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Robust nonexistence of equilibrium with incomplete markets

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Abstract

We construct a pure exchange economy with spot and real security markets for which there does not exist a competitive equilibrium. Moreover, we show that the problem of nonexistence is robust to small perturbations of the endowments of the consumers. The result is driven by a lack of strict convexity of preferences.

JEL classification: D0; D5

Keywords: General equilibrium; Nonexistence; Incomplete markets

1. Introduction

Hart (1975) exhibited an example of an economy with forward and spot markets that does not have a competitive equilibrium. Polemarchakis and Ku (1990) credit Kreps (1979) with first observing the nongeneric nature of this example. Subsequent work by various authors has provided generic existence results, showing that Hart's example was indeed degenerate.[1] The canonical model under consideration in these papers is a two-period economy with one state of the world in the first period and a finite number of states in the second. There are spot markets in each period and state; additionally, there are real asset markets in the first period.[2] Two cases may be differentiated: the "easy" case of as many assets as states

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[1] For an introduction to general equilibrium with incomplete markets, see Geanakoplos (1990).

[2] Models with financial assets have also been considered, e.g. Werner (1985). Since the payout of a financial asset is exogenously fixed, the problems associated with the dependence of asset payoffs on spot market prices do not arise.

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(McManus, 1984; Repullo, 1986; Magill and Shafer, 1990), and the “hard” case of fewer assets than states, addressed by Duffie and Shafer (1985). All these papers assume that the preferences of the consumers are smooth and strictly convex, and they prove that for generic endowments and real asset structures equilibria exist in these models. Furthermore, Magill and Shafer (1990) show that these incomplete market equilibria are allocation equivalent to the contingent market equilibria if the asset structure is “regular”, and that for two-period models, assets are generically regular if there are at least as many assets as states.

In this paper, we show using an example that, if preferences are not strictly convex, then for a fixed asset structure the problem of nonexistence of equilibria can be robust to perturbations in the endowments of the consumers. The model we construct has two time periods, with two states of nature in the second period; there are two assets that are traded in the first period. We fix the asset structure and the preferences of the consumers, and vary the endowments of the consumers. It is well known (cf. Geanakoplos, 1990) that if such an economy had an equilibrium price vector at which the monetary returns from the two assets are different, then there exists an equilibrium of the associated Arrow–Debreu economy—i.e. one where all commodity markets open in the first period and there are no asset markets—that achieves the same allocation. In our example, the unique Arrow–Debreu equilibrium prices result in the payoffs from the two assets being the same for an open set of endowments. Thus, if these economies were to have an equilibrium, it would have to be a pure spot market equilibrium. But the spot market equilibria of these economies involve prices where the assets have different monetary returns. This then implies that these economies have no equilibria.

The asset structure here is such that for generic prices the returns from the two assets are different. It follows from Magill and Shafer (1990) that if the preferences of the consumers are smooth and strictly convex, then for generic endowments the equilibria of the incomplete market economy coincide with the equilibria of the Arrow–Debreu economy. The preferences in our model, while strictly monotonic and convex, are, however, not strictly convex. Consequently, there exists an open set of endowments for which the unique Arrow–Debreu equilibrium price vector lies in the nongeneric set where the assets are payoff equivalent.[3]

2. The example

There are two time periods, date 0 and date 1. There is only one state of nature in date 0, while in date 1 there are two, which are denoted states 1 and 2. (For simplicity in notation, we sometimes refer to date 0 as state 0.) In state $i = 0$ and 2, there is only one consumption good, x_i; in state 1, there are two goods, x_1 and y_1. There are two consumers, denoted A and B. Neither consumer cares about consumption in state 0. Their utility functions are given by:

$$U^{\mathrm{A}}(x_1, y_1, x_2) = \min\{2x_1 + y_1, x_1 + 2y_1\} + x_2$$

[3] Robust nonexistence has also been shown by Polemarchakis and Ku (1990) and Momi (2001). The former consider options for which there also exist open sets of endowments such that the asset return matrix drops rank. In response, Bottazzi (1995, 2002) derives conditions/characterizations of assets that have smooth payoffs, thus guaranteeing endowment-generic existence. Momi (2001) considers a production economy with incomplete stock markets and short sales under the Drèze criterion. In this case, it is the production side of the economy that causes equilibrium prices not to vary smoothly with endowment changes.

and

$$U^{\mathrm{B}}(x_1, y_1, x_2) = \min\{1.5(x_1 + y_1), 3x_1 + y_1\} + 2x_2$$

respectively. Consumer j's endowment vector is $(w^j_{x_1}, w^j_{y_1}, w^j_{x_2})$. (We will assume that both consumers have a zero endowment of good x_0.)

In state 1, there are spot markets for the two commodities. (Given the monotonicity of preferences in the amount of x_2, neither consumer would be willing to trade in state 2, even if there were a market.) In state 0, consumers can trade in two assets. If a consumer buys one unit of asset 1, it entitles him to the bundle $(1, 0)$ in state 1 and one unit of x_2 in state 2. Asset 2, on the other hand, would give him the bundle $(0, 2)$ in state 1 and one unit of x_2 in state 2. The only way for consumers to transfer income across the states in date 1 is through the asset markets. We will assume that both consumers have a zero endowment of the assets. Letting θ_i $(i = 1, 2)$ denote the quantity of asset i, the budget $B^j(p, q)$ of consumer j given asset prices $q = (q_1, q_2)$ and spot prices p_{x_1} and p_{y_1} for the two goods in state 1, is the set of all $(x_1, y_1, x_2, \theta_1, \theta_2)$ s.t.:

$$q_1\theta_1 + q_2\theta_2 = 0 \tag{1}$$

$$p_{x_1}x_1 + p_{y_1}y_1 = p_{x_1}(w^j_{x_1} + \theta_1) + p_{y_1}(w^j_{y_1} + 2\theta_2) \tag{2}$$

$$x_2 = w^j_{x_2} + \theta_1 + \theta_2 \tag{3}$$

Using the fact that $\theta_2 = -\theta_1 q_1/q_2$, we can rewrite the budget equations as:

$$p_{x_1}x_1 + p_{y_1}y_1 = p_{x_1}w^j_{x_1} + p_{y_1}w^j_{y_1} + \left(p_{x_1} - 2\frac{q_1}{q_2}p_{y_1}\right)\theta_1 \tag{4}$$

$$x_2 = w^j_{x_2} + \left(1 - \frac{q_1}{q_2}\right)\theta_1 \tag{5}$$

If $p_{x_1} - 2p_{y_1}q_1/q_2$ and $1 - q_1/q_2$ have the same sign, or if exactly one of them is zero, then both consumers can achieve an arbitrarily high utility level by buying one of the assets and selling the other. Thus, the utility maximization problem of the consumers has a well-defined solution only if: (i) both these expressions have opposite signs; or (ii) $q_1 = q_2$ and $p_{x_1} = 2p_{y_1}$.

An equilibrium of this incomplete market economy is a set of spot and asset prices (p, q), allocations and portfolios $(x^j_1, y^j_1, x^j_2, \theta^j_1, \theta^j_2)$ for each consumer j s.t., at these prices, the allocation and portfolio choices of both consumers maximize their utility subject to the budget constraints, and the spot and asset markets clear. Without loss of generality, we will normalize prices such that $q_2 = p_{y_1} = 1$. It follows from the observation above that if (q_1, p_{x_1}) is part of an equilibrium price vector, then either $(p_{x_1} - 2q_1)(1 - q_1)$ is negative, or $q_1 = 1$ while $p_{x_1} = 2$.

For the above economy with incomplete markets, there is an associated Arrow–Debreu economy where instead of asset and spot markets, we have markets for all three date 1 commodities opening in date 0. If the Arrow–Debreu economy has an equilibrium price vector $(\tilde{p}_{x_1}, 1, \tilde{p}_{x_2})$ with $\tilde{p}_{x_1} \neq 2$, then there exists an equilibrium in the incomplete market economy that achieves the same allocation with the equilibrium prices being $p_{x_1} = \tilde{p}_{x_1}$

and $q = (2 + \tilde{p}_{x_2})^{-1}(\tilde{p}_{x_1} + \tilde{p}_{x_2})$, since the set of consumption bundles that are feasible in the two scenarios are the same. For the same reason, we also have the following converse: if there exists an equilibrium of the incomplete market economy where $p_{x_1} \neq 2$ (and thus $q_1 \neq 1$), then there exists an Arrow–Debreu equilibrium that achieves the same allocation and has the prices $\tilde{p}_{x_1} = p_{x_1}$ and $\tilde{p}_{x_2} = (q_1 - 1)^{-1}(p_{x_1} - 2q_1)$.

Consider now the economy where the initial endowments of A and B are $(3, 10, 20)$ and $(9, 10, w^{\mathrm{B}}_{x_2})$, respectively.[4] We claim that this economy does not have an equilibrium. To prove this, assume to the contrary that there exists an equilibrium. If $p_{x_1} \neq 2$ in this equilibrium, then by the previous paragraph there exists an Arrow–Debreu equilibrium where the price of x_1 is also p_{x_1}. However, it is easily checked that the unique Arrow–Debreu equilibrium price vector is given by $\tilde{p}_{x_1} = 2$ and $\tilde{p}_{x_2} = 1.4$. Therefore, any equilibrium of the incomplete market economy has $p_{x_1} = 2$. But, that implies that $q_1 = 1$ in equilibrium, i.e. that consumers cannot transfer wealth across states. If $p_{x_1} = 2$ in equilibrium, then it must be an equilibrium price of the Edgeworth box economy generated by considering just state 1. However, the unique equilibrium price vector for this latter economy has $p_{x_1} = 1$. Thus, we conclude that the incomplete market economy has no equilibrium.

Finally, we claim that nonexistence of equilibrium persists in an open neighborhood of the endowment vector $((3, 10, 20), (9, 10, w^{\mathrm{B}}_{x_2}))$. Indeed, locally the unique Arrow–Debreu equilibrium has $\tilde{p}_{x_1} = 2$ and $\tilde{p}_{x_2} = (w^{\mathrm{A}}_{x_2})^{-1}(2w^{\mathrm{B}}_{x_1} + w^{\mathrm{B}}_{y_1})$, while the Edgeworth box economy generated by state 1 has $p_{x_1} = 1$, and the previous argument applies. Thus, there exists an open set of endowments for which the incomplete markets economy does not have an equilibrium.

Remark 1. The positive result of Magill and Shafer (1990) assumes that preferences are *both* smooth and strictly convex. The preferences here violate both these assumptions. However, it is clear that the example would work even if we "smoothed" the kink points in the indifference curves, as long as they still contained "flat" segments. (Thus, one could construct a similar counterexample for preferences that are smooth but not analytic.) The problem for existence is that, if preferences are not strictly convex, small variations in the endowments need not cause variations in the equilibrium prices and hence might not restore the rank of the asset return matrix.[5] Thus, it is the lack of strict convexity, and not the lack of smoothness, that drives the result.

Remark 2. The above result is robust only to perturbations of the endowment. Any change to the real asset structure for given preferences and endowments would restore the generic existence of equilibrium.[6] On the other hand, for any real asset structure with two assets there exist preferences and an open set of endowments for which no equilibrium exists.

[4] State 2 endowment for consumer B is a free parameter here.

[5] In this sense, the result is similar to that of Polemarchakis and Ku. In their case, a change in the endowments does not affect the returns of the put and call options sufficiently.

[6] This is different compared to Polemarchakis and Ku. There the asset structure, as indexed by exercise prices, may be varied. This is due to the fact that if an option is out of the money, say, a close by option is also out of the money.

Remark 3. Using the Balasko structure theorem (1988) one can show that the positive result of Magill and Shafer (1990) is restored for strictly convex preferences along with an additional assumption like semi-algebraicity of the graph of the Walrasian equilibrium correspondence. The structure theorem says that the graph E of the equilibrium correspondence is homeomorphic to a space that has the same dimension as the space of endowments if preferences are smooth and strictly convex. The homeomorphism works even if preferences are merely strictly convex. A ready implication of this homeomorphism is the following. For any lower-dimensional set P of prices, the subset of E consisting of endowments and equilibria with prices lying in P is a lower-dimensional subset of E. Hence, the set of economies that have an equilibrium in P is also lower-dimensional.[7] Fix now an asset structure where there are at least as many assets as the number of states and such that for generic prices, that is prices outside a lower-dimensional set, the return matrix has full rank. Then, the set of endowments for which the incomplete market economy does not have an equilibrium is a subset of those for which the corresponding Arrow–Debreu equilibrium prices all lie in this exceptional set. By the above observation, therefore, we have existence for generic endowments.

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[7] The semi-algebraicity assumption ensures this; without such a regularity assumption, one could have a pathological projection from E to the space of endowments.