsymbol{x}}$ be any point in $L(\boldsymbol{c}_t)$ and $\epsilon > 0$, we want to prove that there is $\mathbb{m}_{\boldsymbol{y}_t} \in \boldsymbol{c}_t \backslash \mathbb{m}_{\boldsymbol{x}_t}$ such that $d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t}) < \epsilon$, where $\boldsymbol{y}_t \in \boldsymbol{\Pi}_t \backslash \{\boldsymbol{x}_t\}$.

Base case: Since $\mathbb{m}_{\boldsymbol{x}} \in L(\boldsymbol{c}_t)$, $\boldsymbol{x}_t \in L(\boldsymbol{\Pi}_t)$. Let $y_t^s = x_t^s$ for $s = 1, 2, \ldots, A$, $y_t^{A+1} = x_t^{A+1} - \delta$, and $y_t^{A+2} = x_t^{A+2} + \delta$. By choosing δ satisfying the following conditions:

$$d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t}) < \epsilon \text{ and } \delta < \min\left\{x_t^{A+1}, 1 - x_t^{A+2}\right\},$$

[18]We use $x_t^i(n=k)$ to denote the i^{th} element in $\boldsymbol{x}_t$ for $n=k$.

so that we have $\sum_{i=1}^{A+2} y_t^s = 1$ and $y_t^s > 0$ for $s = 1, 2, \ldots, A+2$ imply that $\boldsymbol{y}_t \in \boldsymbol{\Pi}_t \backslash \{\boldsymbol{x}_t\}$. Therefore, we can find $\mathbb{m}_{\boldsymbol{y}_t} \in \mathcal{B}(\mathbb{m}_{\boldsymbol{x}_t}, \epsilon)$ such that $\mathbb{m}_{\boldsymbol{y}_t} \in \boldsymbol{c}_t$, and thus, $\mathbb{m}_{\boldsymbol{x}_t} \in L(\boldsymbol{c}_t)$.

Induction step: Let $k \in \mathbb{Z}_+$ be given and suppose our statement is true for $n = k$. Then, for every $\boldsymbol{x}_t(n = k+1)$ such that $x_t^{A+k+2} = \eta > 0$ and $x_t^i(n = k+1) = x_t^i(n = k) - \lambda_i \eta$ for $i = 1, 2, \ldots, A+k+1$, where $\sum_{i=1}^{A+1} \lambda_i = 1$ and $\lambda_i > 0$. By setting $y_t^{A+k+2} = \eta$ and $y_t^i(n = k+1) = y_t^i(n = k) - \lambda_i \eta > 0$ $i = 1, 2, \ldots, A+k+1$, we have $d_1(\mathbb{m}_{\boldsymbol{x}_t(n=k+1)}, \mathbb{m}_{\boldsymbol{y}_t(n=k+1)}) = d_1(\mathbb{m}_{\boldsymbol{x}_t(n=k)}, \mathbb{m}_{\boldsymbol{y}_t(n=k)}) < \epsilon$. If $x_t^{A+k+2} = 0$, by choosing η and $\lambda_i \geq 0$ $i = 1, 2, \ldots, A+k+1$ such that $y_t^{A+k+2} = \eta > 0$, $y_t^i(n = k+1) = y_t^i(n = k) - \lambda_i \eta > 0$, and $\sum_{i=1}^{A+1} \lambda_i = 1$, and satisfies the condition that $d_1(\mathbb{m}_{\boldsymbol{x}_t(n=k+1)}, \mathbb{m}_{\boldsymbol{y}_t(n=k+1)}) < \epsilon$ so that we have $\sum_{i=1}^{A+k+2} y_t^s = 1$ and $y_t^s > 0$ for $s = 1, 2, \ldots, A+k+2$ imply that $\boldsymbol{y}_t \in \boldsymbol{\Pi}_t \backslash \{\boldsymbol{x}_t\}$. Therefore, we can find $\mathbb{m}_{\boldsymbol{y}_t} \in \mathcal{B}(\mathbb{m}_{\boldsymbol{x}_t}, \epsilon)$ such that $\mathbb{m}_{\boldsymbol{y}_t} \in \boldsymbol{c}_t$, and thus, $\mathbb{m}_{\boldsymbol{x}_t} \in L(\boldsymbol{c}_t)$.

Since $\partial \boldsymbol{c}_t \subset L(\boldsymbol{c}_t)$, every $\mathbb{m}_{\boldsymbol{x}_t} \in \partial \boldsymbol{c}_t$ is an element in $L(\boldsymbol{c}_t)$. Therefore, for $\epsilon > 0$, there is at least one point in $\mathcal{B}(\mathbb{m}_{\boldsymbol{x}_t}, \epsilon)$ that is also an element of $\boldsymbol{c}_t$. Now, consider $\mathbb{m}_{\boldsymbol{y}_t} \in \mathcal{B}(\mathbb{m}_{\boldsymbol{x}_t}, \epsilon)$ in that $y_t^1 = x_t^1 + \delta$ $y_t^2 = x_t^2 - \delta$, and $y_t^s = x_t^s$ for $s = 3, 4, \ldots, A+2$, where $\delta < \min\{1 - x_t^1, x_t^2\}$ and satisfies $d_1(\mathbb{m}_{\boldsymbol{x}}, \mathbb{m}_{\boldsymbol{y}}) < \epsilon$. Then, $y_t^s > 0$ for $s = 1, 2, \ldots, A+1$, and $y_t^s = 0$ for $s = A+2, A+3, \ldots, S$, implying that $\boldsymbol{y}_t \notin \boldsymbol{\Pi}_t$, and thus, $\mathbb{m}_{\boldsymbol{y}_t} \notin \boldsymbol{c}_t$. Hence, $\mathbb{m}_{\boldsymbol{x}_t} \in \partial \boldsymbol{c}_t$.

A.11 Proof of Lemma 2.4.1

Proof. Suppose that $(m_{t,n}^1, m_{t,n}^2, \ldots, m_{t,n}^{A+1}) \to (m_t^1, m_t^2, \ldots, m_t^{A+1}) \in \partial \boldsymbol{c}_t$ with $\sum_{s=1}^{A+1} \pi_{t,n}^s = 1$ and $1/\pi_{t,n}^s > 0$ for $s = 1, 2, \ldots, A+1$. If it were $1/\pi_t^1 = 0$, then $\pi_{t,n}^1 \to \infty$, but $\pi_{t,n}^1 \leq \sum_{s=1}^{A+1} \pi_{t,n}^s = 1$, so that's impossible. Similarly, we cannot have $1/\pi_{t,n}^s = 0$ for any $s = 2, 3, \ldots, A+1$. Then, $\lim_{n\to\infty} \sum_{s=1}^{A+1} \pi_{t,n}^s = \sum_{s=1}^{A+1} \pi_t^s = 1$ so that $(m_{t,n}^1, m_{t,n}^2, \ldots, m_{t,n}^{A+1}) \in \partial \boldsymbol{c}_t$, and $\partial \boldsymbol{c}_t$ is closed. Since $\partial \boldsymbol{c}_t$ is a non-empty subspace of $\bar{\boldsymbol{c}}_t$, taking $\mathbb{m}_{\boldsymbol{x}_t} \in \boldsymbol{c}_t$, there exists a closed ball $\mathcal{B} = \mathcal{B}(\mathbb{m}_{\boldsymbol{x}_t}, \epsilon)$ such that $\mathcal{B} \bigcap \partial \boldsymbol{c}_t$ is a non-empty compact set. So the function $\mathbb{m}_{\boldsymbol{y}_t} \mapsto d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t})$ defined on $\mathcal{B} \bigcap \partial \boldsymbol{c}_t$ must achieve a minimum. That is, there is some $\mathbb{m}_{\boldsymbol{y}_t} = \mathbb{m}_{\boldsymbol{y}}^* \in \mathcal{B} \bigcap \partial \boldsymbol{c}_t$, which minimizes $d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t})$. Further, for $\mathbb{m}_{\boldsymbol{y}_t} \in \partial \boldsymbol{c}_t \backslash \mathcal{B}$, we have $d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t}) > \epsilon \geq d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t}^*)$, so it minimizes the distance on the whole of $\partial \boldsymbol{c}_t$. Moreover, since $\mathbb{m}_{\boldsymbol{y}_t}^* \notin \boldsymbol{c}_t$, $d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t}^*) > 0$. □

A.12 Proof of Theorem 2.8

Proof. Let $\rho \in [0, 1]$ and $\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t} \in \boldsymbol{c}_t$. The affine combination of $\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t}$ is

$$\begin{aligned}\rho\, \mathbb{m}_{\boldsymbol{x}_t} + (1-\rho)\, \mathbb{m}_{\boldsymbol{y}_t} &= \{\boldsymbol{m}_t : \rho\, \mathbb{E}_{\boldsymbol{x}_t}[\boldsymbol{r}_t \boldsymbol{m}_t] + (1-\rho)\, \mathbb{E}_{\boldsymbol{y}_t}[\boldsymbol{r}_1 \boldsymbol{m}_t] = \rho \mathbf{1}_{A+1} + (1-\rho) \mathbf{1}_{A+1}\} \\ &= \{\boldsymbol{m}_t : \rho\, \mathbb{E}_{\boldsymbol{x}_t}[\boldsymbol{r}_t \boldsymbol{m}_t] + (1-\rho)\, \mathbb{E}_{\boldsymbol{y}_t}[\boldsymbol{r}_t \boldsymbol{m}_t] = \mathbf{1}_{A+1}\} \in \boldsymbol{c}_t.\end{aligned}$$

Hence, $\boldsymbol{c}_t$ is *convex.*

Then let $\mathbb{m}_{\boldsymbol{x}_t} \in \boldsymbol{c}_t$ with $\boldsymbol{x}_t \in \boldsymbol{\Pi}_t$. There is $\eta > 0$ such that

$$\eta = \min_{\mathbb{m}_{\boldsymbol{y}_t} \in \boldsymbol{c}_t} \{d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t})\},$$

where $\boldsymbol{y}_t \in \partial\boldsymbol{\Pi}_t$. Then, since $\boldsymbol{c}_t = \bar{\boldsymbol{c}}_t \backslash \partial \boldsymbol{c}_t$, by choosing $\epsilon < \eta$, we have $\mathcal{B}(\mathbb{m}_{\boldsymbol{x}_t}, \epsilon) \subset \boldsymbol{c}_t$. Hence, $\boldsymbol{c}_t$ is *open* in $(\bar{\boldsymbol{c}}_t, d_1)$.

Next, we prove that $\boldsymbol{c}_t$ is bounded in $(\bar{\boldsymbol{c}}_t, d_1)$. First, let $\boldsymbol{\kappa}_{\boldsymbol{v}_t}$ and $\boldsymbol{\kappa}_{\boldsymbol{u}_t}$ denote the coefficient vectors of $\boldsymbol{v}_t(\boldsymbol{x}_t)$ and $\boldsymbol{u}_t(\boldsymbol{x}_t)$, respectively, for $\boldsymbol{x}_t \in \bar{\boldsymbol{\Pi}}_t$. Given that $\boldsymbol{w}_t \in W(\boldsymbol{x}_t, \boldsymbol{y}_t)$, we have

$$\begin{aligned}
\mathcal{D} &= \sum_{s_{\boldsymbol{x}_t}=1}^{S_{\boldsymbol{x}_t}} \sum_{s_{\boldsymbol{y}_t}=1}^{S_{\boldsymbol{y}_t}} w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} x_t^{s_{\boldsymbol{x}_t}} d_2(m_t^{s_{\boldsymbol{x}_t}}, m_t^{s_{\boldsymbol{y}_t}}) \\
&= \sum_{s_{\boldsymbol{x}_t}=1}^{S_{\boldsymbol{x}_t}} \sum_{s_{\boldsymbol{y}_t}=1}^{S_{\boldsymbol{y}_t}} w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} x_t^{s_{\boldsymbol{x}_t}} (|\kappa_{\boldsymbol{v}_t}^{s_{\boldsymbol{x}_t}} / x_t^{s_{\boldsymbol{x}_t}} - \kappa_{\boldsymbol{v}_t}^{s_{\boldsymbol{y}_t}} / y_t^{s_{\boldsymbol{y}_t}}| + |\kappa_{\boldsymbol{u}_t}^{s_{\boldsymbol{x}_t}} / x_t^{s_{\boldsymbol{x}_t}} - \kappa_{\boldsymbol{u}_t}^{s_{\boldsymbol{y}_t}} / y_t^{s_{\boldsymbol{y}_t}}|) \\
&= \sum_{s_{\boldsymbol{x}_t}=1}^{S_{\boldsymbol{x}_t}} \sum_{s_{\boldsymbol{y}_t}=1}^{S_{\boldsymbol{y}_t}} \left(|w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} \kappa_{\boldsymbol{v}_t}^{s_{\boldsymbol{x}_t}} - w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} \kappa_{\boldsymbol{v}_t}^{s_{\boldsymbol{y}_t}} x_t^{s_{\boldsymbol{x}_t}} / y_t^{s_{\boldsymbol{y}_t}}| + |w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} \kappa_{\boldsymbol{u}_t}^{s_{\boldsymbol{x}_t}} - w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} \kappa_{\boldsymbol{u}_t}^{s_{\boldsymbol{y}_t}} x_t^{s_{\boldsymbol{x}_t}} / y_t^{s_{\boldsymbol{y}_t}}|\right).
\end{aligned}$$

Since $\sum_{s_{\boldsymbol{x}_t}=1}^{S_{\boldsymbol{x}_t}} w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} x_t^{s_{\boldsymbol{x}_t}} = y_t^{s_{\boldsymbol{y}_t}}$, for every $s_{\boldsymbol{x}_t} = 1, 2, \ldots, S_{\boldsymbol{x}_t}, s_{\boldsymbol{y}_t} = 1, 2, \ldots, S_{\boldsymbol{y}_t}$, $w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} x_t^{s_{\boldsymbol{x}_t}} \in [0, y_t^{s_{\boldsymbol{y}_t}}]$. Thus, $w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} \kappa_{\boldsymbol{v}_t}^{s_{\boldsymbol{y}_t}} x_t^{s_{\boldsymbol{x}_t}} / y_t^{s_{\boldsymbol{y}_t}} \in [0, \kappa_{\boldsymbol{v}_t}^{s_{\boldsymbol{y}_t}}]$ and $w_{s_{\boldsymbol{y}_t} s_{\boldsymbol{x}_t}} \kappa_{\boldsymbol{u}_t}^{s_{\boldsymbol{y}_t}} x_t^{s_{\boldsymbol{x}_t}} / y_t^{s_{\boldsymbol{y}_t}} \in [0, \kappa_{\boldsymbol{u}_t}^{s_{\boldsymbol{y}_t}}]$. Therefore, $\mathcal{D}$ is bounded, implying that there is $0 < R < \infty$ such that $d_1(\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t}) \le R$ for all $\mathbb{m}_{\boldsymbol{x}_t}, \mathbb{m}_{\boldsymbol{y}_t} \in \boldsymbol{c}_t$. Hence, $\boldsymbol{c}_t$ is *bounded.*

Last, to show that $\boldsymbol{c}_t$ is not compact, we just need one example of an open cover that has no finite open subcovers. Let $\{\boldsymbol{G}_{t,n}\} = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \boldsymbol{\Pi}_{t,n}, n \in \mathbb{N}\}$, where

$$\{\boldsymbol{\Pi}_{t,n}\} = \left\{\left[\pi_t^1, \pi_t^2, ..., \pi_t^{S-1}, 1 - \sum_{s=1}^{S-1} \pi_t^s\right]^\top \in (\mathbb{R})_{++}^S : \sum_{s=1}^{S-1} \pi_t^s \in \left(\frac{1}{n}, \frac{n-1}{n}\right), n \in \mathbb{N}\right\}.$$

Notice that, if this gives us an invalid segment such as $(1, 0)$, we treat it as an empty element. Here, for any $b = \sum_{s=1}^{S-1} \pi_t^s \in (0, 1)$, the Archimedean Property provides an $n \in \mathbb{N}$ such that $n > \max\left\{\frac{1}{b}, \frac{1}{1-b}\right\}$. Then,

$$\begin{aligned}
&nb > 1 \text{ and } n - nb > 1 \\
&\Rightarrow 1 < nb < n - 1 \\
&\Rightarrow b \in \left(\frac{1}{n}, \frac{n}{n-1}\right).
\end{aligned}$$

Thus, every element of $\mathbf{\Pi}_t$ is in $\{\mathbf{\Pi}_{t,n}\}$ for some $n \in \mathbb{N}$, and therefore, every element of $\boldsymbol{c}_t$ is in $\{\boldsymbol{G}_{t,n}\}$ for some $n \in \mathbb{N}$, suggesting that $\boldsymbol{c}_t \subset \bigcup_{n=1}^{\infty}\{\boldsymbol{G}_{t,n}\}$. Moreover, since for any $n \in \mathbb{N}$, $\boldsymbol{G}_{t,n}$ has a neighbourhood centred at $\boldsymbol{G}_{t,n}$ with radius $\epsilon > 0$ lying in the set, $\{\boldsymbol{G}_{t,n}\}$ is an open cover of $\boldsymbol{c}_t$. Let $k, l \in \mathbb{N}$ such that $k > l > 2$, we have

$$\frac{1}{k} < \frac{1}{l} < 1 - \frac{1}{l} < 1 - \frac{1}{k} \Rightarrow \{\mathbf{\Pi}_{t,l}\} \subset \{\mathbf{\Pi}_{t,k}\} \Rightarrow \{\boldsymbol{G}_{t,l}\} \subset \{\boldsymbol{G}_{t,k}\}.$$

Therefore, for any finite $m \in \mathbb{N}$, $\bigcup_{n=1}^{m}\{\boldsymbol{G}_{t,n}\} = \{\boldsymbol{G}_{t,m}\} = \{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \mathbf{\Pi}_{t,m}, m \in \mathbb{N}\}$. However, for any $m \in \mathbb{N}$, there exists $\mathbf{\Pi}_{t,m+1} \notin \{\mathbf{\Pi}_{t,m}\}$, while $\mathbf{\Pi}_{t,m+1} \in \mathbf{\Pi}_t$. Thus, there exists $\{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \mathbf{\Pi}_{t,m+1}\} \notin \{\boldsymbol{G}_{t,m}\}$, while $\{\mathbb{m}_{\boldsymbol{\pi}_t}, \boldsymbol{\pi}_t \in \mathbf{\Pi}_{t,m+1}\} \in \boldsymbol{c}_t$. Therefore, $\{\boldsymbol{G}_{t,n}\}$ is an open cover of $\boldsymbol{c}_t$ that does not have a finite subcover. Hence, $\boldsymbol{c}_t$ is *not compact.* □

A.13 Proof of Proposition 3.1

Proof. Let $\boldsymbol{\phi}_t = \left(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)\right)$ be any point in $L(\mathbf{\Phi}_t)$. Let $\epsilon > 0$, we want to prove that there is $\boldsymbol{\phi}_t' = \left(\boldsymbol{\mu}_t^{B'}, \boldsymbol{\sigma}_t^{B'}, \boldsymbol{\mu}_t^{J'}, \boldsymbol{\sigma}_t^{J'}, \boldsymbol{v}'(dx)\right) \in \mathbf{\Phi}_t \backslash \{\boldsymbol{\phi}_t\}$ such that $d(\boldsymbol{\phi}_t, \boldsymbol{\phi}_t') < \epsilon$. Let $\boldsymbol{\mu}_t^{B'} = \boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^{B'} = \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^{J'} = \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^{J'} = \boldsymbol{\sigma}_t^J$, and $\boldsymbol{v}'(dx) = \boldsymbol{v}_t(dx) + \delta$, whence Equation 3.2 and 3.3 hold. Then, we have $\boldsymbol{v}'(dx) > \mathbf{0}_A$, implying that $\boldsymbol{\phi}_t' \in \mathbf{\Phi}_t \backslash \{\boldsymbol{\phi}_t\}$ and $d(\boldsymbol{\phi}_t, \boldsymbol{\phi}_t') < \epsilon$. Hence, $\boldsymbol{\phi}_t \in L(\mathbf{\Phi}_t)$.

Since $\partial \mathbf{\Phi}_t \subset L(\mathbf{\Phi}_t)$, every $\boldsymbol{\phi}_t \in \partial \boldsymbol{\phi}_t$ is an element in $L(\mathbf{\Phi}_t)$. Therefore, for $\epsilon > 0$, there is at least one point in $\mathcal{B}(\mathbf{\Phi}_t, \epsilon)$ that is also an element of $\boldsymbol{\phi}_t$. Now, consider $\boldsymbol{\phi}_t' = (\boldsymbol{\mu}_t^{B'}, \boldsymbol{\sigma}_t^{B'}, \boldsymbol{\mu}_t^{J'}, \boldsymbol{\sigma}_t^{J'}, \boldsymbol{v}'(dx)) \in \mathcal{B}(\mathbf{\Phi}_t, \epsilon)$ in that $d(\boldsymbol{\mu}_t^B, \boldsymbol{\mu}_t^{B'}) < \epsilon$, $\boldsymbol{\sigma}_t^{B'} = \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^{J'} = \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^{J'} = \boldsymbol{\sigma}_t^J$, $\boldsymbol{v}'(dx) = \boldsymbol{v}_t(dx)$, and Equation 3.3 holds. Then, $\boldsymbol{v}_t(dx) = 0$, implying that $\boldsymbol{\phi}_t \notin \mathbf{\Phi}_t$. Hence, $\boldsymbol{\phi}_t \in \partial \mathbf{\Phi}_t$. □

A.14 Proof of Theorem 3.1

Proof. Let $\boldsymbol{c}(\boldsymbol{\phi}_t)$ be any point in $L(\boldsymbol{C}_t)$, and thus, $\boldsymbol{\phi}_t = \left(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)\right) \in L(\mathbf{\Phi}_t)$. Let $\epsilon > 0$, we want to prove that there is $\mathbb{m}(\boldsymbol{\phi}_t') \in \boldsymbol{c}_t \backslash \mathbb{m}(\boldsymbol{\phi}_t)$ such that $d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}_t')) < \epsilon$, where $\boldsymbol{\phi}_t' = \left(\boldsymbol{\mu}_t^{B'}, \boldsymbol{\sigma}_t^{B'}, \boldsymbol{\mu}_t^{J'}, \boldsymbol{\sigma}_t^{J'}, \boldsymbol{v}_t'(dx)\right) \in \mathbf{\Phi}_t \backslash \{\boldsymbol{\phi}_t\}$. Let $\boldsymbol{\mu}_t^{B'} = \boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^{B'} = \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^{J'} = \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^{J'} = \boldsymbol{\sigma}_t^J$, and $\boldsymbol{v}_t'(dx) = \boldsymbol{v}_t(dx) + \boldsymbol{\delta}$. By choosing $\boldsymbol{\delta}$ satisfying $d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}_t')) < \epsilon$, and Equation 3.2 and 3.3 hold, we have $\boldsymbol{v}_t'(dx) > \mathbf{0}_A$ implying that $\boldsymbol{\phi}_t' \in \mathbf{\Phi}_t \backslash \{\boldsymbol{\phi}_t\}$. Therefore, $\mathbb{m}(\boldsymbol{\phi}_t') \in \mathcal{B}(\mathbb{m}(\boldsymbol{\phi}_t), \epsilon)$ such that $\mathbb{m}(\boldsymbol{\phi}_t') \in \boldsymbol{c}_t \backslash \mathbb{m}(\boldsymbol{\phi}_t)$, and thus, $\mathbb{m}(\boldsymbol{\phi}_t) \in L(\boldsymbol{c}_t)$.

Since $\partial \boldsymbol{c}_t \subset L(\boldsymbol{c}_t)$, every $\mathbb{m}(\boldsymbol{\phi}_t) \in \partial \boldsymbol{c}_t$ is an element in $L(\boldsymbol{c}_t)$. Therefore, for $\epsilon > 0$, there is at least one point in $\mathcal{B}(\mathbb{m}(\boldsymbol{\phi}_t), \epsilon)$ that is also an element of $\boldsymbol{c}_t$. Now, consider $\mathbb{m}(\boldsymbol{\phi}_t') \in \mathcal{B}(\mathbb{m}(\boldsymbol{\phi}_t), \epsilon)$ in that $|\boldsymbol{\mu}_t^B - \boldsymbol{\mu}_t^{B'}| < \boldsymbol{\delta}$, $\boldsymbol{\sigma}_t^{B'} = \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^{J'} = \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^{J'} = \boldsymbol{\sigma}_t^J$, $\boldsymbol{v}_t'(dx) = \boldsymbol{v}_t(dx)$, and $\boldsymbol{\delta}$ is chosen to satisfy

$d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}_t(\boldsymbol{\phi}'_t)) < \epsilon$ and Equation 3.3 holds. Then, $\boldsymbol{v}'(dx) = 0$, implying that $\boldsymbol{\phi}'_t \notin \boldsymbol{\Phi}_t$, and thus, $\mathbb{m}_t(\boldsymbol{\phi}'_t) \notin \boldsymbol{c}_t$. Hence, $\mathbb{m}(\boldsymbol{\phi}_t) \in \partial \boldsymbol{c}_t$. □

A.15 Proof of Lemma 3.1.1

Since $\partial \boldsymbol{c}_t$ is a non-empty closed subspace of $\bar{\boldsymbol{c}}_t$, taking $\mathbb{m}(\boldsymbol{\phi}_t) \in \boldsymbol{c}_t$, there exists a closed ball $\mathcal{B} = \mathcal{B}(\mathbb{m}(\boldsymbol{\phi}_t), \epsilon)$ such that $\mathcal{B} \cap \partial \boldsymbol{c}_t$ is a non-empty compact set. So the function $\mathbb{m}(\boldsymbol{\phi}'_t) \mapsto d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}'_t))$ defined on $\mathcal{B} \cap \partial \boldsymbol{c}_t$ must achieve a minimum. That is, there is some $\mathbb{m}(\boldsymbol{\phi}'_t) = \mathbb{m}(\boldsymbol{\phi}^*_t) \in \mathcal{B} \cap \partial \boldsymbol{c}_t$, which minimizes $d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}'_t))$. Further, for $\mathbb{m}(\boldsymbol{\phi}'_t) \in \partial \boldsymbol{c}_t \backslash \mathcal{B}$, we have $d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}'_t)) > \epsilon \geq d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}^*_t))$, so it minimizes the distance on the whole of $\partial \boldsymbol{c}_t$. Moreover, since $\mathbb{m}(\boldsymbol{\phi}^*_t) \notin \boldsymbol{c}_t, d_3(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}^*_t)) > 0$.

A.16 Proof of Theorem 3.2

Proof. Let $\rho \in [0,1]$ and $\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}'_t) \in \boldsymbol{c}_t$. Recall that the set $\mathbb{m}(\boldsymbol{\phi}_t)$ contains all the SDF that prices the payoff over a small time interval dt, and satisfies the asset pricing formula $\mathbb{E}[\boldsymbol{R}_t \boldsymbol{m}(\boldsymbol{\phi}_t)] = \mathbf{1}_A$, where $\boldsymbol{R}_t$ is the gross return vector from $t - dt$ to t. Then, the affine combination of $\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}'_t)$ can be expressed in the form

$$\begin{aligned} \rho\, \mathbb{m}(\boldsymbol{\phi}_t) + (1-\rho)\, \mathbb{m}(\boldsymbol{\phi}'_t) &= \left\{ \boldsymbol{m}(\boldsymbol{\phi}''_t) : \rho\, \mathbb{E}[\boldsymbol{R}_t \boldsymbol{m}(\boldsymbol{\phi}_t)] + (1-\rho)\, \mathbb{E}[\boldsymbol{R}_1 \boldsymbol{m}(\boldsymbol{\phi}'_t)] = \rho \mathbf{1}_A + (1-\rho) \mathbf{1}_A \right\} \\ &= \left\{ \boldsymbol{m}(\boldsymbol{\phi}''_t) : \rho\, \mathbb{E}[\boldsymbol{R}_t \boldsymbol{m}(\boldsymbol{\phi}_t)] + (1-\rho)\, \mathbb{E}[\boldsymbol{R}_1 \boldsymbol{m}(\boldsymbol{\phi}'_t)] = \mathbf{1}_A \right\} \in \boldsymbol{c}_t. \end{aligned}$$

Hence, $\boldsymbol{c}_t$ is *convex.*

Let $\mathbb{m}(\boldsymbol{\phi}_t) \in \boldsymbol{c}_t$ with $\boldsymbol{\phi}_t = \left(\boldsymbol{\mu}^B_t, \boldsymbol{\sigma}^B_t, \boldsymbol{\mu}^J_t, \boldsymbol{\sigma}^J_t, \boldsymbol{v}_t(dx) \right) \in \boldsymbol{\Phi}_t$. There is $\eta > 0$ such that

$$\eta = \min_{\mathbb{m}(\boldsymbol{\phi}'_t) \in \partial \boldsymbol{c}_t} \left\{ d_3 \left(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}'_t) \right) \right\},$$

where $\boldsymbol{\phi}'_t \in \partial \boldsymbol{\phi}_t$. Then, since $\boldsymbol{c}_t = \bar{\boldsymbol{c}}_t \backslash \partial \boldsymbol{c}_t$, by choosing $\epsilon < \eta$, we have $\mathcal{B}(\mathbb{m}(\boldsymbol{\phi}_t), \epsilon) \subset \boldsymbol{c}_t$. Hence, $\boldsymbol{c}_t$ is *open* in $(\bar{\boldsymbol{c}}_t, d_3)$.

Next, we prove that $\boldsymbol{c}$ is bounded in $(\bar{\boldsymbol{c}}, d_3)$. Suppose there is a positive upper bound $R < \infty$ and let $(\mathbb{m}(\boldsymbol{\phi}^*_t), \mathbb{m}(\boldsymbol{\phi}'^*_t), w^*_t) = \arg\max_{\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}_t(\boldsymbol{\phi}'_t) \in \boldsymbol{c}_t} d_3 \left(\mathbb{m}(\boldsymbol{\phi}_t), \mathbb{m}(\boldsymbol{\phi}'_t) \right)$. Then, any divergence of $(\boldsymbol{\phi}_t, \boldsymbol{\phi}'_t)$ from $\boldsymbol{\phi}_t{}^*, (\boldsymbol{\phi}'^*_t)$ can be offset by the corresponding change in the optimal transport plan w^*_t, as $w^*_t \in W(P(\boldsymbol{\phi}_t), P(\boldsymbol{\phi}'_t))$ is a function of $(\boldsymbol{\phi}_t, \boldsymbol{\phi}'_t)$ with

$$W(P(\boldsymbol{\phi}_t), P(\boldsymbol{\phi}'_t)) \coloneqq \left\{ w_t : \int w_t dP(\boldsymbol{\phi}'_t) = P(\boldsymbol{\phi}_t), \int w_t dP(\boldsymbol{\phi}_t) = P(\boldsymbol{\phi}'_t) \right\}.$$

Hence, $\boldsymbol{c}_t$ is *bounded.*

Lastly, to show that $\boldsymbol{c}_t$ is not compact, we just need one example of an open cover that has no finite open subcovers. Let $\{\boldsymbol{G}_{t,n}\} = \{\mathbb{m}(\boldsymbol{\phi}_t), \boldsymbol{\phi}_t \in \boldsymbol{\Phi}_{t,n}, n \in \mathbb{N}\}\}$, where

$$\{\boldsymbol{\Phi}_{t,n}\} = \left\{\left(\boldsymbol{\mu}_t^B, \boldsymbol{\sigma}_t^B, \boldsymbol{\mu}_t^J, \boldsymbol{\sigma}_t^J, \boldsymbol{v}_t(dx)\right) : \boldsymbol{v}_t(dx) > \frac{1}{n}, n \in \mathbb{N}\right\}.$$

Thus, every element of $\boldsymbol{\phi}_t$ is in $\{\boldsymbol{\Phi}_{t,n}\}$ for some $n \in \mathbb{N}$, and therefore, every element of $\boldsymbol{c}_t$ is in $\{\boldsymbol{G}_{t,n}\}$ for some $n \in \mathbb{N}$, suggesting that $\boldsymbol{G}_t \subset \bigcap_{n=1}^{\infty}\{\boldsymbol{G}_{t,n}\}$. Moreover, since for any $n \in \mathbb{N}$, $\boldsymbol{G}_{t,n}$ has a neighbourhood centred at $\boldsymbol{G}_{t,n}$ with radius $\epsilon > 0$ lying in the set, $\{\boldsymbol{G}_{t,n}\}$ is an open cover of $\boldsymbol{c}_t$. Let $k, l \in \mathbb{N}$ such that $k > l > 2$, we have

$$\frac{1}{k} < \frac{1}{l} \Rightarrow \{\boldsymbol{\Phi}_{t,l}\} \subset \{\boldsymbol{\Phi}_{t,k}\} \Rightarrow \{\boldsymbol{G}_{t,l}\} \subset \{\boldsymbol{G}_{t,k}\}.$$

Hence, for any finite $m \in \mathbb{N}$, $\bigcup_{n=1}^{m}\{\boldsymbol{G}_{t,n}\} = \{\boldsymbol{G}_{t,m}\} = \{\mathbb{m}(\boldsymbol{\phi}_t), \boldsymbol{\phi}_t \in \Phi_{t,m}, m \in \mathbb{N}\}$. However, for any $m \in \mathbb{N}$, there exists $\Phi_{t,m+1} \notin \{\boldsymbol{\Phi}_{t,m}\}$, while $\{\mathbb{m}(\boldsymbol{\phi}_t), \boldsymbol{\phi}_t \in \boldsymbol{\Phi}_{t,m+1}\} \in \boldsymbol{c}_t$. Therefore, $\{\boldsymbol{G}_{t,n}\}$ is an open cover of $\boldsymbol{c}_t$ that does not have a finite subcover. Hence, $\boldsymbol{c}_t$ is *not compact.* □

A.17 Proof of Theorem 3.3

Proof. Let $t_0 \in [0,1]$, $\epsilon > 0$, we show that for any $t \in [0,1]$ such that $|t - t_0| < \delta$, we have

$$\begin{aligned}
&\left|MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right) - MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t_0]}\right)\right| \\
&= \left|\mathbb{E}_t\left[\min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right)\right] - \mathbb{E}_{t_0}\left[\min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right)\right]\right| \\
&= \left|\frac{1}{t}\int_0^t \min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right) di - \frac{1}{t_0}\int_0^{t_0} \min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right) di\right| \\
&\le \left|\frac{1}{t}\int_0^t \min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right) di - \frac{1}{t}\int_0^{t_0} \min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right) di\right| \qquad \text{(A.1)} \\
&+ \left|\frac{1}{t}\int_0^{t_0} \min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right) di - \frac{1}{t_0}\int_0^{t_0} \min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right) di\right| \\
&= \frac{1}{t}\left|\int_{t_0}^t \min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right) di\right| + \left|\frac{1}{t} - \frac{1}{t_0}\right|\left|\int_0^{t_0} \min_{\boldsymbol{\phi}_i'\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i), \mathbb{m}(\boldsymbol{\phi}_i')\right) di\right| \\
&< \epsilon.
\end{aligned}$$

Thus, by choosing $\delta = \delta(t_0, \epsilon) > 0$ satisfying Equation A.1, we have

$$|MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right) - MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t_0]}\right)| < \epsilon,$$

and therefore, $MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right)$ is *continuous* on the time interval $[0,1]$.

Next, we prove that $MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right)$ is not monotonic. Let

$$\begin{aligned}
F(t) &= MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right) \\
&= \mathbb{E}_t\left[\min_{\boldsymbol{\phi}'_i\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i),\mathbb{m}(\boldsymbol{\phi}'_i)\right)\right] \\
&= \frac{1}{t}\int_0^t \min_{\boldsymbol{\phi}'_i\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i),\mathbb{m}(\boldsymbol{\phi}'_i)\right) di.
\end{aligned}$$

Then,

$$\begin{aligned}
F'(t) &= -t^{-2}\int_0^t \min_{\boldsymbol{\phi}'_i\in\partial\boldsymbol{\Phi}_i} d_3\left(\mathbb{m}(\boldsymbol{\phi}_i),\mathbb{m}(\boldsymbol{\phi}'_i)\right) di + t^{-1}\min_{\boldsymbol{\phi}'_t\in\partial\boldsymbol{\Phi}_t} d_3\left(\mathbb{m}(\boldsymbol{\phi}_t),\mathbb{m}(\boldsymbol{\phi}'_t)\right) \\
&= t^{-1}\left(\min_{\boldsymbol{\phi}'_t\in\partial\boldsymbol{\Phi}_t} d_3\left(\mathbb{m}(\boldsymbol{\phi}_t),\mathbb{m}(\boldsymbol{\phi}'_t)\right) - F(t)\right).
\end{aligned}$$

Therefore, whether the sign of $F'(t)$ depends on the difference between the sub-period market incompleteness at t and the average of sub-periods market incompleteness up to t, which is not strictly increasing nor decreasing. Hence, $MI\left(\{\boldsymbol{\phi}_i\}_{i\in[0,t]}\right)$ is *not monotonic.*

□