Boundedly Rational Demand*

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Abstract

Evidence suggests that consumers do not perfectly optimize, contrary to a critical assumption of classical consumer theory. We propose a model in which consumer types can vary in both their preferences and their choice behavior. Given data on demand and the distribution of prices, we identify the set of possible values of the consumer surplus based on minimal rationality conditions: every type of consumer must be no worse off than if they either always bought the good or never did. We develop a procedure to narrow the set of surplus values using richer datasets and provide bounds on counterfactual demands.

Keywords: Behavioral welfare, bounded rationality, consumer theory, revealed preferences

JEL codes: D11, D6, D9

1 Introduction

A key implicit assumption of the standard approach to analyzing consumer demand and welfare is that consumers perfectly optimize. Yet it is clear from a number of empirical studies—if not from introspection alone—that this assumption does not generally hold in practice. For example, simply changing the way that prices are presented to consumers can have significant effects on demand (Chetty, Looney, and Kroft, 2009; Finkelstein, 2009).[1]

Failures to optimize perfectly can occur for a variety of reasons. The consumer may not be fully attentive to the price of a good they buy, perhaps because it is a habitual purchase, or because

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[1]See also Ito (2014) for empirical evidence that consumers do not correctly account for marginal electricity pricing and Feldman, Katuščák, and Kawano (2016) for related evidence regarding marginal tax rates. Dickson and Sawyer (1990) document that more than half of the supermarket shoppers they surveyed were unable to accurately report the price of an item immediately after placing it in their cart. Taubinsky and Rees-Jones (2018) and Tipoe (2021) find that there is significant heterogeneity in attention to prices.

prices involve complexities that require effort to understand. Similarly, consumers may not always be aware of the price of a good they do *not* buy. Alternatively, they may simply make stochastic errors, as is typically assumed in random utility models. Varying salience of certain features of the product, such as whether it is on sale, may also influence consumers' choices.

How are we to make inferences about a consumer's preferences and welfare in the absence of optimal choices? We propose an approach grounded in minimal assumptions about the consumer's rationality. We focus on a setting with unit demand in which the data consist of a (downward-sloping) demand curve indicating the probability of purchase at each price together with a distribution of prices. Our rationality conditions require only that no type of consumer would be better off if they switched to either always buying or never buying the good.

We consider an analyst who seeks to rationalize the data with a *model* describing a distribution of types of the consumer, with each type specifying the consumer's value together with their demand curve.[2,3] A model rationalizes the data if, at each price, the observed purchase probability is equal to the expected demand across the types in the model. We study two main questions. First, what can the analyst infer from the observed data about the surplus the consumer receives from participating in the market for this good? Second, what can the analyst predict about the counterfactual demand if the consumer were able to fully optimize? For each of these questions, we first obtain bounds using simple datasets then show how to narrow these bounds with richer data.

Relative to our setting, the standard assumption of optimal choice (coupled with quasilinear preferences) simplifies the analysis in two ways. First, each type has a threshold demand—buying the good if and only if the price is below that type's value for the good—and thus the value can be directly inferred from its demand. Second, any demand curve admits a unique decomposition into threshold demands of individual consumer types; thus the distribution of values can be directly inferred from the overall demand curve. Calculating the total consumer surplus is then simply a matter of adding up these values (weighted by the corresponding purchase probabilities) and subtracting the expenditure. In contrast, in our setting, types with the same value may differ in their demand, corresponding to differences in attention or sophistication. Thus values cannot be directly inferred from demands, and demands need not take a simple threshold form.

There are generally many different models that can rationalize the data. First, the analyst must consider various decompositions of the overall demand into demands of individual types. Second, for each type, given its demand, there is a range of incentive-compatible values, i.e., values for which our rationality conditions are satisfied. In light of this flexibility, it is not possible to pin down the surplus exactly. For instance, the analyst can assume perfect optimization and rationalize the demand in the standard way to obtain the usual consumer surplus. At the opposite extreme, the analyst can attribute all stochasticity in behavior to errors by assuming a single type whose

[2]One special case of particular interest—to which it turns out that all of our results apply verbatim—involves a consumer who is Bayes-rational but observes a noisy signal of the price. See Kocourek, Steiner, and Stewart (2022) for details.

[3]While we describe the model in terms of a single consumer with stochastic preferences and choice behavior, an equivalent interpretation is that there is a continuum of consumers each with fixed preferences and choice behavior.

demand matches the observed demand. Or the analyst can employ a richer model with many types that may or may not be optimizing.

We characterize the levels of consumer surplus (and counterfactual demands) across all rationalizations of the data. The levels of surplus consistent with the data comprise an interval ranging from 0 to an upper bound that has a simple mathematical structure akin to that of the standard consumer surplus. Just as the standard surplus is the area between the price line and the inverse demand up to the quantity demanded, the upper bound is the area between the price line and an "elevated" inverse demand up to the quantity demanded. As the name suggests, this elevated demand—which depends on both the observed demand and the price distribution—lies above the observed demand.

Each feasible level of surplus can be obtained with a simple model: as in the standard approach, the observed demand is decomposed into threshold demands of individual types. However, the value of each type need not be equal to the price at the corresponding threshold. For the upper bound on surplus, this value is equal to the expectation of all prices conditional on *exceeding* the threshold. In other words, each type's value is the expectation of the price conditional on that type's not buying the good. The aforementioned elevated demand identifies, for each quantity, the value of the type responsible for the marginal change in demand, that is, the type whose threshold lies at the corresponding price on the original demand curve. Other levels of surplus can be obtained from a similar model except with lower values for each type; for the lower bound, the values are chosen so as to make them indifferent between their behavior and abstaining from the market.

The main object of our analysis is the random variable describing the value a randomly chosen type assigns to the good, which we refer to as the *stochastic value*. Our bounds on surplus and counterfactual demand are based on bounds on the stochastic value with respect to various stochastic orders. For the upper bound on consumer surplus, we make use of the increasing convex order (ICX), which can be viewed as the analogue of second-order stochastic dominance (SOSD) for a decision-maker who is risk-loving instead of risk averse;[4] for the lower bound on surplus, we use SOSD; and for the bounds on counterfactual demand, we use first-order stochastic dominance (FOSD).

Our proofs feature two distinct steps. In the first step, we identify how the relevant stochastic order is related to the bound we seek to establish. For example, fixing the demand curve, we provide an upper bound on surplus for each stochastic value that is increasing with respect to the ICX order. The second, more substantive step involves identifying a bound on stochastic values consistent with the observed data with respect to the relevant stochastic order.

For a given decomposition of the observed demand, finding bounds on the stochastic value is straightforward: our rationality conditions state that each type's value lies between its *buying price expectation* and its *non-buying price expectation*, that is, between the expected price conditional on this type buying the good or conditional on not buying the good, respectively. Otherwise, this type would either benefit from switching to always buying the good or to never buying it. Increasing

[4]Thus whereas SOSD favors higher means and smaller spreads, ICX favors higher means and larger spreads.

types' values within these bounds generates an increase with respect to each of the stochastic orders we consider. The main challenge in establishing bounds on the stochastic value involves the need to consider all possible decompositions of the observed demand. As it turns out, the extremes with respect to the ICX and SOSD orders are achieved by decomposing the demand into threshold demands. The extremes with respect to FOSD correspond to a different decomposition.

Bounds on the stochastic value are particularly useful with richer data. In section 7, we consider datasets comprising two or more market regimes that may differ in the distribution of prices and/or the consumer's behavior at any given price. For example, it could be that, as in Chetty, Looney, and Kroft (2009), sales taxes are included in the posted price in one regime but not included in the other. The analyst considers all rationalizations of the datasets in which the value of each type is fixed across regimes (though its demand may vary).

We propose a simple procedure for narrowing the bounds on surplus or counterfactual demand within each regime using the data from the other regimes. This procedure involves taking the collection of bounds on the stochastic value across regimes and combining them to obtain a common tighter bound. To compute this combined bound, we first translate the stochastic values into functions that admit standard consumer-theoretic interpretations. To illustrate, consider the upper bound on consumer surplus. The key idea is to associate each stochastic value with the convex function describing the consumer surplus at each price if the consumer were to fully optimize. We show that increases in the stochastic value with respect to the ICX order correspond to increases in the associated consumer surplus function; upper bounds with respect to the ICX order therefore correspond to upper bounds on this function. The bounds arising from each regime can be combined by taking the convex closure of the minimum of these functions. Mapping this convex closure back to a stochastic value yields a new upper bound with respect to the ICX order, and from there an upper bound on consumer surplus.

Our setup abstracts away from any costs associated with decision-making, such as attention costs. Section 9 addresses how our results extend in the presence of such costs. In brief, as long as choice behavior that is independent of the price incurs no cost, all of our bounds apply to the consumer's surplus net of these costs except for the lower bound with multiple regimes.

In a similar spirit to Bernheim and Rangel (2009), we propose a revealed-preference approach to measuring the welfare of a decision-maker who may not be perfectly rational. Empirical studies of behavioral welfare typically assume, either explicitly or implicitly, that certain observed choices reveal the decision-maker's true preferences.[5] These preferences can then be used to assess the welfare associated with other choices that could be suboptimal. For example, Chetty, Looney, and Kroft (2009) and Taubinsky and Rees-Jones (2018) recover true preferences from consumer choices when a tax is made salient and use these to measure the welfare loss arising from mistakes that occur when the taxes are not salient; Bronnenberg et al. (2015) identify the true preferences from experts' choices and use these to evaluate non-experts' welfare. An alternative approach, taken by Gruber and Köszegi (2001), is to use a structural model that relates true preferences to choice

[5]See Bernheim and Taubinsky (2018) for a survey.

behavior. Relative to these empirical studies, we make much weaker assumptions about the extent to which true preferences can be inferred from the data, requiring only that behavior satisfies certain minimal rationality conditions.

In the special case of our model in which errors are due to imperfect information, our work can be viewed as combining revealed preference with information design, where the design has the goal of maximizing or minimizing the surplus or counterfactual demand consistent with the observed data.[6] Bergemann, Brooks, and Morris (2022) identify bounds on counterfactual behavior in abstract games. Theorem 4 in the present paper concerns counterfactual behavior in a more specific setting, but unlike in Bergemann, Brooks, and Morris (2022), the distribution of preferences in our model is not known to the analyst. Bergemann, Brooks, and Morris (2015, 2017) identify the range of surplus values that can be attained for given preferences as information varies in a monopolistic market or a first-price auction. Condorelli and Szentes (2020, 2022) characterize the range of surplus values consistent with partial knowledge of demand in settings with market power on the supply side. Regarding revealed preference, we are closest to the branch of the literature that uses choice data to jointly identify preferences and information, as in Masatlioglu, Nakajima, and Ozbay (2012) and Manzini and Mariotti (2014).

When considering bounds on surplus using data from multiple regimes, we represent random variables as convex functions to construct bounds with respect to the ICX or SOSD order. A similar technique has been used in Bayesian persuasion problems by Gentzkow and Kamenica (2016) and Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017). Müller and Scarsini (2006) establish lattice properties of these orders using the same transformation. This technique has a natural interpretation in our context: the convex function that represents a given stochastic value maps each price to the consumer surplus that would arise under that stochastic value under perfect optimization.

Varian (1985) and Kang and Vasserman (2022) study a complementary problem of identifying bounds on consumer surplus. In their models, consumers perfectly optimize but there are gaps in the demand observed by the analyst. Kang and Vasserman (2022) discuss how to interpret their problem in terms of concavification, along the lines of Kamenica and Gentzkow (2011). Sandomirskiy and Ushchev (2022) use a different concavification problem to identify bounds on consumer welfare across possible disaggregations of an observed aggregate demand curve. Our problem can also be viewed as one of concavification—a connection we discuss in section 5—but in a different space and with a different constraint. In all three cases, the space over which concavification occurs is very large, making standard techniques inapplicable. Allen and Rehbeck (2021) provide bounds on surplus from finite data sets for consumers who approximately optimize. In contrast, we focus on idealized infinite data with different bounded rationality assumptions that are not nested with theirs.

[6]While information design problems typically place no restrictions on the information structure, we impose an implicit restriction to ensure that each type has monotone demand.

2 Setup

An analyst observes *data* (Q, F) describing the stochastic purchasing behavior of a consumer with unit demand together with the distribution of prices. The demand function $Q : [\underline{p}, \overline{p}] \longrightarrow [0, 1]$, which we assume is non-increasing, specifies the probability $Q(p)$ of purchase at each price p; we denote by $P(q)$ the inverse demand associated with $Q(p)$.[7] Prices are distributed according to the continuous distribution $F(p)$ with support $[\underline{p}, \overline{p}]$, where $\underline{p} \geq 0$. As is standard when measuring consumer welfare, we assume that the analyst observes the choke price, i.e., $Q(\overline{p}) = 0$; similarly, we assume that $Q(\underline{p}) = 1$.

The demand Q is an aggregation of many choices made by the consumer across which both her valuations and her behavior may vary. In each such choice, the consumer faces a take-it-or-leave-it offer at a random price $\mathbf{p}$ drawn according to F. (We denote random variables in bold and their realizations with the corresponding non-bold symbol; all probabilities and expectations are evaluated with respect to these bold variables unless otherwise stated.) The consumer has a stochastic type $\mathbf{i}$ with support $I \subset \mathbb{R}$. Each type i specifies the consumer's value v_i for the good together with a non-increasing demand $Q_i(p)$.

We interpret the data as describing the choices of a single individual whose value for the good may be changing and whose behavior also varies due to unobserved factors such as attention or salience. We allow for the possibility that these factors are related to the value since, for example, the consumer may be more attentive to the price when their value is lower. An alternative interpretation of the data is that they combine choices made by a large population of consumers, with each type corresponding to a distinct individual whose value and behavior are fixed.

We assume that the consumer's value is independent of the price. This assumption is natural for individual-level data, such as scanner data, where the individual is negligible from the perspective of the seller. If consumers' values vary systematically over time—as, for instance, would be expected for a seasonal product—then the analyst should split the data into periods within which independence of values and prices is plausible. The tools developed in section 7 can be applied to the separated data. If, however, the analyst fails to account for temporal variation that affects the seller's pricing strategy, then our results may not apply.

In the standard approach, each type would purchase the good whenever the price is below her value. The demand Q then admits a unique disaggregation into threshold demands for each type with a threshold equal to the type's value. At any given price, summing the utilities across all types gives the standard measure of consumer surplus. In our approach, allowing for behavior that is not determined entirely by a type's value for the good leads to two complications: first, the analyst must consider many different ways of disaggregating the demand into demands of individual types; second, once the demand has been disaggregated, each type's demand does not pin down her value.

We impose a minimal rationality restriction on the types that the analyst can use to rationalize the data: we require that no type can be worse off than she would be if she never bought the

[7] That is, $P(q) := \inf\{p : Q(p) \leq q\}$. Analogously, given any inverse demand function $\tilde{P}$, we define the corresponding demand function $\tilde{Q}(p) = \inf\{q : \tilde{P}(q) \leq p\}$.

good (regardless of its realized price), nor can she be worse off than if she always bought the good. Letting $s_i = \mathrm{E}\left[(v_i - \mathbf{p})\, Q_i(\mathbf{p})\right]$ denote type i's expected surplus, this restriction corresponds to the pair of incentive compatibility constraints

$$s_i \geq 0 \tag{1}$$

$$\text{and} \qquad s_i \geq v_i - \mathrm{E}[\mathbf{p}]. \tag{2}$$

This requirement imposes discipline on the relationship between a type's value and its demand.

There are several reasons to expect that a consumer's demand may not perfectly reflect her value. It could be that she does not always check the price of the good, or does so only if she notices that it is on sale; the posted price may not include taxes that the consumer does not accurately compute; or the consumer may make random errors in assessing the value of the good, as in a random utility model. In each of these cases, the consumer's attentiveness and likelihood of making a mistake could depend on the current value. The analyst therefore allows for types' demands to vary along with their values in a general way, imposing only that no type makes systematic errors such that they would be better off either always buying or never buying.

The analyst seeks to explain the observed choices with a *model* that consists of a distribution M of types $i \in I$ together with a specification $(v_i, Q_i)_{i \in I}$ of values and demand functions for each type satisfying (1) and (2). We say that a given model *rationalizes* data (Q, F) if $Q(p) = \mathrm{E}\left[Q_{\mathbf{i}}(p)\right]$ for all p. Given a model that rationalizes the data, the (ex ante) *consumer surplus* is $s = \mathrm{E}\left[s_{\mathbf{i}}\right]$. In general, data can be rationalized by many different models which in turn yield different values of surplus. We say that surplus $s \in \mathbb{R}$ is *consistent with the data* if there exists a model that rationalizes (Q, F) and generates surplus s.

Example 1. The analyst observes the linear demand function $Q(p) = 1 - p$ and prices uniformly distributed on $[0, 1]$. There are many possible models that rationalize this data. For instance, it could be that, as in the standard analysis, the consumer always makes the optimal decision: her value $v_{\mathbf{i}}$ is uniformly distributed on $[0, 1]$ and each type i demands the good precisely when $p < v_i$. For any realized price p, this consumer receives surplus $(1 - p)^2/2$ (corresponding to the area between the demand curve and the price). The expected consumer surplus for this model is therefore $s = \mathrm{E}\left[(1 - \mathbf{p})^2/2\right] = 1/6$.

Alternatively, the data can be rationalized by a model with stochastic choices. Perhaps the simplest such rationalization features a consumer with a single type. For each price realization p, the consumer purchases the good with probability $Q(p)$, which trivially generates the observed aggregate demand. Inequalities (1) and (2) place limits on this type's value, v: it must be at least $\underline{v} = 1/3$ to ensure that she does not prefer to abstain from buying, and at most $\overline{v} = 2/3$ to ensure that she does not prefer to always buy. Taking $v = \overline{v}$ leads to a surplus of $\mathrm{E}\left[(\overline{v} - \mathbf{p})\, Q(\mathbf{p})\right] = 1/6$; taking $v = \underline{v}$ leads to a surplus of $\mathrm{E}\left[(\underline{v} - \mathbf{p})\, Q(\mathbf{p})\right] = 0$. Using values of v in between these two extremes, any surplus in $[0, 1/6]$ can be obtained.[8]

[8]That the upper bound of 1/6 is equal to the standard surplus is a coincidence that does not generally hold outside

More complex models can yield additional values of the surplus. Consider two equally likely types, 1 and 2, with respective demands

$$Q_1(p) = \begin{cases} 2(1-p) & \text{if } p \geq 1/2 \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad Q_2(p) = \begin{cases} 0 & \text{if } p \geq 1/2 \\ 1-2p & \text{otherwise.} \end{cases}$$

Since $(Q_1 + Q_2)/2 = Q$, these individual demands generate the observed total demand. Given each type's demand Q_i, inequalities (1) and (2) place restrictions on the values of the form $v_i \in [\underline{v}_i, \overline{v}_i]$. Using the maximal values gives surplus $\frac{1}{2}\sum_i \overline{v}_i \, \mathrm{E}\left[Q_i(\mathbf{p})\right] - \mathrm{E}[\mathbf{p}Q(\mathbf{p})] = 2/9$ (whereas the minimal values again give a surplus of 0).[9] △

We see from this example that it is possible to obtain values of the surplus consistent with the data that exceed the standard consumer surplus. This observation may seem surprising since the consumer in the standard model perfectly optimizes while in other models she does not. While it is true that introducing imperfections in decision-making can only lower the surplus *given the consumer's preferences*, allowing for these imperfections expands the range of preferences that can rationalize the data.

In the example, by splitting the aggregate demand across two types, we obtained higher levels of surplus than can be obtained with only one type. The question remains, however, as to whether further disaggregation of the two demands—or some other splitting—can expand the range of attainable surpluses. As we show in the next sections, it turns out that a "maximal" disaggregation of the demand can rationalize the highest value of the surplus (1/4 for Example 1).

3 Bounds on Consumer Surplus

3.1 Bounds for a single dataset

We identify tight bounds on the consumer surplus consistent with the observed data. To formulate the result, we define, for arbitrary demand $\tilde{Q}$ and (possibly unrelated) inverse demand $\hat{P}$, the functional

$$\mathcal{CS}\left(\tilde{Q}, \hat{P}; p\right) := \int_0^{\tilde{Q}(p)} \left(\hat{P}(q) - p\right) dq. \tag{3}$$

When applied to the observed demand function Q and its inverse demand P, $\mathcal{CS}\left(Q, P; p\right)$ returns the standard consumer surplus. In the standard case, when the consumer always chooses optimally, the inverse demand is equal to the her marginal benefit of consumption at each q. If the consumer does not always choose optimally, the inverse demand is not generally equal to the marginal benefit. Nonetheless, if $\tilde{Q}$ is the demand and $\hat{P}$ the marginal benefit of consumption, then $\mathcal{CS}\left(\tilde{Q}, \hat{P}; p\right)$ is the consumer surplus (at price p).

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of this example. On the other hand, 0 is a tight lower bound regardless of the data as there are always models in which the consumer is indifferent whenever she buys the good.

[9] Type 1 buys with probability 3/4 and has maximal value $\overline{v}_1 = 5/6$ and type 2 buys with probability 1/4 and has maximal value $\overline{v}_2 = 11/18$, giving a surplus of $1/2 \cdot 5/6 \cdot 3/4 + 1/2 \cdot 11/18 \cdot 1/4 - 1/6 = 2/9$.

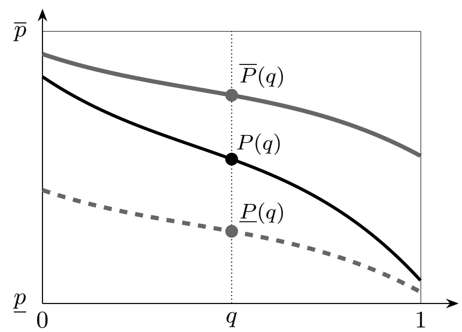


Figure 1: Elevated and lowered inverse demands for a particular inverse demand $P(q)$ with uniformly distributed prices. In this case, $\overline{P}(q)$ is the midpoint between $P(q)$ and 1. Likewise, $\underline{P}(q)$ is the midpoint between $P(q)$ and 0.

For any data (Q, F), we provide tight bounds on the consumer surplus using $\mathcal{CS}(Q, \hat{P}; p)$ for appropriate choices of $\hat{P}$. Accordingly, define the *elevated* and *lowered* inverse demands to be

$$\overline{P}(q) := \mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \geq P(q)\right]$$
$$\text{and} \qquad \underline{P}(q) := \mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \leq P(q)\right],$$

respectively. These two functions are non-increasing and satisfy $\overline{P}(q) \geq P(q) \geq \underline{P}(q)$ for all q; see Figure 1 for an illustration.

Theorem 1. *Consumer surplus s is consistent with data (Q, F) if and only if*

$$0 = \mathrm{E}\left[\mathcal{CS}\left(Q, \underline{P}; \mathbf{p}\right)\right] \leq s \leq \mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right)\right].$$

As explained above, $\mathcal{CS}\left(\tilde{Q}, \hat{P}; p\right)$ is the surplus obtained by a consumer who demands $\tilde{Q}(p)$ at each p and receives marginal benefit $\tilde{P}(q)$ at each q. The upper bound in Theorem 1 corresponds to the "highest possible" marginal benefit function consistent with the data, in a sense that is made precise in the proof of the theorem in section 5. To obtain the marginal benefit function $\overline{P}$, we construct a model rationalizing the data in which the analyst associates every marginal increase in demand with a different consumer type and takes each type's value to be the largest one that is consistent with its own demand. A similar construction with lower values can generate any surplus between 0 and the upper bound.

Example 2. To illustrate the theorem, consider the data from Example 1. The empirical inverse demand is $P(q) = 1 - q$, the elevated inverse demand is $\overline{P}(q) = (1 + P(q))/2 = 1 - q/2$, and $\mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right)\right] = 1/4$. Theorem 1 therefore indicates that surplus levels consistent with the data are precisely those in the interval $[0, 1/4]$. In this case, the consumer surplus can be up to $3/2$ times as large as the standard surplus. △

A special case of our environment that may be of particular interest arises when the consumer is Bayesian but observes only a noisy signal of the price of the good. The noise in this signal could result from inattention; for example, the consumer may assume that the price of the good exceeds her willingness to pay unless she notices that it is on sale (in which case she checks the price). In this case, a type of the consumer can be described by a value and an information structure: together, these two elements determine the type's demand function. Bayesian optimality implies that (1) and (2) are satisfied for each type. Conversely, for each type (v_i, Q_i) satisfying (1) and (2), there exists an information structure for which a Bayesian type with value v_i would have demand Q_i.[10] Therefore, Theorem 1 applies as written to this special case; in particular, Bayesian optimality does not narrow the bounds relative to those obtained with our minimal rationality assumptions.

Instead of welfare, the analyst may be interested in predicting the consumer's demand in some counterfactual market regime. In section 8, we derive tight bounds on the demand that would arise in a counterfactual market with a deterministic price, or, equivalently, in which the consumer always chooses optimally.

3.2 Bounds for multiple datasets

The bounds on consumer surplus can be narrowed if the analyst observes the consumer's choices under varying market conditions, which we refer to as *regimes*. We assume that the consumer's valuations are fixed across regimes, but the regimes may differ in the distribution of prices or in the purchasing behavior of each type at any given price (or both). For example, one such regime may correspond to a publicly announced "sale" associated with a low distribution of prices while another corresponds to the same market in the absence of a sale; the sale announcement may affect the consumer's stochastic choice at each price through changes in attention or salience. Alternatively, the regimes may differ only in how prices are presented to consumers, as in the empirical studies of Chetty, Looney, and Kroft (2009) and Finkelstein (2009).

Accordingly, suppose the analyst observes a separate dataset for each regime $k = 1, \ldots, K$. We provide here a brief overview of how the analyst can combine these datasets to obtain a lower upper bound on the surplus within each regime than that in Theorem 1. A similar construction leads to nontrivial positive lower bounds. Section 7 describes both constructions in detail.

Let $Q_k(p)$ denote the observed demand in regime k and $\overline{P}_k(q)$ the elevated inverse demand computed using only the data from regime k. While, as noted above, $\overline{P}_k$ describes the highest marginal benefits consistent with the data in regime k, it may be that the model yielding these marginal benefits is inconsistent with behavior in the other regimes, for example because the demand in the other regimes is too low to be generated by a consumer with values this high. We would therefore like to combine the elevated inverse demands across regimes to obtain a "lower" inverse demand $P_*(q)$ such that $\mathrm{E}_k\left[\mathcal{CS}\left(Q_k, P_*; \mathbf{p}\right)\right]$ is an upper bound on the surplus, where $\mathrm{E}_k[\cdot]$ refers to the expectation given the distribution of prices in regime k. (Note that the upper bound in Theorem 1 for regime k is $\mathrm{E}_k\left[\mathcal{CS}\left(Q_k, \overline{P}_k; \mathbf{p}\right)\right]$.)

[10] Given a type (v_i, Q_i), take the binary information structure that generates a "buy" signal with probability $Q_i(p)$ and an "abstain" signal otherwise. By (1) and (2), following the action recommended by the signal is incentive compatible.

How should the elevated inverse demands be combined? The answer to this question depends on the appropriate ordering of the marginal benefit functions. It turns out that this ordering can be captured by comparing the standard consumer surplus $CS_k(p)$ obtained if a consumer with inverse demand $\overline{P}_k$ perfectly optimizes. Each such consumer surplus must be convex since the inverse demand is downward-sloping. Let $CS_*(p)$ be the highest convex function that lies below every $CS_k(p)$ at each p. We show that differentiating $CS_*(p)$ gives a demand curve whose inverse is the desired marginal benefit function P_*; see Theorem 2 for details.

4 Preliminaries

The distribution of the consumer's value of the good plays a central role in our analysis. As we explain in this section, when the value is viewed as a random variable, stochastic orders on the value are relevant for comparisons of demand and of consumer surplus.

Given a model, let $\mathbf{v} := v_{\mathbf{i}}$ be the consumer's *stochastic value* of the good. Thus $\mathbf{v}$ is a random variable partially describing the model, disregarding types' demands. Let $Q^{\mathrm{s}}(p;\mathbf{v}) := \Pr(\mathbf{v} > p)$. For any price p, $Q^{\mathrm{s}}(p;\mathbf{v})$ is the probability with which the consumer would buy the good in the standard model (except possibly at atoms of $\mathbf{v}$).[11] Therefore, we refer to $Q^{\mathrm{s}}(p;\mathbf{v})$ as the *standard demand function* for $\mathbf{v}$. We use the superscript s throughout to indicate elements relating to the standard model.

Note that the standard demand function is the complementary distribution function of $\mathbf{v}$. Likewise, the standard inverse demand function $P^{\mathrm{s}}(q;\mathbf{v})$—which is the inverse to the demand $Q^{\mathrm{s}}(p;\mathbf{v})$—is the complementary quantile function of $\mathbf{v}$.

In light of the connection between the distribution of the stochastic value and the demand, first-order stochastic dominance comparisons of $\mathbf{v}$ correspond to rankings of the associated standard demands. Indeed, the following statements are equivalent: (i) $\mathbf{v}'$ first-order stochastically dominates $\mathbf{v}$; (ii) $Q^{\mathrm{s}}(p;\mathbf{v}') \geq Q^{\mathrm{s}}(p;\mathbf{v})$ for all p; and (iii) $P^{\mathrm{s}}(q;\mathbf{v}') \geq P^{\mathrm{s}}(q;\mathbf{v})$ for all q.[12]

Our bounds on consumer surplus make use of the so-called convex order. Gentzkow and Kamenica (2016) and Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017) identify a mapping from random variables to convex functions that is useful for making comparisons with respect to this order. Define the function $CS^{\mathrm{s}}(\cdot;\mathbf{v}) : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$CS^{\mathrm{s}}(p;\mathbf{v}) := \int_p^{\infty} Q^{\mathrm{s}}\left(p';\mathbf{v}\right) dp'. \tag{4}$$

This mapping has a natural interpretation in our context: it is the standard consumer surplus at price p for a consumer with stochastic value $\mathbf{v}$.[13] Observe that $CS^{\mathrm{s}}(p;\mathbf{v})$ is convex in p because Q^{s}

[11] If $\mathbf{v}$ has an atom at p, then the demand in the standard model lies in the closed interval between the left and right limits of $Q^{\mathrm{s}}(\cdot;\mathbf{v})$ at p.

[12] We use first-order stochastic dominance order in section 8 where we study bounds on counterfactual demand that would arise in the absence of imperfections in choice.

[13] Gentzkow and Kamenica (2016) and Kolotilin et al. (2017) map each random variable to the integral of the *lower* tail of its distribution function. For our purposes, the relevant integral is over the *upper* tail of the *complementary*

is downward-sloping.

In addition to its economic interpretation, the function $CS^{\mathrm{s}}(\cdot;\mathbf{v})$ characterizes the convex order on $\mathbf{v}$. Given two real-valued random variables $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{y}$ *dominates* $\mathbf{x}$ *in the increasing convex order*, denoted by $\mathbf{y} \succeq_{\mathrm{icx}} \mathbf{x}$, if there exists a random variable $\mathbf{z}$ such that $\mathbf{z}$ first-order stochastically dominates $\mathbf{x}$ and $\mathbf{y}$ is a mean-preserving spread of $\mathbf{z}$.

Lemma 1. *For any $\mathbf{v}'$ and $\mathbf{v}$, $\mathbf{v}' \succeq_{\mathrm{icx}} \mathbf{v}$ if and only if $CS^{\mathrm{s}}(p;\mathbf{v}') \geq CS^{\mathrm{s}}(p;\mathbf{v})$ for every price p.*

Proof. The result follows from Theorem 4.A.2 of (Shaked and Shanthikumar, 2007) together with the fact that $Q^{\mathrm{s}}(p;\mathbf{v})$ is the complementary distribution function of $\mathbf{v}$. □

The increasing convex order is closely related to second-order stochastic dominance, denoted here by $\succeq_{\mathrm{sosd}}$.[14] (Indeed, $\mathbf{y} \succeq_{\mathrm{icx}} \mathbf{x}$ if and only if $-\mathbf{x} \succeq_{\mathrm{sosd}} -\mathbf{y}$.) Roughly speaking, both orders favor higher values, but the increasing convex order favors spreads while second-order stochastic dominance disfavors them. Lemma 1 is essentially the analogue for the increasing convex order of the usual characterization of SOSD in terms of integrals of the lower tails of distribution functions.

Given a stochastic value $\mathbf{v}$ and demand $\tilde{Q}(p)$, a special role in our analysis will be played by $\mathcal{CS}\left(\tilde{Q}, P^{\mathrm{s}}(\cdot;\mathbf{v});p\right)$. This quantity is the highest possible surplus a consumer with value $\mathbf{v}$ and demand $\tilde{Q}(p)$ can achieve, which is attained when the measure $\tilde{Q}(p)$ of the highest types are the ones that buy the good. We abuse notation and write $\mathcal{CS}(\tilde{Q},\mathbf{v};p)$ for $\mathcal{CS}\left(\tilde{Q}, P^{\mathrm{s}}(\cdot;\mathbf{v});p\right)$ throughout.

The next result suggests how the increasing convex order can be useful outside of the standard model, for a consumer who makes imperfect choices.

Lemma 2. *For every demand function $\tilde{Q}$ and any price p, $\mathcal{CS}(\tilde{Q},\mathbf{v};p)$ is nondecreasing in $\mathbf{v}$ with respect to the increasing convex order.*

Clearly, a first-order stochastic dominance increase of values increases the consumer surplus $\mathcal{CS}(\tilde{Q},\mathbf{v};p)$. A mean-preserving spread of the values also increases this surplus because it is computed under the assumption that, for each p, it is the measure $\tilde{Q}(p)$ of types with the highest values that buy. The gross surplus at a given price is therefore proportional to the mean value conditional on being among these buying types, which increases with a mean-preserving spread. The proof of this lemma—and those of other results not proved in the main text—may be found in the appendix.

5 Proof of Theorem 1

We begin by identifying the set of values consistent with the demand of a given type. We say that a value v_i is *consistent with demand* Q_i if v_i together with Q_i satisfy inequalities (1) and (2). Given

distribution function.

[14] Recall that $\mathbf{y}$ second-order stochastically dominates $\mathbf{x}$ if there exists $\mathbf{z}$ such that $\mathbf{x}$ is a mean-preserving spread of $\mathbf{z}$ and $\mathbf{y}$ first-order stochastically dominates $\mathbf{z}$.

Q_i, let

$$\underline{v}_i := \mathrm{E}\left[\mathbf{p} \mid Q_i(\mathbf{p}) \geq \mathbf{q}\right] = \mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \leq P_i(\mathbf{q})\right],$$
$$\overline{v}_i := \mathrm{E}\left[\mathbf{p} \mid Q_i(\mathbf{p}) \leq \mathbf{q}\right] = \mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \geq P_i(\mathbf{q})\right],$$

where $\mathbf{q} \sim U[0,1]$ and P_i is the inverse demand associated with Q_i. Since type i buys with probability $Q_i(p)$ at each price p, $\underline{v}_i$ and $\overline{v}_i$ are, respectively, the expected price conditional on the event that consumer of type i does or does not make a purchase. Accordingly, we refer to $\underline{v}_i$ as the *buying price expectation* and to $\overline{v}_i$ as the *non-buying price expectation*. Note that, since Q_i is downward sloping, $\underline{v}_i \leq \overline{v}_i$.

Lemma 3. *A value v_i is consistent with Q_i if and only if $\underline{v}_i \leq v_i \leq \overline{v}_i$.*

This result follows directly from inequalities (1) and (2); we omit the details of the proof.

We divide Theorem 1 into its sufficiency and necessity claims. To prove sufficiency, we first show by construction that each surplus between 0 and $\mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right)\right]$ is consistent with the data. For the necessity claim, we prove that no other levels of surplus are consistent with the data. For the latter, we show that the stochastic value associated with the construction yielding the upper bound on surplus provides an upper bound with respect to the increasing convex order. (For necessity, it suffices to consider only the upper bound: since the lower bound on surplus is 0, it follows trivially from (1) that no lower surplus can be obtained.)

To prove the sufficiency claim, we construct for each $s \in \left[0, \mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right)\right]\right]$ a model that generates surplus s. Let the type $\mathbf{i}$ be uniformly distributed on $[0,1]$ and let each realization i generate the demand function $Q_i(p) = \mathbb{1}_{p \leq P(i)}$. Thus type i always buys when the price is below $P(i)$ and never buys at prices above $P(i)$. Note that the average demand across all types is equal to the observed demand Q, as needed for the model to rationalize the data:

$$\mathrm{E}\left[Q_{\mathbf{i}}(p)\right] = \Pr\left(P(\mathbf{i}) \geq p\right) = \Pr\left(\mathbf{i} \leq Q(p)\right) = Q(p).$$

By Lemma 3, a value v_i is consistent with demand Q_i if $\underline{v}_i \leq v_i \leq \overline{v}_i$. Due to the choice of Q_i, we have $\underline{v}_i = \underline{P}(i)$ and $\overline{v}_i = \overline{P}(i)$. Since type i buys if and only if $i \leq Q(p)$, taking $v_i = \overline{v}_i$ for all i gives ex ante surplus $\mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right)\right]$; we refer to this model as the *upper threshold model*. At the other extreme, taking $v_i = \underline{v}_i$ for all i gives $\mathrm{E}\left[\mathcal{CS}\left(Q, \underline{P}; \mathbf{p}\right)\right] = 0$ since the value $\underline{P}(i)$ of each type i is equal to its expected expenditure. For any $s \in \left(0, \mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right]\right)\right)$, taking $v_i = \lambda \overline{P}(i) + (1-\lambda)\underline{P}(i)$ with $\lambda = s / \mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right)\right]$ yields surplus s. This completes the proof that lying in the given interval is sufficient for s to be consistent with the data.

We now shift our attention to the other direction, namely, that lying in the given interval is a necessary condition for consistency of surplus with the data. We say that a stochastic value $\mathbf{v}$ is *consistent with data* (Q, F) if there exists a model satisfying $\mathbf{v} \overset{d}{=} v_{\mathbf{i}}$ that rationalizes (Q, F). We provide an upper bound on stochastic values consistent with the data with respect to the increasing convex order. Let $\overline{\mathbf{v}} := \overline{P}(\mathbf{i})$ and $\underline{\mathbf{v}} := \underline{P}(\mathbf{i})$ for $\mathbf{i} \sim U[0,1]$. Thus $\overline{\mathbf{v}}$ and $\underline{\mathbf{v}}$ are, respectively, the

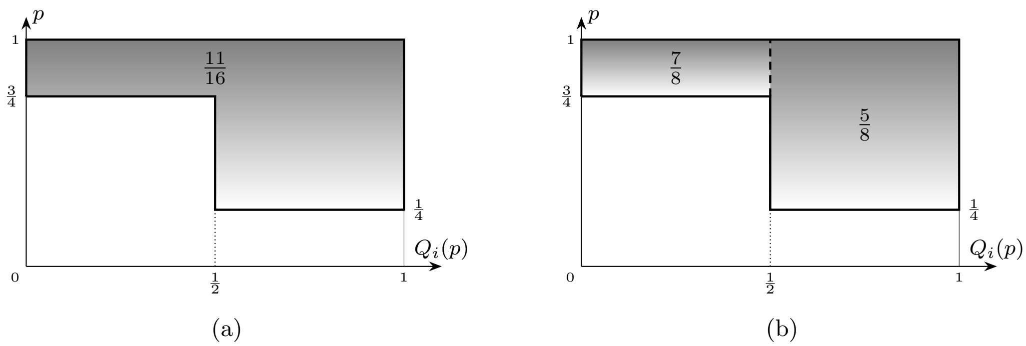


Figure 2: Example illustrating the effect of splitting demand into threshold demands. (a) For $\mathbf{p} \sim U[0,1]$, the non-buying price expectation associated with demand Q_i is the expected price across the shaded region with both coordinates uniformly distributed, which is $\frac{11}{16}$. (b) Splitting type i into two (equally likely) types with threshold demands gives non-buying price expectations of $\frac{7}{8}$ and $\frac{5}{8}$. The average of these two expectations conditional on not buying is $\frac{7}{8} \cdot \frac{1}{4} + \frac{5}{8} \cdot \frac{3}{4} = \frac{11}{16}$, whereas the unconditional average is $\frac{7}{8} \cdot \frac{1}{2} + \frac{5}{8} \cdot \frac{1}{2} = \frac{3}{4} > \frac{11}{16}$.

stochastic values associated with the upper threshold model and the corresponding model for the lower bound constructed above. For the proof of Theorem 1, we make use only of $\overline{\mathbf{v}}$; $\underline{\mathbf{v}}$ is needed in subsections 6 and 7.2.

The following lemma is the core technical insight underlying the necessity part of Theorem 1.

Lemma 4. *If a stochastic value* $\mathbf{v}$ *is consistent with data* (Q, F)*, then* $\overline{\mathbf{v}} \succeq_{\text{icx}} \mathbf{v}$*.*

The proof of this lemma, which is in the appendix, starts by considering an arbitrary model rationalizing the data with values v_i and demands $Q_i(p)$ for each type i. We then amend the model in two steps such that (i) each step leads to an increase in the stochastic value with respect to the increasing convex order and (ii) in combination, the two steps transform the original stochastic value $\mathbf{v} = v_{\mathbf{i}}$ to $\overline{\mathbf{v}}$.

In the first step, we replace the value v_i of each type i with i's non-buying price expectation $\overline{v}_i$ (given the demand Q_i). Since, by Lemma 3, $\overline{v}_i \geq v_i$, this replacement leads to a first-order stochastic dominance increase in the stochastic value and hence also to an increase with respect to the increasing convex order.

In the second step, we decompose the demand of each type into demands of the form $Q_j(p) = \mathbb{1}_{p \leq \rho}$ for some ρ; we refer to such functions as *threshold demands.* More specifically, we replace each type i with a stochastic type $\mathbf{j}$ such that each realization j has a threshold demand and the average demand across $\mathbf{j}$ is Q_i. (If Q_i is itself a threshold demand, then such a decomposition is trivial.) We assign to each j the value $\overline{v}_j$ equal to its non-buying price expectation. Replacing each i with the corresponding $\mathbf{j}$ clearly increases the spread in the values. For this change to be an increase with respect to the increasing convex order, it suffices to show that it also increases the means, i.e., that $\mathrm{E}[\overline{v}_{\mathbf{j}}] \geq \overline{v}_i$ for each i. To see why the last inequality holds, notice that, by the Law of Iterated Expectations, the expected non-buying price expectation *conditional on not buying* is unaffected by the decomposition of the demand $Q_i(p)$. Since higher values of $\overline{v}_j$ are associated with lower probabilities of not buying, when compared to the conditional expectation, the relative weight assigned to higher values $\overline{v}_j$ in the unconditional expectation is larger, as claimed. See Figure 2 for an illustration.

Taken together, the two steps transform the original model into one in which all types have threshold demands and values equal to their non-buying price expectations, and the average demand is Q. The associated stochastic value is $\overline{\mathbf{v}}$, as needed for the proof of Lemma 4.

We now establish an interim upper bound on consumer surplus that holds for each realization p of the random price $\mathbf{p}$. We say that a function $s(p)$ is an *interim consumer surplus consistent with the data* if there exists a model that rationalizes the data for which

$$s(p) = \mathrm{E}\left[\left(v_{\mathbf{i}} - p\right) Q_{\mathbf{i}}(p)\right].$$

Lemma 5. *If $s(p)$ is an interim consumer surplus consistent with the data, then $s(p) \leq \mathcal{CS}\left(Q, \overline{P}; p\right)$.*

Proof. First note that, for a given stochastic value $\mathbf{v}$, the consumer can suffer from two types of losses relative to optimal behavior: (i) the probability of purchase at a given price may not be optimal, i.e., $Q(p)$ may differ from $Q^{\mathrm{s}}(p; \mathbf{v})$, and (ii) the set of types purchasing the good at a given price may not be those with the highest values. Starting from any model that rationalizes the data, reallocating demands across types to eliminate this latter loss (ignoring incentive compatibility) gives an upper bound $\mathcal{CS}\left(Q, \mathbf{v}; p\right)$ on the interim surplus at each p for models with stochastic value $\mathbf{v}$. Therefore, the surplus $s(p)$ generated by any such model satisfies

$$s(p) \leq \mathcal{CS}\left(Q, \mathbf{v}; p\right) \leq \mathcal{CS}\left(Q, \overline{\mathbf{v}}; p\right) = \mathcal{CS}\left(Q, \overline{P}; p\right),$$

where the middle inequality follows from Lemmas 2 and 4. □

Since the interim surplus is bounded from above by $\mathcal{CS}\left(Q, \overline{P}; p\right)$ for each price p, the ex ante surplus is bounded by $\mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right)\right]$, as needed. This concludes the proof of Theorem 1.

6 Related Results

Theorem 1 has connections with several other questions.

First, Lemmas 1 and 4 together have immediate implications for the counterfactual consumer surplus that would arise under optimal choice.

Corollary 1. *Given data (Q, F), the consumer surplus that would arise if the consumer chose optimally is no greater than $CS^{\mathrm{s}}\left(p; \overline{\mathbf{v}}\right)$.*

Second, as in Bayesian persuasion, the upper bound on consumer surplus can be viewed as the value of a concavification problem. Just as Kamenica and Gentzkow (2011) split the prior belief into posterior beliefs under a Bayes-plausibility constraint, we split the aggregate demand $Q(p)$ into individual types' demands $Q_i(p)$ under the constraint $\mathrm{E}\left[Q_{\mathbf{i}}\right] = Q$. The objective in the persuasion problem is to maximize the expected value across posteriors. Likewise, our objective is to maximize the expected surplus $\mathrm{E}\left[s\left(Q_{\mathbf{i}}\right)\right]$, where $s\left(Q_i\right)$ is the highest surplus for type i consistent with demand Q_i. (Due to the high dimensionality of this problem, standard concavification techniques are not sufficient to identify a solution.) Threshold demands can be viewed as analogous to degenerate posteriors insofar as neither can be further split. Since the upper threshold model splits the original demand into threshold demands, it is analogous to full disclosure in the persuasion problem. The optimality of this model is nontrivial since the objective function $s(Q_i)$ is not convex.[15]

Finally, all of the results stated so far have symmetric counterparts regarding a different welfare measure. Consumer surplus s captures the consumer's benefit from freely choosing whether to buy relative to not having the option to buy the good. Define the *complementary consumer surplus*

$$\widehat{s} \coloneqq \mathrm{E}\left[\left(1 - Q_{\mathbf{i}}(\mathbf{p})\right)\left(\mathbf{p} - v_{\mathbf{i}}\right)\right],$$

which captures the consumer's benefit from freely choosing whether to buy relative to being forced to buy the good, i.e., not having the option *not* to buy the good. One can think of the complementary surplus as the gain relative to universal provision of the good financed by a tax equal to the average price.

Whereas consumer surplus is maximized when the consumer's value is high, complementary consumer surplus is maximized when the consumer's value is low. In both cases, however, greater spreads in values are associated with higher (complementary) surplus. Consequently, the relevant ranking of stochastic values for the complementary surplus is (the reverse of) second-order stochastic dominance: a lower bound with respect to $\succeq_{\text{sosd}}$ provides an upper bound on $\widehat{s}$.

Just as $\overline{\mathbf{v}}$ is the highest and the most spread out stochastic value consistent with the data, $\underline{\mathbf{v}}$ is the lowest and the most spread out such stochastic value. More precisely, $\underline{\mathbf{v}}$ is a lower bound with respect to $\succeq_{\text{sosd}}$ on all $\mathbf{v}$ consistent with the data. While the central step of the proof of Lemma 4 was to show that a decomposition into threshold demands induces a mean-*increasing* spread of the *non-buying* price expectations, a symmetric argument implies that the same decomposition induces a mean-*decreasing* spread of the *buying* price expectations.

By analogy to the functional $\mathcal{CS}$, let

$$\widehat{\mathcal{CS}}\left(\tilde{Q}, \hat{P}; p\right) \coloneqq \mathrm{E}\left[\int_{\tilde{Q}(p)}^{1}\left(p - \hat{P}(q)\right) dq\right].$$

Note that, just as $\mathrm{E}\left[\mathcal{CS}\left(Q, \underline{P}; \mathbf{p}\right)\right] = 0$, $\mathrm{E}\left[\widehat{\mathcal{CS}}\left(Q, \overline{P}; \mathbf{p}\right)\right] = 0$. Complementary consumer surplus $\widehat{s}$ is consistent with data (Q, F) if and only if

$$0 = \mathrm{E}\left[\widehat{\mathcal{CS}}\left(Q, \overline{P}; \mathbf{p}\right)\right] \leq \widehat{s} \leq \mathrm{E}\left[\widehat{\mathcal{CS}}\left(Q, \underline{P}; \mathbf{p}\right)\right];$$

this is the mirror image of Theorem 1. Since the proof of this result is analogous to that of

[15] To see this, consider $Q_i(p) = 1$ for $p < \overline{p} - \varepsilon$ and $Q_i(p) = 0$ otherwise. For small ε, $s(Q_i) \approx \overline{p} - \mathrm{E}[p]$ and $s(Q_i/2) \approx 0 < s(Q_i)/2$.

Theorem 1, we omit the details.

Example 3. Consider the same data as in Example 1, namely, $Q(p) = 1 - p$ and $\mathbf{p} \sim U[0,1]$. In this case, the empirical inverse demand is $P(q) = 1 - q$, the lowered inverse demand is $\underline{P} = P(q)/2 = (1-q)/2$, giving an upper bound on complementary consumer surplus of $\mathrm{E}\left[\widehat{\mathcal{CS}}(Q, \underline{P}; \mathbf{p})\right] = 1/4$. △

7 Multiple Datasets

We now extend the model to allow for the possibility that the analyst observes demand in two or more market regimes that may differ in the distribution of prices or in the consumer's behavior (or both), as discussed in subsection 3.2. The analyst observes a *profile* of datasets $\left(Q^k, F^k\right)$, $k = 1, \ldots, K$, where $Q^k(p)$ and $F^k(p)$ are, respectively, the probability that the consumer makes a purchase at each price p and the distribution of prices in regime k and each (Q^k, F^k) satisfies the assumptions on data made in section 2. The consumer has a stochastic type $\mathbf{i}$, with each realization i specifying her value v_i for the good and her (nonincreasing) demand function $Q_i^k(p)$ in each regime. The distribution of types and the value of each type are the same across all regimes. A model for the analyst consists of a distribution of types together with a specification of $\left(v_i, Q_i^1, \ldots, Q_i^K\right)_i$ for each type.

We say that a model *rationalizes the profile of datasets* $\left(Q^k, F^k\right)_k$ if, for each regime k, it rationalizes dataset (Q^k, F^k) when each type i has demand Q_i^k. In particular, within each regime, we impose the basic rationality assumptions described by (1) and (2). A stochastic value $\mathbf{v}$ is *consistent with the profile of datasets* $\left(Q^k, F^k\right)_k$ if there exists a model that rationalizes this profile and satisfies $\mathbf{v} \stackrel{d}{=} v_{\mathbf{i}}$.

Example 4. Consider two regimes. The data in regime 1 consist of the linear demand $Q^1(p) = 1 - p$ and uniform price distribution $\mathbf{p} \sim U[0,1]$, as in Example 1. The data in regime 2 consist of the step-function demand $Q^2(p) = \mathbb{1}_{p \leq 2/3}$ and uniform price distribution $\mathbf{p} \sim U[2/3 - \varepsilon, 2/3 + \varepsilon]$, where $0 < \varepsilon \leq 1/3$. These two regimes are jointly rationalizable, for example by a single type with value $2/3$ and demand $Q^k(p)$ for $k = 1, 2$. △

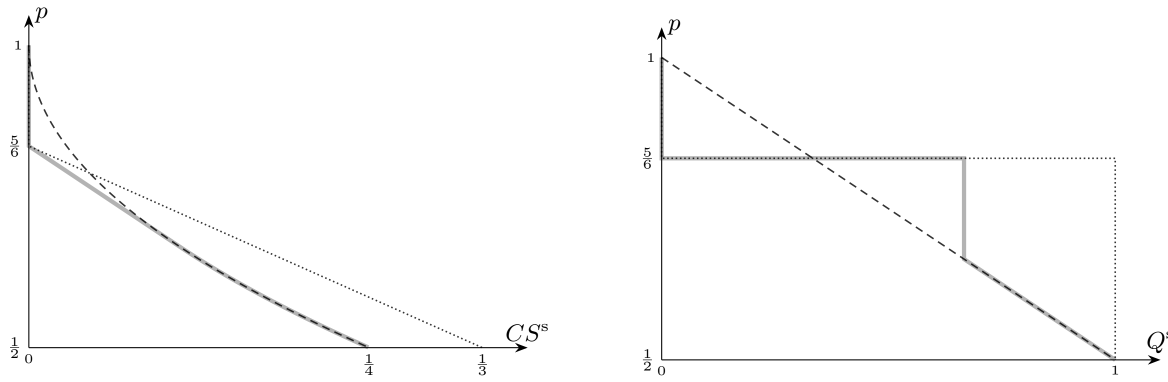
Given a model, the *interim consumer surplus in regime k at price p* is

$$s^k(p) = \mathrm{E}\left[Q_{\mathbf{i}}^k(p)\,(v_{\mathbf{i}} - p)\right].$$

We write $s^k = \mathrm{E}\left[s^k(\mathbf{p})\right]$ for the ex ante surplus in regime k. Consumer surplus $s^k(p)$ in regime k is *consistent with the profile of datasets* $(Q^k, F^k)_k$ if there exists a model that rationalizes this profile and generates surplus $s^k(p)$ in regime k (and analogously for the ex ante surplus).

7.1 Upper bound

The next result provides an upper bound on the surplus within each regime that generally improves upon the bounds that can be obtained for each regime separately. The basic idea is to derive the



(a) (b)

Figure 3: Convexification for the two regimes described in Example 4 with $\varepsilon = 1/3$. Note that the graphs depict only prices $p \in [1/2, 1]$ since the convexification is trivial for $p < 1/2$. (a) Standard consumer-surplus functions: (dashed) $CS^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^1\right)$, (dotted) $CS^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^2\right)$, (thick) the convexification $CS^{\mathrm{s}}_*(p)$. (b) Standard demands associated with stochastic values: (dashed) $Q^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^1\right)$, (dotted) $Q^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^2\right)$, (thick) $Q^{\mathrm{s}}_*(p)$.

upper bounds on the stochastic value with respect to the increasing convex order when considering each regime separately and to combine them in such a way as to generate a tighter bound. The approach therefore requires combining bounds on random variables with respect to this stochastic order. To do so, building on ideas of Gentzkow and Kamenica (2016) and Kolotilin et al. (2017), we exploit the connection described in section 4 between random variables and convex functions—in this case, the stochastic value and the standard consumer surplus. According to Lemma 1, comparisons of stochastic values in the increasing convex order correspond to comparisons of the standard consumer surplus. Using this connection, we find the largest random variable that satisfies the bounds on the stochastic value across all of the regimes by finding the largest convex function lying below the corresponding bounds on the standard consumer surplus.

Let $\overline{\mathbf{v}}^k$ be the upper bound on stochastic values consistent with the data for regime k with respect to the increasing convex order as in Lemma 4.[16] For each k, the bound $\overline{\mathbf{v}}^k$ corresponds to the convex function $CS^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^k\right)$. The upper bound using data across all regimes therefore corresponds to the largest convex function that lies below each $CS^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^k\right)$. Accordingly, let $CS^{\mathrm{s}}_*(p)$ denote the convex closure of the function $\min_k CS^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^k\right)$.[17] By extension of the terminology of Kamenica and Gentzkow (2011), we refer to CS^{s}_* as the *convexification* of $\min_k CS^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^k\right)$.

To map CS^{s}_* back to a stochastic value, recall from section 4 that $CS^{\mathrm{s}}(p; \mathbf{v})$ is the integral of the upper tail of the standard demand $Q^{\mathrm{s}}(p; \mathbf{v})$, which is the complementary distribution function of $\mathbf{v}$. Define the demand function $Q^{\mathrm{s}}_*(p) = -\partial_- CS^{\mathrm{s}}_*(p)$, where ∂_- denotes the left derivative. Note that

[16] That is, $\overline{\mathbf{v}}^k = \overline{P}^k(\mathbf{i})$ with $\mathbf{i} \sim U[0,1]$, where the elevated demand for regime k is $\overline{P}^k(i) = \mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \geq P^k(i)\right]$ with $\mathbf{p} \sim F^k$ and P^k is the inverse demand to Q^k.

[17] Recall that the convex closure of a function $g(p)$ is the function that maps each p to $\inf\{s : (p,s) \in \mathrm{co}(g)\}$, where $\mathrm{co}(g)$ denotes the convex hull of the graph of the function g. In the terminology of convex analysis, CS^{s}_* is the biconjugate function to $\min_k CS^{\mathrm{s}}\left(p; \overline{\mathbf{v}}^k\right)$.

	Bound from Theorem 1	Bound from Theorem 2
Regime 1	$\mathrm{E}_1\left[\mathcal{CS}\left(Q^1,\overline{\mathbf{v}}^1;\mathbf{p}\right)\right]=0.25$	$\mathrm{E}_1\left[\mathcal{CS}\left(Q^1,\overline{\mathbf{v}}_*;\mathbf{p}\right)\right]\approx 0.238$
Regime 2	$\mathrm{E}_2\left[\mathcal{CS}\left(Q^2,\overline{\mathbf{v}}^2;\mathbf{p}\right)\right]\approx 0.167$	$\mathrm{E}_2\left[\mathcal{CS}\left(Q^2,\overline{\mathbf{v}}_*;\mathbf{p}\right)\right]\approx 0.125$

Table 1: Upper bounds on consumer surplus for the two regimes from Example 4 with the parameter $\varepsilon = 1/3$.

$1-Q^{\mathrm{s}}_*$ is a distribution function and let $\overline{\mathbf{v}}_*$ be a stochastic value associated with this distribution.[18] See Figure 3 for an illustration.

The following result is the main step underlying the upper bound for multiple regimes.

Lemma 6. *If a stochastic value* $\mathbf{v}$ *is consistent with the profile of datasets, then* $\overline{\mathbf{v}}_* \succeq_{\mathrm{icx}} \mathbf{v}$.

Proof. Follows from Theorem 3.2 of Müller and Scarsini (2006). □

A direct argument is as follows. If $\mathbf{v}$ is consistent with the profile of datasets, then it is consistent with each dataset separately; thus $\overline{\mathbf{v}}^k \succeq_{\mathrm{icx}} \mathbf{v}$ for each regime k. By Lemma 1, $\min_k CS^{\mathrm{s}}\left(p;\overline{\mathbf{v}}^k\right) \geq CS^{\mathrm{s}}(p;\mathbf{v})$. Since $CS^{\mathrm{s}}(p;\mathbf{v})$ is convex in p, $CS^{\mathrm{s}}(p;\mathbf{v})$ is no greater than the convexification of $\min_k CS^{\mathrm{s}}\left(p;\overline{\mathbf{v}}^k\right)$. Finally, again by Lemma 1, $\overline{\mathbf{v}}_* \succeq_{\mathrm{icx}} \mathbf{v}$.

Combining Lemmas 2 and 6 leads to the following upper bound on the consumer surplus within each regime.

Theorem 2. *If the interim consumer surplus* $s^k(p)$ *in regime* k *is consistent with the profile of datasets, then* $s^k(p) \leq \mathcal{CS}\left(Q^k,\overline{\mathbf{v}}_*;p\right)$.

As an immediate consequence, the ex ante consumer surplus in regime k consistent with the profile of datasets is bounded from above by $\mathrm{E}_k\left[\mathcal{CS}\left(Q^k,\overline{\mathbf{v}}_*;\mathbf{p}\right)\right]$ where the expectation is with respect to the distribution of prices in regime k. See Table 1 for an illustration.

Proof of Theorem 2. Given a stochastic value $\mathbf{v}$, the interim consumer surplus $s^k(p)$ in regime k is at most $\mathcal{CS}\left(Q^k,\mathbf{v};p\right)$ because this is the surplus associated with having the measure $Q^k(p)$ of types with the highest values buy at p. By Lemma 2, $\mathcal{CS}\left(Q^k,\mathbf{v};p\right)$ is nondecreasing in $\mathbf{v}$ with respect to the increasing convex order. Finally, by Lemma 6, any stochastic value $\mathbf{v}$ consistent with the profile of datasets is bounded by $\overline{\mathbf{v}}_*$ in the increasing convex order. □

Lemma 6, when combined with Lemma 1, also provides an upper bound on the counterfactual consumer surplus that would arise if the consumer perfectly optimized.

Corollary 2. *Given a profile of datasets* $\left(Q^k,F^k\right)_k$, *the consumer surplus that would arise if the consumer chose optimally is no greater than* $CS^{\mathrm{s}}_*(p)$.

[18] Since CS^{s}_* is convex, its left derivative exists, and Q^{s}_* is nonincreasing and left-continuous. Additionally, $CS^{\mathrm{s}}_*(p) = 0$ for $p > \overline{p}$ and $CS^{\mathrm{s}}_*(p)$ has slope -1 for $p < \underline{p}$; hence, $\lim_{p\to-\infty} Q^{\mathrm{s}}_*(p) = 1$ and $\lim_{p\to+\infty} Q^{\mathrm{s}}_*(p) = 0$. Thus, $1-Q^{\mathrm{s}}_*$ is a distribution function.

7.2 Lower bound

An analogous construction to that for the upper bound can be used to obtain a nontrivial lower bound on surplus using data from multiple regimes. Given a stochastic value $\mathbf{v}$ and demand $Q(p)$, we can compute a lower bound on surplus by supposing that the measure $Q(p)$ of the *lowest* types purchase the good at each p (as opposed to the highest types we used for the upper bound). Under this assignment, roughly speaking, lower means and greater spreads of the stochastic value both reduce the lower bound on surplus. Consequently, the relevant ordering of stochastic values is $\succeq_{\text{sosd}}$ (as opposed to $\succeq_{\text{icx}}$ for the upper bound).

To represent the second-order stochastic dominance order, we define the *complementary standard consumer surplus*

$$\widehat{CS}^{\text{s}}(p;\mathbf{v}) := \int_{-\infty}^{p} \left(1 - Q^{\text{s}}\left(p';\mathbf{v}\right)\right) dp'$$

and note that it is nondecreasing and convex in p. By the well known characterization of Hadar and Russell (1969) and Rothschild and Stiglitz (1970), the ranking of stochastic values $\mathbf{v}$ with respect to $\succeq_{\text{sosd}}$ implies the opposite ranking of $\widehat{CS}^{\text{s}}(p;\mathbf{v})$, and the converse also holds provided the latter ranking is consistent across all p. Following the analogous construction to that for the upper bound, let $\widehat{CS}^{\text{s}}_{*}(p)$ be the convexification of $\min_k \widehat{CS}^{\text{s}}\left(p;\underline{\mathbf{v}}^k\right)$, where, for each k, $\underline{\mathbf{v}}^k$ is the lower bound on stochastic values with respect to $\succeq_{\text{sosd}}$ consistent with the dataset (Q^k, F^k) (see section 6 for the definition of $\underline{\mathbf{v}}^k$). Let $\underline{\mathbf{v}}_*$ be the stochastic value associated with $\widehat{CS}^{\text{s}}_{*}$.[19] Along the same lines as in Lemma 6, $\underline{\mathbf{v}}_*$ is a lower bound with respect to $\succeq_{\text{sosd}}$ on stochastic values $\mathbf{v}$ consistent with the profile of datasets.

Let $\widehat{P}(q;\mathbf{v}) := P^{\text{s}}(1-q;\mathbf{v})$ denote the qth lowest quantile of $\mathbf{v}$.

Theorem 3. *If the interim consumer surplus $s^k(p)$ in regime k is consistent with the profile of datasets, then $s^k(p) \geq \mathcal{CS}\left(Q^k, \widehat{P}\left(\cdot, \underline{\mathbf{v}}_*\right); p\right)$.*

Once again, taking expectations with respect to the price in each regime gives a lower bound on the ex ante surplus in that regime.

To understand this result, consider a consumer with stochastic value $\mathbf{v}$. According to the data for regime k, a measure $Q^k(p)$ of types buy at each price p. Selecting the types with the lowest values generates surplus $\mathcal{CS}\left(Q^k, \widehat{P}(;\mathbf{v}); p\right)$ in regime k; this lower bound is nondecreasing in $\mathbf{v}$ with respect to second-order stochastic dominance. Finally, because stochastic values consistent with the profile of datasets are bounded from below with respect to $\succeq_{\text{sosd}}$ by $\underline{\mathbf{v}}_*$, the bound on $s^k(p)$ from the theorem applies.

Example 5. To illustrate the lower bound, consider the regimes from Example 4 with $\varepsilon = 1/3$. In this case, $\underline{\mathbf{v}}_1$ is uniformly distributed on $[0, 1/2]$ and $\underline{\mathbf{v}}_2$ is almost surely equal to $1/2$. Thus $\underline{\mathbf{v}}_2$ second-order stochastically dominates $\underline{\mathbf{v}}_1$, making the convexification trivial with $\underline{\mathbf{v}}_* = \underline{\mathbf{v}}_2$. The lower bound from Theorem 3 on the ex ante consumer surplus in regime 1 is therefore $1/2 \cdot 1/2 - 1/6 = 1/12$ and the lower bound in regime 2 is 0. △

[19] That is, let $1 - Q^{\text{s}}\left(\cdot;\underline{\mathbf{v}}_*\right)$ be the right derivative of $\widehat{CS}^{\text{s}}$, observe that it is a distribution function, and let $\underline{\mathbf{v}}_*$ be a random variable with this distribution.

7.3 Tightness of the bounds

Theorem 1 provides tight bounds on consumer surplus for data from a single market regime; for each value within the bounds, we have constructed a model for which the surplus is equal to that value. While the bounds on surplus in Theorems 2 and 3 are generally tighter within each regime than the bounds obtained from the data in that regime alone, they are not themselves tight bounds.

Example 6. To illustrate, consider the upper bound for the two regimes from Example 4 with $\varepsilon \leq 1/6$. In this case, $CS^{\mathrm{s}}(p; \overline{\mathbf{v}}_2) \leq CS^{\mathrm{s}}(p; \overline{\mathbf{v}}_1)$ for all p, making the convexification trivial: $CS^{\mathrm{s}}_*(p) \equiv CS^{\mathrm{s}}(p; \overline{\mathbf{v}}_2)$. Therefore, $\overline{\mathbf{v}}_* \overset{d}{=} \overline{\mathbf{v}}_2$ almost surely takes on the value $2/3 + \varepsilon/2$, which is the non-buying price expectation for regime 2. However, this value is not consistent with the data for regime 1 since the non-buying price expectation of at least some types must be no more than $2/3$ (the non-buying price expectation for demand $Q^1(p)$). Thus, the upper bound on consumer surplus in regime 1 constructed in Theorem 2 is not attainable in this case. △

If the analyst observes only one market regime, then, to determine the range of values of consumer surplus, it suffices to consider types with simple threshold demands. With multiple regimes, decompositions into threshold demands are not generally sufficient; it can happen that the regimes are not jointly rationalizable by any model with threshold demands (but can be rationalized by other models).[20] The constructions in Theorems 2 and 3 circumvent this complication by using a bound on surplus in each regime based on threshold demands. The upside of this approach is that the combined bound is simple. The downside is that the combined bound need not correspond to a model that rationalizes the profile of datasets, and hence the bound is not generally tight.

8 Bounds on Counterfactual Demand

Returning to the original model in which the analyst observes a single dataset (Q, F) that may result from imperfect optimization, we now consider the counterfactual demand that would arise if instead the consumer were to perfectly optimize and purchase precisely when her value v_i exceeds the price p. These bounds apply equally to a counterfactual market with a fixed, deterministic price where our minimal rationality conditions imply that the consumer would choose optimally.

As for consumer surplus, bounds on counterfactual demand correspond to bounds on the consumer's stochastic value, albeit with respect to a different stochastic order: while the increasing convex order and second-order stochastic dominance provide the relevant bounds for consumer surplus, the bounds for counterfactual demand correspond to first-order stochastic dominance.

To state these bounds, define the *doubly elevated* and *doubly lowered* inverse demands, respec-

[20] Example 4 provides one such example when $\varepsilon < 1/6$. If the demand from regime 1 is decomposed into threshold demands, then a nonzero mass of types must have thresholds below $1/3 - 2\varepsilon$. The non-buying price expectation of such types is less than $2/3 - \varepsilon$. Therefore, these types would not buy at any price that occurs in regime 2, contradicting that $Q^2(p) = 1$ for $p \leq 2/3$.

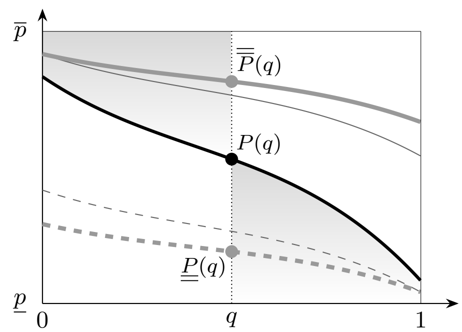


Figure 4: Doubly elevated and doubly lowered inverse demands, $\overline{\overline{P}}$ (thick grey) and $\underline{\underline{P}}$ (thick dashed), for given inverse demand P (black). For each q, $\overline{\overline{P}}(q)$ is the expected price conditional on $\mathbf{p}$ and $\mathbf{q}$ lying in the upper-left grey area. Similarly, $\underline{\underline{P}}(q)$ is the expected price conditional on the lower-right grey area. For comparison, the thin grey and thin dashed curves depict $\overline{P}$ and $\underline{P}$, respectively.

tively, by

$$\overline{\overline{P}}(q) := \mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \geq P(\mathbf{q}), \mathbf{q} \leq q\right]$$
$$\text{and} \qquad \underline{\underline{P}}(q) := \mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \leq P(\mathbf{q}), \mathbf{q} \geq q\right],$$

where $\mathbf{q} \sim U[0,1]$ and $\mathbf{p} \sim F$. Both functions are non-increasing. Relative to the elevated and lowered inverse demands $\overline{P}$ and $\underline{P}$, these inverse demands are further elevated and lowered, i.e., $\overline{\overline{P}}(q) \geq \overline{P}(q)$ and $\underline{\underline{P}}(q) \leq \underline{P}(q)$ for all q. To see this, observe that $\overline{\overline{P}}(q)$ is a convex combination of $\overline{P}(q')$ across $q' \in [0,q]$ and $\overline{P}$ is non-increasing; a symmetric argument shows that $\underline{\underline{P}}(q) \leq \underline{P}(q)$. See Figure 4 for an illustration.

Theorem 4. *For every stochastic value* $\mathbf{v}$ *consistent with data* (Q,F)*, the standard inverse demand function satisfies*

$$\underline{\underline{P}}(q) \leq P^{\mathrm{s}}(q;\mathbf{v}) \leq \overline{\overline{P}}(q)$$

for all q*. These bounds are tight in the sense that for each* q*, there exists a stochastic value* $\mathbf{v}$ *consistent with the data such that* $P^{\mathrm{s}}(q;\mathbf{v}) = \overline{\overline{P}}(q)$*, and similarly for* $\underline{\underline{P}}(q)$*.*

We sketch the argument for the upper bound; the argument for the lower bound is analogous. For each q, given any stochastic value $\mathbf{v}$, the standard inverse demand $P^{\mathrm{s}}(q;\mathbf{v})$ is a particular quantile of $\mathbf{v}$ (namely, the $(1-q)$th quantile). The model that maximizes the counterfactual inverse demand at q among those rationalizing the data is therefore the one that maximizes this quantile. Accordingly, bounds on counterfactual demand correspond to bounds on stochastic values with respect to first-order stochastic dominance.

How can we maximize a given quantile of $\mathbf{v}$ (among stochastic values consistent with the data)? Recall that the highest value compatible with a type's demand is its non-buying price expectation. It turns out that this non-buying price expectation is maximized when no other type has a higher value and the demand of this type is as large as possible. Accordingly, to maximize the value at the $(1-q)$th quantile, we use a model in which the type with the highest value has measure q and demand $\min\{Q(p)/q, 1\}$. By construction, the non-buying price expectation of this type is exactly $\overline{\overline{P}}(q)$. To see that the bound is tight, note that such a type can be part of a model that rationalizes the data (in which the remaining measure $1-q$ of types generate the residual demand).

As with consumer surplus, data from multiple market regimes can be used to tighten the bounds on counterfactual demand. Assuming, as in section 7, that preferences are stable across regimes, a tighter bound can be obtained by simply taking the minimum and maximum, respectively, of the upper and the lower bounds from Theorem 4 across all of the regimes.

9 Decision-Making Costs

By defining the surplus $s_i = \mathrm{E}\left[(v_i - \mathbf{p})Q_i(\mathbf{p})\right]$ to be the expected value less the price, we have implicitly assumed that there are no other costs associated with decision-making. We now consider bounds on consumer welfare in the presence of such costs, which we assume are unobservable to the analyst. This more general approach encompasses models such as those in rational inattention where the consumer obtains information at a cost before making a purchasing decision.

To this end, we distinguish between the surplus s_i that we have considered until this point and the *net expected utility* $\tilde{s}_i$ that accounts for any decision costs; thus $\tilde{s}_i \leq s_i$. In this framework, our basic rationality assumption is that no type can improve its net expected utility by switching to either always or never buying. In addition, we assume that either always or never buying the good minimizes decision costs. Normalizing this minimal decision cost to 0, these assumptions are described by the inequalities

$$\tilde{s}_i \geq 0 \tag{5}$$

$$\text{and} \qquad \tilde{s}_i \geq v_i - \mathrm{E}[\mathbf{p}]. \tag{6}$$

Since $s_i \geq \tilde{s}_i$, inequalities (1) and (2) follow from (5) and (6). In particular, Lemma 1 applies, constraining each type's value to lie between its buying and non-buying price expectations. Consequently, all of the bounds we have identified on the stochastic value $\mathbf{v}$—either for a single dataset (Lemma 4) or for multiple datasets (Lemma 6 and the analogous lower bound from subsection 7.2)—still apply. Moreover, all of our bounds on surplus or counterfactual demand also still apply.

What about bounds on the net expected utility? Let $\tilde{s} = \mathrm{E}[\tilde{s}_\mathbf{i}]$ denote the ex ante net expected utility. The main result from Theorem 1 applies to this case:

$$0 \leq \tilde{s} \leq \mathrm{E}\left[\mathcal{CS}\left(Q, \overline{P}; \mathbf{p}\right)\right].$$

The lower bound follows immediately from (5). The upper bound holds because $\tilde{s} \leq s$ and the upper bound holds for s. Similarly, the tighter upper bound in Theorem 2 using multiple datasets applies. However, the tighter lower bound in Theorem 3 using multiple datasets does *not* apply to the net expected utility; decision costs could lower the expected utility below these bounds.

10 Discussion

If the analyst does not know whether the consumer engages in optimal choice behavior, the consumer surplus cannot be point identified from price and demand data. Nonetheless, weak rationality assumptions impose significant restrictions on the levels of surplus consistent with the data. Identification of the consumer surplus can be further sharpened by combining data from market regimes with varying priors or consumer demands.

Two relevant questions related to this project remain open. First, the bounds we provide under multiple regimes are not tight; our bounds rely on separate rationalizations for each regime, whereas, in principle, identification of the surplus can be tightened by simultaneously rationalizing the profile of datasets. Second, in the interest of generality, we have imposed minimal structure on the relationship between the consumer's value and her demand. Depending on the context, there may be additional structure that could be used to narrow the bounds on surplus or counterfactual demand.

A Proofs

Proof of Lemma 2. Note that

$$\mathcal{CS}\left(\tilde{Q}, \mathbf{v}; p\right) = \mathcal{CS}\left(\tilde{Q}, P^{\mathrm{s}}(\cdot; \mathbf{v}); p\right) = \int_0^{\tilde{Q}(p)} P^{\mathrm{s}}(q; \mathbf{v}) dq - \tilde{Q}(p)p.$$

Consider any $\mathbf{v}$ and $\mathbf{v}'$ such that $\mathbf{v}' \succeq_{\text{icx}} \mathbf{v}$. Since the expenditure $\tilde{Q}(p)p$ does not depend on the stochastic value, it suffices to prove that $\int_0^{q^*} P^{\mathrm{s}}\left(q; \mathbf{v}'\right) dq \geq \int_0^{q^*} P^{\mathrm{s}}(q; \mathbf{v}) dq$ for each $q^* \in [0,1]$. Fix q^*. For $p = P^{\mathrm{s}}\left(q^*; \mathbf{v}'\right)$,

$$\begin{aligned}
\int_0^{q^*} P^{\mathrm{s}}\left(q; \mathbf{v}'\right) dq &= CS^{\mathrm{s}}\left(p; \mathbf{v}'\right) + pq^* \\
&\geq CS^{\mathrm{s}}\left(p; \mathbf{v}\right) + pq^* \\
&\geq \int_0^{q^*} (P^{\mathrm{s}}(q; \mathbf{v}) - p) dq + pq^* \\
&= \int_0^{q^*} P^{\mathrm{s}}(q; \mathbf{v}) dq;
\end{aligned}$$

the first inequality follows from Lemma 1 while the second follows from the observation that $CS^{\mathrm{s}}\left(p; \mathbf{v}\right) = \max_{q'} \int_0^{q'} (P^{\mathrm{s}}(q; \mathbf{v}) - p) dq$. □

Proof of Lemma 4. Step 1: Consider a model such that each type i has value v_i and demand

$Q_i(p)$. Let $\mathbf{v} = v_{\mathbf{i}}$ be the associated stochastic value. Let $\mathbf{v}' = \overline{v}_{\mathbf{i}}$, where $\overline{v}_i = \mathrm{E}\left[\mathbf{p} \mid \mathbf{q} \geq Q_i(\mathbf{p})\right]$ for $\mathbf{q} \sim U[0,1]$ denotes the non-buying price expectation associated with demand Q_i. By Lemma 3, $\overline{v}_i \geq v_i$ for each i. Thus, $\mathbf{v}' \succeq_{\text{icx}} \mathbf{v}$ (because $\mathbf{v}'$ first-order stochastically dominates $\mathbf{v}$).

Step 2: For each type i, define a random variable $\overline{\mathbf{v}}_i$ as follows. Let $P_i(q)$ be the inverse demand to demand Q_i, let $\overline{P}_i(q) = \mathrm{E}[\mathbf{p} \mid \mathbf{p} \geq P_i(q)]$ be the elevated demand of type i, and define the stochastic value $\overline{\mathbf{v}}_i = \overline{P}_i(\mathbf{q})$ for $\mathbf{q} \sim U[0,1]$. Finally, let $\mathbf{v}'' = \overline{\mathbf{v}}_{\mathbf{i}}$; thus $\mathbf{v}''$ is a spread of $\mathbf{v}'$ that replaces $v'_i = \overline{v}_i$ with $\overline{\mathbf{v}}_i$ for each i.

We will show that $\mathbf{v}'' \succeq_{\text{icx}} \mathbf{v}'$ (and hence $\mathbf{v}'' \succeq_{\text{icx}} \mathbf{v}$). It suffices to show that $\overline{v}_i \leq \mathrm{E}\left[\overline{\mathbf{v}}_i\right]$ for each i. Indeed, for $\mathbf{q} \sim U[0,1]$, by the Law of Iterated Expectations,

$$\begin{aligned}
\overline{v}_i &= \mathrm{E}\left[\mathbf{p} \mid \mathbf{q} \geq Q_i(\mathbf{p})\right] \\
&= \mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \geq P_i\left(\mathbf{q}\right)\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \geq P_i\left(\mathbf{q}\right), \mathbf{q}\right] \mid \mathbf{p} \geq P_i\left(\mathbf{q}\right)\right] \\
&= \mathrm{E}\left[\overline{P}_i\left(\mathbf{q}\right) \mid \mathbf{p} \geq P_i\left(\mathbf{q}\right)\right] \\
&= \mathrm{E}\left[\overline{P}_i\left(\mathbf{q}\right) \mid \mathbf{q} \geq Q_i(\mathbf{p})\right].
\end{aligned}$$

Since $\mathbf{q}$ conditional on $\mathbf{q} \geq Q_i(\mathbf{p})$ first-order stochastically dominates $\mathbf{q}$ itself and $\overline{P}_i\left(q\right)$ is nonincreasing, it follows that

$$\overline{v}_i \leq \mathrm{E}\left[\overline{P}_i\left(\mathbf{q}\right)\right] = \mathrm{E}\left[\overline{\mathbf{v}}_i\right],$$

as needed.

Step 3: We conclude by proving that $\mathbf{v}'' \overset{d}{=} \overline{\mathbf{v}}$. Consider any p at which Q is continuous and let $\tilde{v} = \mathrm{E}[\mathbf{p} \mid \mathbf{p} \geq p]$. For any $j \in [0,1]$,

$$j = \Pr\left(\overline{\mathbf{v}} \geq \tilde{v}\right) \quad \Longrightarrow \quad \overline{P}(j) = \tilde{v} \quad \Longrightarrow \quad P(j) = p \quad \Longrightarrow \quad j = Q(p).$$

Hence $\Pr(\overline{\mathbf{v}} \geq \tilde{v}) = Q(p)$. Likewise, $\Pr\left(\overline{\mathbf{v}}_i \geq \tilde{v}\right) = Q_i(p)$ for almost all i (i.e., for all i except those for which Q_i is discontinuous at p), and thus

$$\Pr\left(\mathbf{v}'' \geq \tilde{v}\right) = \Pr\left(\overline{\mathbf{v}}_{\mathbf{i}} \geq \tilde{v}\right) = \mathrm{E}\left[Q_{\mathbf{i}}(p)\right] = Q(p) = \Pr\left(\overline{\mathbf{v}} \geq \tilde{v}\right)$$

for all $\tilde{v}$ from a dense subset of the support of $\overline{\mathbf{v}}$ and $\mathbf{v}''$, as needed. □

Proof of Theorem 3. Consider a model consistent with the profile of datasets and let $\mathbf{v}$ be its associated stochastic value. Recall that $\widehat{P}(q;\mathbf{v}) = P^{\mathrm{s}}\left(1-q;\mathbf{v}\right)$ is the qth lowest quantile of $\mathbf{v}$. Note that

$$s^k(p) \geq \mathcal{CS}\left(Q^k, \widehat{P}(\cdot;\mathbf{v}); p\right)$$

for each k since the right-hand side is the expected consumer surplus if the measure $Q^k(p)$ of types with the lowest values buy at price p.

For any price p and any two stochastic values $\mathbf{v}$ and $\mathbf{v}'$ such that $\mathbf{v} \succeq_{\text{sosd}} \mathbf{v}'$ and any demand function $\tilde{Q}$, we claim that

$$\mathcal{CS}\left(\tilde{Q}, \widehat{P}(\cdot;\mathbf{v});p\right) \geq \mathcal{CS}\left(\tilde{Q}, \widehat{P}(\cdot;\mathbf{v}');p\right).$$

The proof of this claim is analogous to that of Lemma 2. In particular, we may disregard expenditures since they depend only on the first and the last arguments of $\mathcal{CS}$. It suffices to prove that if $\mathbf{v} \succeq_{\text{sosd}} \mathbf{v}'$, then $\int_0^{q^*} \widehat{P}(q;\mathbf{v})\,dq \geq \int_0^{q^*} \widehat{P}(q;\mathbf{v}')\,dq$ for every q^*. Fixing q^* and letting $p = \widehat{P}(q^*;\mathbf{v}')$, we have

$$\begin{aligned}\int_0^{q^*} \widehat{P}(q;\mathbf{v}')\,dq &= q^*p - \widehat{CS}^{\text{s}}(p;\mathbf{v}') \\ &\leq q^*p - \widehat{CS}^{\text{s}}(p;\mathbf{v}) \\ &\leq q^*p - \int_0^{q^*}\left(p - \widehat{P}(q;\mathbf{v})\right)dq \\ &= \int_0^{q^*} \widehat{P}(q;\mathbf{v})\,dq;\end{aligned}$$

the first inequality follows from the integral condition for $\mathbf{v} \succeq_{\text{sosd}} \mathbf{v}'$ and the second from the fact that $\widehat{CS}^{\text{s}}(p;\mathbf{v}) = \max_{\hat{q}} \int_0^{\hat{q}}\left(p - \widehat{P}(q;\mathbf{v})\right)dq$.

Therefore, if a stochastic value $\mathbf{v}$ is consistent with the profile of datasets, then

$$s^k(p) \geq \mathcal{CS}\left(Q^k, \widehat{P}(\cdot;\mathbf{v});p\right) \geq \mathcal{CS}\left(Q^k, \widehat{P}(\cdot;\underline{\mathbf{v}}_*);p\right)$$

since $\mathbf{v} \succeq_{\text{sosd}} \underline{\mathbf{v}}_*$. □

Proof of Theorem 4. We prove only the upper bound; the argument for the lower bound is analogous.

For any $q \in (0,1]$ consider a demand function $\tilde{Q}$ that attains values in $[0,q]$, i.e., a nonincreasing function from $[\underline{p},\overline{p}]$ onto $[0,q]$. Let

$$\begin{aligned}\tilde{v}\left(\tilde{Q};q\right) &:= \mathrm{E}\left[\mathbf{p} \mid \mathbf{q} \geq \tilde{Q}(\mathbf{p})\right] \\ \text{and} \qquad w\left(\tilde{Q};q\right) &:= q\Pr\left(\mathbf{q} \geq \tilde{Q}(\mathbf{p})\right)\end{aligned}$$

for $\mathbf{q} \sim U[0,q]$ and $\mathbf{p} \sim F$. To interpret these two functions, consider a model with type distribution M and a subset $I' \subseteq I$ of types such that $\Pr(\mathbf{i} \in I') = q$ and $\tilde{Q}(p) = \int_{i \in I'} Q_i(p)dM(i)$. Then $\tilde{v}\left(\tilde{Q};q\right)$ is the expected price conditional on a type randomly drawn from I' not making a purchase and $w\left(\tilde{Q};q\right)$ is the probability that a randomly drawn type lies in I' and does not buy.

Note the following recursion. For any $q_a, q_b \in (0,q]$ such that $q_a + q_b = q$ and any two demands

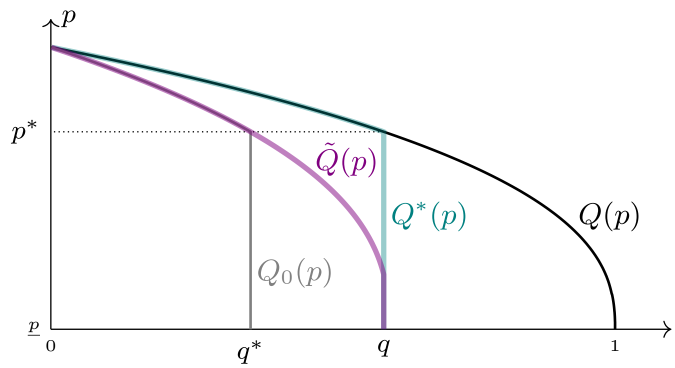


Figure 5: Illustration of the definitions of Q^*, $\tilde{Q}$, and Q_0.

Q_a and Q_b that attain values in $[0, q_a]$ and $[0, q_b]$, respectively, such that $Q_a + Q_b = \tilde{Q}$,

$$\tilde{v}\left(\tilde{Q}; q\right) = \frac{w\left(Q_a; q_a\right)\tilde{v}\left(Q_a; q_a\right) + w\left(Q_b; q_b\right)\tilde{v}\left(Q_b; q_b\right)}{w\left(Q_a; q_a\right) + w\left(Q_b; q_b\right)}. \tag{7}$$

Given any model and a subset I' of types such that $\Pr\left(\mathbf{i} \in I'\right) = q$, let $v_* := \inf_{i \in I'} v_i$. To establish the upper bound, it suffices to show for each q that the supremum of v_* across all models that rationalize the data and subsets I' such that $\Pr\left(\mathbf{i} \in I'\right) = q$ is at most $\overline{\overline{P}}(q)$.

Fix a model with type distribution M on I and types (v_i, Q_i) that rationalizes the data. Fix a set I' of types such that $\Pr\left(\mathbf{i} \in I'\right) = q$. Let $\tilde{Q}(p) = \int_{i \in I'} Q_i(p) dM(i)$ be the demand generated by the types in I'. Note that

$$\inf_{i \in I'} v_i \leq \tilde{v}\left(\tilde{Q}; q\right)$$

since, by Lemma 3, $v_i \leq \overline{v}_i$ for each type i, where $\overline{v}_i$ is the non-buying price expectation associated with the demand Q_i of type i and $\tilde{v}\left(\tilde{Q}; q\right)$ is a convex combination of $\overline{v}_i$ across $i \in I'$.

Let $Q^*(p) := \min\{Q(p), q\}$ and observe that $\overline{\overline{P}}(q) = \tilde{v}\left(Q^*; q\right)$. It suffices to show that

$$\tilde{v}\left(\tilde{Q}; q\right) \leq \tilde{v}\left(Q^*; q\right) \tag{8}$$

for all q and all demands $\tilde{Q}$ that can be generated by a subset I' of types from a model that rationalizes the data and satisfies $\Pr\left(\mathbf{i} \in I'\right) = q$. For all such demands $\tilde{Q}$, both $\tilde{Q}$ and $Q(p) - \tilde{Q}(p)$ are nonnegative and nonincreasing because they are the demands induced by types in I' and $I \setminus I'$, respectively.

Let $\tilde{Q}(p)$ be any demand function attaining values in $[0, q]$ such that $Q(p) - \tilde{Q}(p)$ is nonnegative and nonincreasing. Let $p^* := P\left(q\right)$ and $q^* := \tilde{Q}\left(p^*\right)$. Since $\tilde{Q}(p) \leq Q^*(p) \leq q$ for all p, we have that $q^* \leq q$. Define the demand function $Q_0(p) := \min\left\{\tilde{Q}(p), q^*\right\}$ that attains values in $[0, q^*]$ and let $Q_1(p) := Q^*(p) - Q_0(p)$ and $Q_2(p) := \tilde{Q}(p) - Q_0(p)$. See Figure 5 for an illustration.

Note that $Q_1(p)$ is nonincreasing: it is equal to $q - q^*$ for $p \leq p^*$ and to $Q(p) - \tilde{Q}(p)$ for $p \geq p^*$. The function $Q_2(p)$ is also nonincreasing since it is equal to $\tilde{Q}(p) - q^* \geq 0$ for $p \leq p^*$ and to 0 for $p > p^*$. Let P_0, P_1, and P_2 be the inverse demand functions associated with Q_0, Q_1, and Q_2, respectively. Note that P_0 and P_1 lie above P_2 in the strong sense that both P_0 and P_1 only attain values above p^* while P_2 only attains values below p^*.

Recall that $\tilde{v}(\tilde{Q}; q)$ can be written as $\mathrm{E}\left[\mathbf{p} \mid \mathbf{p} \geq \tilde{P}(\mathbf{q})\right]$ for $\mathbf{q} \sim U[0, q]$, where $\tilde{P}$ is the inverse demand to $\tilde{Q}$; similarly, $w(\tilde{Q}; q)$ can be written as $q \Pr\left(\mathbf{p} \geq \tilde{P}(\mathbf{q})\right)$. It follows that $\tilde{v}(Q_1; q - q^*) \geq \tilde{v}(Q_2; q - q^*)$, $\tilde{v}(Q_0; q^*) \geq \tilde{v}(Q_2; q - q^*)$, and $w(Q_2; q - q^*) \geq w(Q_1; q - q^*)$. Finally, since $Q^* = Q_0 + Q_1$ and $\tilde{Q} = Q_0 + Q_2$, we have from (7) that

$$\tilde{v}(Q^*; q) = \frac{w(Q_0; q^*)\,\tilde{v}(Q_0; q^*) + w(Q_1; q - q^*)\,\tilde{v}(Q_1; q - q^*)}{w(Q_0; q^*) + w(Q_1; q - q^*)}$$
$$\text{and} \qquad \tilde{v}\left(\tilde{Q}; q\right) = \frac{w(Q_0; q^*)\,\tilde{v}(Q_0; q^*) + w(Q_2; q - q^*)\,\tilde{v}(Q_2; q - q^*)}{w(Q_0; q^*) + w(Q_2; q - q^*)}.$$

Therefore,

$$\begin{aligned}
\tilde{v}(Q^*; q) &= \tilde{v}(Q_0; q^*) + \frac{w(Q_1; q - q^*)}{w(Q_0; q^*) + w(Q_1; q - q^*)}\left(\tilde{v}(Q_1; q - q^*) - \tilde{v}(Q_0; q^*)\right) \\
&\geq \tilde{v}(Q_0; q^*) + \frac{w(Q_1; q - q^*)}{w(Q_0; q^*) + w(Q_1; q - q^*)}\left(\tilde{v}(Q_2; q - q^*) - \tilde{v}(Q_0; q^*)\right) \\
&\geq \tilde{v}(Q_0; q^*) + \frac{w(Q_2; q - q^*)}{w(Q_0; q^*) + w(Q_2; q - q^*)}\left(\tilde{v}(Q_2; q - q^*) - \tilde{v}(Q_0; q^*)\right) \\
&= \tilde{v}\left(\tilde{Q}; q\right),
\end{aligned}$$

which establishes inequality (8), as needed. □

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