A Threshold Stochastic Volatility Model with Realized Volatility

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Abstract

Rapid development in the computer technology has made the financial transaction data visible at an ultimate limit level. The realized volatility, as a proxy for the "true" volatility, can be constructed using the high frequency data. This paper extends a threshold stochastic volatility specification proposed in So, Li and Lam (2002) by incorporating the high frequency volatility measures. Due to the availability of the volatility time series, the parameters' estimation can be easily implemented via the standard maximum likelihood estimation (MLE) rather than using the simulated Bayesian methods. In the Monte Carlo section, several mis-specification and sensitivity experiments are conducted. The proposed methodology shows good performance according to the Monte Carlo results. In the empirical study, three stock indices are examined under the threshold stochastic volatility structure. Empirical results show that in different regimes, the returns and volatilities exhibit asymmetric behavior. In addition, this paper allows the threshold in the model to be flexible and uses a sequential optimization based on MLE to search for the "optimal" threshold value. We find that the model with a flexible threshold is always preferred to the model with a fixed threshold according to the log-likelihood measure. Interestingly, the "optimal" threshold is found to be stable across different sampling realized volatility measures.

Keywords Realized Volatility; Threshold Stochastic Volatility Model; Leverage Effect; High Frequency Data.

JEL Classification C01, C51.

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1 Introduction

Recent empirical studies have established many stylized facts for the financial asset returns. As well known, the return time-series exhibit significantly non-Gaussian behavior, such as heavy tail. In addition, the volatility is clustered, known as the volatility persistency, and tends to be negatively correlated with the past return, known as the leverage effect. To capture these empirical phenomena, a lot of models have been developed in the past two decades. The benchmark class of models is the (Generalized) Autoregressive Conditional Heteroskedasticity (ARCH/GARCH) family, which was firstly proposed by Engle (1982) and Bollerslev (1986). The standard ARCH/GARCH specification allows the volatility to be time-varying. The conditional volatility is modeled as a function of the past squared mean return innovations and the past conditional volatilities. Alternatively, another seminal model, called Stochastic Volatility (SV) model, was introduced by Taylor (1986). Rather than assuming the volatility to be deterministic in the ARCH/GARCH, the SV model allows the volatility to evolve with a stochastic process. Theoretically, the SV model is more flexible than the ARCH/GARCH specification since a new innovation term is embedded in the latent volatility process.

The SV specification has an intuitive appeal and realistic modeling structure, however, the estimation for the SV parameters is proved to be more challenging in the literature. In particular, the likelihood function implied from the SV structure involves a sequence of integrals with a dimension equal to the sample size. As a consequence, direct estimation approach based on the exact likelihood seems impossible especially when the sample size is large. As noticed, the main difficulty in the SV estimation is because the volatility series is latent and needs to be integrated out in order to construct the objective likelihood function.[1] Despite of many different estimation methods proposed in the literature, we realize that if the volatility can be incorporated into the SV model as an observed sequence, the estimating process is very straightforward via the standard maximum likelihood estimation (MLE). Naturally, the realized volatility is considered in this paper. Barndorff-Nielsen and Shephard (2002), Andersen, Bollerslev, Diebold and Labys (2003), Meddahi (2002), among others[2], have established some theoretical foundations for realized volatility construction using the high frequency data. In particular,

[1]Different approaches have been devised for estimating SV parameters. Broto and Ruiz (2004) document a survey for the recent development of the estimation methodology under the SV specification.

[2]A recent survey paper, McAleer and Medeiros (2008), has documented an excellent review of rapidly expanding literature on realized volatility.

Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2001) propose using the sum of the squared intra-daily returns as a proxy measure for the corresponding daily volatility. This measure provides a consistent estimator of the latent volatility under an ideal market condition. A few papers have utilized the realized volatility in the estimation of the SV models, such as Takahashi, Omori and Watanabe (2009), Xu and Li (2010) and etc.

Furthermore, recently empirical research has indicated that the financial asset returns and volatilities exhibit asymmetric behavior in different regimes (e.g. bear/bull markets). Li and Lam (1995) have detected the significantly asymmetric movements of the conditional mean structure corresponding to the rise and fall of the previous-day market. In addition, Liu and Maheu (2008) have found strong empirical evidence of asymmetry in the volatility regimes. To accommodate these asymmetric effects, So, Li and Lam (2002) extend the standard SV model into a threshold framework, in which the latent volatility dynamic is determined by the sign of the lagged return. They also detected the significant asymmetric behavior in the variance persistence based on their sample data.

In this paper, we extend the threshold stochastic volatility specification proposed in So, Li and Lam (2002) by incorporating the high frequency volatility measures. The proposed model is applied to three stock indices, including Standard and Poor's 500 (S&P 500), Dow Jones Industrial Average (DJIA) and France CAC 40 Index (PX1). The realized volatilities are constructed using different sampling-frequency data. The empirical results show that the returns and volatilities exhibit asymmetric behavior in different regimes. In addition, instead of fixing the threshold values, we allow the thresholds to be flexible for each sample. We find that the flexible threshold model is always preferred according to the log-likelihood measure and more interestingly, the "optimal" threshold is stable across different sampling realized volatility measures.

The rest of the paper is organized as follows. Section 2 details the model specification and presents the constructions of the realized volatility measures. Section 3 conducts several sensitivity and mis-specification Monte Carlo experiments. Section 4 presents the empirical data and discusses the applications and empirical results. Section 5 concludes the paper. All the tables and the figures are collected in the Appendix.

2 Threshold SV Model Specification and Realized Volatility Measures

In the standard discrete-time SV model, there are two processes describing the dynamics of the returns and volatilities. The model structure is given as follows,

$$
\begin{aligned}
x_t &= \exp(h_t/2)e_t && (1)\\
h_{t+1} &= \lambda + \alpha h_t + v_{t+1} && (2)
\end{aligned}
$$

In the above set-up, x_t is the continuously compounded return time series, which can be constructed using the logarithmic closing price differences. Assuming unit variance on the innovation (e_t) of the return process, $\exp(h_t)$ characterizes the conditional variance at time t. The log-volatility, h_t, is normally assumed to follow an AR(1) process. In general, to capture the leverage effect, we allow certain correlation structure between the innovations from the return and volatility processes. Typically, following Harvey and Shephard (1996) and Yu (2005), the bivariate structure is assumed as follows:

$$
\begin{pmatrix} e_t \\ v_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\sigma_v \\ \rho\sigma_v & \sigma_v^2 \end{pmatrix}\right) \qquad (3)
$$

The asymmetric relationship between the return and the future volatility can be accommodated in the correlation coefficient parameter, ρ. Empirically, this correlation is found to be significantly negative, which suggests that the return volatility tends to increase generally after observing a drop of the stock price.

Furthermore, as mentioned earlier, recent empirical evidence, such as So, Li and Lam (2002) and Smith (2009), indicates that there seems to exist different behavior in the volatility process. In other words, the volatility responses to the price change quite distinctly. So, Li and Lam (2002) argue that the volatility is on average higher under the influence of the bad news than that of good news. To capture this volatility asymmetry phenomenon, a threshold effect is naturally considered in the volatility autoregressive dynamic. Essentially, the volatility process in (2) is assumed to follow a threshold AR model, which belongs a class of threshold time-series models proposed by Tong and Lim (1980).

This paper follows the basic threshold SV model structure from So, Li and Lam (2002) with the volatility threshold triggered by the observed return time series. In particular, the volatility process in (2) follows a threshold autoregressive framework, specified as follows,

$$h_{t+1} = \begin{cases} \lambda_0 + \alpha_0 h_t + v_{0,t+1}, & x_t \leq \gamma \\ \lambda_1 + \alpha_1 h_t + v_{1,t+1}, & x_t > \gamma \end{cases} \tag{4}$$

In (4), the log-volatility exhibits different AR dynamics in the two specified regimes triggered by the previous return. γ is the threshold value for the regimes. In this representation, the parameters switch between the two regimes according to the threshold level of the price change in the lagged period. In So, Li and Lam (2002), their model does not accommodate correlation between the return and volatility. This paper extends the model by allowing that the correlation (or the leverage effect) coefficient also switches between the two regimes. Essentially, extended from (3), a more flexible bivariate structure is assumed:

$$\begin{pmatrix} e_t \\ v_{s_t,t+1} \end{pmatrix} \sim \quad N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{s_{t+1}}\sigma_{v_{s_{t+1}}} \\ \rho_{s_{t+1}}\sigma_{v_{s_{t+1}}} & \sigma^2_{v_{s_{t+1}}} \end{pmatrix} \right) \tag{5}$$

where s_{t+1} is a state variable defined as follows,

$$s_{t+1} = \begin{cases} 0, & x_t \leq \gamma \\ 1, & x_t > \gamma \end{cases} \tag{6}$$

$\sigma^2_{v_0}$ and $\sigma^2_{v_1}$ are the variances of the innovations in the two volatility regimes. Correspondingly, ρ_0 and ρ_1 capture the asymmetric correlations between the return and volatility in the regimes ($x_t \leq \gamma$) and ($x_t > \gamma$), respectively. Therefore, the unknown parameter vector to be estimated is defined as $\theta = (\lambda_0, \lambda_1, \alpha_0, \alpha_1, \sigma_{v_0}, \sigma_{v_1}, \rho_0, \rho_1)'$.

In the standard threshold SV set-up, see So, Li and Lam (2002), there is no correlation assumed between the return and the volatility process. Smith (2009) accommodates a constant correlation among regimes. In this paper, we allow the correlations to be state-dependent switching by the threshold. In addition, in both So, Li and Lam (2002) and Smith (2009), the threshold value is "arbitrarily" set to be zero, which essentially implies that the volatility regimes depend on the sign of the last period return. In this paper, we further allow a flexible threshold value in the model and search for the "optimal" threshold level with respect to certain measure. This potentially increases the goodness-of-fit to the data in practice. We will give more details in the later discussion.

As mentioned, the SV parameters' estimation is challenging since the volatility is an unobserved sequence. Therefore, to estimate the threshold SV model, So, Li and Lam (2002) and Smith (2009) have adopted a simulated Bayesian approach (MCMC) to estimate the parameters and the latent h_t sequence simultaneously. But if the sample size is large, the computational cost could be potentially expensive based on the simulations, because of which allowing a flexible threshold in the SV is even more computationally costly. In addition, in the Bayesian methods, the estimates (for both the model parameters and h_t) could be sensitive to the initial prior information. In this paper, we approach the estimation from a different angle by incorporating the realized volatility measures into the model. Based on the "observed" data of the volatilities, the model estimation can be easily implemented via the standard MLE. Specifically, the exact likelihood function is as follows,

$$\begin{aligned} L(\theta; x, h) &= L_{s_T}(x_T, h_{T+1}|x_{T-1}, h_T, \theta) \times L_{s_{T-1}}(x_{T-1}, h_T|x_{T-2}, h_{T-1}, \theta) \times ... \\ &\times L_{s_2}(x_2, h_3|x_1, h_2, \theta) \times L_{s_1}(x_1, h_2|x_0, \theta) \end{aligned} \tag{7}$$

The likelihood in (7) consists of a serial products of the conditional densities. In essence, at the each time t, given the state determined from the previous observation, the return and the future volatility follow a bivariate Gaussian under (5). Therefore, the conditional density can be specified as,

$$L_{s_t} = \begin{cases} \frac{1}{2\pi\sigma_{v_0}\sigma_{x_t}\sqrt{1-\rho_0^2}} e^{\left(-\frac{1}{2(1-\rho_0^2)}\left[\frac{x_t^2}{\sigma_{x_t}^2} + \frac{(h_{t+1}-\lambda_0-\alpha_0 h_t)^2}{\sigma_{v_0}^2} - 2\rho_0 \frac{x_t(h_{t+1}-\lambda_0-\alpha_0 h_t)}{\sigma_{x_t}\sigma_{v_0}}\right]\right)} & s_t = 0 \\ \frac{1}{2\pi\sigma_{v_1}\sigma_{x_t}\sqrt{1-\rho_1^2}} e^{\left(-\frac{1}{2(1-\rho_1^2)}\left[\frac{x_t^2}{\sigma_{x_t}^2} + \frac{(h_{t+1}-\lambda_1-\alpha_1 h_t)^2}{\sigma_{v_1}^2} - 2\rho_1 \frac{x_t(h_{t+1}-\lambda_1-\alpha_1 h_t)}{\sigma_{x_t}\sigma_{v_1}}\right]\right)} & s_t = 1 \end{cases} \tag{8}$$

where $\sigma_{x_t} = \exp(h_t/2)$. In the objective likelihood function, the volatility is treated as observable. In this paper, we use the realized volatility measure as a proxy for the latent true volatility.

Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2001) propose the construction of the realized volatility, formulated as the sum of squared intra-day returns over a certain interval. Specifically, let $p_{d,t}$ be the logarithmic price at a certain sampling frequency interval on day t. Consequently, the continuously compounded return with D observations on day t is defined as, $r_{d,t} = 100(p_{d,t} - p_{d-1,t})$, where $d = 1, 2, ..., D$ and $t = 1, 2, ..., T$. This simple estimator of the daily volatility, denoted as RV, can be constructed by summing up the intra-day squared intra-day returns during the market open period, i.e.,

$$(\text{RV})_t = \sum_{d=1}^{D} r_{d,t}^2 \tag{9}$$

This measure provides a consistent estimator of the latent volatility in an ideal market condition. However, as Hansen and Lunde (2005) argued, the over-night information is ignored in this simple construction. Consequently, they propose an estimator by incorporating the "overnight" effects into the measurement and assign "optimal" weights for the overnight component and the intra-day component. In other words, a linear combination of the overnight squared return and $(\mathrm{RV})_t$ is used to form a mean-square-error "optimal" realized volatility measure for the whole day volatility. This paper denotes this estimator as RV*, which is given by,

$$(\mathrm{RV}^*)_t = \omega_1 \cdot z_{1,t}^2 + \omega_2 \cdot (\mathrm{RV})_t \tag{10}$$

where $z_{1,t}$ is the return over the inactive period, which measures the close-to-open price change (in logarithm). $\omega_1 = (1-\phi)\mu_0/\mu_1$ and $\omega_2 = \phi\mu_0/\mu_2$ with $\phi = (\mu_2^2\eta_1^2 - \mu_1\mu_2\eta_{12})/(\mu_2^2\eta_1^2 + \mu_1^2\eta_2^2 - 2\mu_1\mu_2\eta_{12})$. $\eta_1^2 = \mathrm{var}(z_{1,t}^2)$, $\eta_2^2 = \mathrm{var}((\mathrm{RV})_t)$ and η_{12} is the covariance of $z_{1,t}^2$ and $(\mathrm{RV})_t$. Parameters μ_0, μ_1 and μ_2 are computed as the mean of $(z_{1,t}^2 + (\mathrm{RV})_t)$, $z_{1,t}$ and $(\mathrm{RV})_t$ respectively.

Following Hansen and Lunde (2005) and Xu and Li (2010), which have established some supportive empirical evidence of RV* measure, in this paper, we incorporate the RV* into the proposed threshold SV model estimation. In addition, we consider the sampling frequencies at 5-minute, 10-min, 15-min and 30-min intervals, which are commonly used for the constructions of the realized volatility measures in the literature.

With the observed return and the constructed volatility inputs, the model estimation is implemented via the standard MLE. In essence, we maximize the objective function based on (7), i.e., $\hat{\theta} = \mathrm{argmax}[\log(L(\theta; x, h))]$. The standard errors of the estimates are computed in the usual way by evaluating the expectation of the second derivatives of the log-likelihood function. Given the feasibility of the computation, in this paper, we also consider a flexible threshold value on γ (rather than arbitrarily fixing γ at 0). Formally, the model is estimated sequentially for each possible point of the threshold variable. We search the whole sample domain of x_t and choose the "optimal" one which yields the highest log-likelihood value, i.e., $\hat{\gamma}* = \mathrm{argmax}[\log(L(\gamma))]$ and $\hat{\theta}* = \hat{\theta}(\hat{\gamma}*)$.

3 Sensitivity and Model Mis-Specification Analysis via Monte Carlo Simulations

In this section, we conduct the sensitivity and model mis-specification analysis under the Monte Carlo environment to investigate the performance of the proposed model and its estimation. In each Monte Carlo experiment, the sample size is 1000 and the simulation is replicated 1000 times in the first group of experiments and 500 times in the second group of the experiments.[3]

In the first group of the simulations, we investigate the model estimation sensitivity to different parameter configurations. Some benchmark parameter values are taken from the simulation section in So, Li and Lam (2002).[4] The parameter values change accordingly in other cases compared to the benchmark in this group. Specifically, the benchmark data generating process (DGP) is specified as follows,

$$\underline{\text{DGP 1.}} \quad \begin{aligned} x_t &= \exp(h_t/2)e_t \\ h_{t+1} &= \begin{cases} -0.5 + 0.6h_t + v_{0,t+1}, & s_{t+1} = 0 \quad (x_t \leq 0) \\ -1.0 + 0.9h_t + v_{1,t+1}, & s_{t+1} = 1 \quad (x_t > 0) \end{cases} \end{aligned}$$

$$\begin{pmatrix} e_t \\ v_{s_{t+1},t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{s_{t+1}}\sigma_{v_{s_{t+1}}} \\ \rho_{s_{t+1}}\sigma_{v_{s_{t+1}}} & \sigma^2_{v_{s_{t+1}}} \end{pmatrix} \right)$$

where $\sigma_{v_0} = 1.0$, $\sigma_{v_1} = 0.5$, $\rho_0 = -0.1$ and $\rho_1 = -0.3$.

In the benchmark case, the threshold value of γ is chosen to be zero. In other words, the volatility regime switches depending on the sign of the last period return. The persistent parameter in the regime 1 ($x_t \leq 0$) is smaller than that in the regime 2 ($x_t > 0$) because So, Li and Lam (2002) argues that the volatility exhibits less persistency under the bad news than that under the good news. Furthermore, in the benchmark case it is believed that the variance of the innovation in the volatility equation is higher with the

[3]In the second group of the simulations, the computational cost is relatively expensive since the model allows the threshold values to be flexible. Consequently, in each simulation, given the sample size of 1000, the program executes 1000 optimizations to search for the "optimal" threshold value. Due to this reason, we reduce the replication times to be 500 to keep the computational cost at a feasible level while with sufficient numbers of estimates for constructing the reliability measures.

[4]Since in the threshold SV model from So, Li and Lam (2002), there is no correlation assumed between the return and volatility processes, we arbitrarily set the benchmark correlation coefficients with $\rho_0 = -0.1$ and $\rho_1 = -0.3$. We also test other sets of correlation values and find similar results (patterns) as those reported in this paper.

bad news arrival than that with the good news arrival. Consequently, the simulated data is generated from DGP 1.[5] The threshold stochastic volatility model parameters are replicatedly estimated 1000 times. The simulation results are reported in Table 1.

To investigate the robustness of our proposed methodology under different parameter configurations, we change some subset of the parameter values accordingly compared to the benchmark case. From the experiments 2 to 4, we individually increase the distances between σ_{v_0} and σ_{v_1}, between ρ_0 and ρ_1 and between α_0 and α_1, with the remaining parameters' values unchanged. More specifically, in experiment 2, we only decrease σ_{v_1} from 0.5 to 0.1; in experiment 3, we only increase ρ_1 from -0.3 to -0.9; and in experiment 4, we only decrease α_0 from 0.6 to 0.1. The simulation results are presented in Table 2, 3 and 4, respectively.[6]

Some standard measures, such as mean, bias and root of mean squared error (RMSE), are constructed for Monte Carlo evaluations. In general, for all four experiments (1-4) the means of the estimates are very close to the true parameter values. The bias and RMSE are of small magnitudes, which indicates that our proposed methodology can accurately and stably produce the estimates around their corresponding true values. One interesting comparison worth mentioning here is that both the bias and RMSE for all eight estimates are generally smaller in experiment $2-4$ than those in the benchmark case. The possible reason is that in experiment $2-4$, we differentiate the two regimes in a higher degree than the benchmark case. In other words, more similarities in the two regimes may deteriorate the quality of the estimates, and hence produces bigger bias and RMSE.[7]

Furthermore, we carry out a reliability test to examine the asymptotic behavior of the estimates. One standard evaluation is to examine the asymptotic distributions of the estimates via the Kolmogorov - Smirnov (K-S) test. The K-S statistics with the associated p-values are presented in the last row for each table. As expected, the distribution for all the estimates exhibit asymptotically normal. More specifically, the Normality can not be rejected

[5] Regarding the initial condition effect, for each simulation, we generate 1500 data points and discard the first 500 to reduce the impact.

[6] The true parameter values are also presented in the descriptions under each table. The changed parameter is highlighted in a bold font.

[7] In the extreme case, if two regimes have exactly the same values for all the parameters, then the estimation would not be identified. This is noted as a common problem in the mixture modeling literature.

at 1% significance level for all the estimates in experiments 1 – 4. In order to visualize the distribution of the estimates, we provide the QQ-plots in Figure 1 for the estimates from the benchmark experiment.[8] All the estimates from the 1000 replications fit well with the 45-degree quantile line against the normal distribution. This reinforces the K-S test results reported in the tables.

In the standard threshold SV model, the regime-switch depending on the sign of the return. In other words, the threshold value is set to be zero. As argued, fixing the threshold might give us mis-leading results. Therefore, we set up the second group of experiments to investigate the mis-specification of threshold effects. The true model parameter values are still set to be the same as the benchmark case, except we allow either a positive or negative threshold value for γ. We then apply both the zero-threshold and flexible-threshold models to the simulated data. Specifically, the data is generated from the following DGP 2,

$$\underline{\text{DGP 2.}} \quad \begin{aligned} x_t &= \exp(h_t/2)e_t \\ h_{t+1} &= \begin{cases} -0.5 + 0.6h_t + v_{0,t+1}, & s_{t+1} = 0 \;\; (x_t \leq \gamma*) \\ -1.0 + 0.9h_t + v_{1,t+1}, & s_{t+1} = 1 \;\; (x_t > \gamma*) \end{cases} \end{aligned}$$

$$\begin{pmatrix} e_t \\ v_{s_{t+1},t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{s_{t+1}}\sigma_{v_{s_{t+1}}} \\ \rho_{s_{t+1}}\sigma_{v_{s_{t+1}}} & \sigma^2_{v_{s_{t+1}}} \end{pmatrix} \right)$$

where $\sigma_{v_0} = 1.0$, $\sigma_{v_1} = 0.5$, $\rho_0 = -0.1$ and $\rho_1 = -0.3$. $\gamma*$ is equal to -0.02 and 0.02 respectively for experiment 5 and 6.

Finally, in the last experiment (experiment 7), we set the true parameter values to be close to the estimated values from the empirical section. The data is generated following DGP 3,

$$\underline{\text{DGP 3.}} \quad \begin{aligned} x_t &= \exp(h_t/2)e_t \\ h_{t+1} &= \begin{cases} -0.1 + 0.8h_t + v_{0,t+1}, & s_{t+1} = 0 \;\; (x_t \leq \gamma*) \\ -0.3 + 0.7h_t + v_{1,t+1}, & s_{t+1} = 1 \;\; (x_t > \gamma*) \end{cases} \end{aligned}$$

$$\begin{pmatrix} e_t \\ v_{s_{t+1},t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{s_{t+1}}\sigma_{v_{s_{t+1}}} \\ \rho_{s_{t+1}}\sigma_{v_{s_{t+1}}} & \sigma^2_{v_{s_{t+1}}} \end{pmatrix} \right)$$

where $\sigma_{v_0} = 0.4$, $\sigma_{v_1} = 0.5$, $\rho_0 = 0.1$ and $\rho_1 = -0.1$. $\gamma*$ is equal to -0.5.

[8]To save space, we do not provide all the distributional plots for the estimates from other experiments. These figures are available upon request.

The simulation results for experiment 5, 6 and 7are reported in Table 5, 6 and 7 correspondingly. In these tables, in addition to presenting the statistical measures of the model estimates, we also construct the same measures for the "optimal" threshold values over the 500 replications. It is worth mentioning that the threshold parameter is not treated as a parameter endogenously in the estimation. Instead, we search the whole sample domain from the simulated returns for the "optimal" threshold that yields the maximum log-likelihood value. Hence, for comparisons, we also report the average of the log-likelihood values for both zero-threshold and flexible-threshold models in the last row of Tables 5, 6 and 7.

In general, the model with a flexible threshold ($\gamma = \gamma^*$) generates uniformly smaller biases and RMSEs than the model with the fixed threshold at zero ($\gamma = 0$). For example, in experiment 5, λ_0 and α_0 are positively biased with relatively big magnitudes in the zero-threshold case. In experiment 7, a relatively big bias for ρ_0 is detected in the $\gamma = 0$ case. Consequently, the associated RMSEs are significantly larger compared to those from the flexible threshold model. We also note that in experiment 5 ($\gamma = 0$), the Normality is rejected at 5% level for the distributions of λ_0, α_0 and σ_{v0}. These Monte Carlo evidences suggest that, in practice, "arbitrarily" fixing the regime threshold at zero may produce significant biases for the parameter estimates, which could be potentially a serious issue for further inferences and applications. As noted from Table 5, 6 and 7, our proposed methodology can identify the true threshold accurately by searching the maximum of the set of the optimized log-likelihood values. On average, the mean of the 500 "optimal" thresholds is fairly close to the true values. The bias and RMSE are small, which indicates that the "optimal" γ is distributed stably around γ^*. Furthermore, as expected, the average log-likelihood values from the flexible threshold SV model are significantly greater than those calculated from the zero-threshold specification. In the following empirical study, we apply both models to the financial data and make further empirical comparisons.

4 Empirical Study

In this section, we apply the proposed model and methodology to three stock indices including S&P 500, DJIA and PX1 covering from 2003 to 2008. At daily level, there are around 1400 data points for each data set. The realized volatilities are constructed based on (10) using the intra-day transaction data.

To summarize the data, we provide the descriptive statistics for the daily returns (DR) and the realized volatilities at 5-min sampling frequency (RV_{5m}) in Table 8.[9] Standard statistics, such as mean (Mean), variance (Var), skewness (Skn), kurtosis (Kurt), minimum (min), maximum (Max) and Jarque-Bera statistic (J-B Stats), are reported. Generally, the statistics show consistency with the common empirical findings in the literature. Both the return and realized volatility exhibit non-Gaussian behavior with relatively large kurtosis and J-B statistics values. The logarithms of the realized volatility are nearly normally distributed, where the kurtosis values are close to 3 and J-B statistics are much smaller. We also provide several descriptive data plots in Figures 2-4 for empirical illustrations. We will give detailed discussions combining with the empirical estimation analysis later in this section.

We apply both the threshold SV models with $\gamma = 0$ and $\gamma = \gamma^*$ to the return and 5-min realized volatility data for S&P 500, DJIA and PX1, respectively. For the flexible threshold SV estimation, we follow the sequential optimization procedure via MLE described in section 2. The standard errors of the estimates are calculated by evaluating the Hessian matrix numerically. The empirical results are presented in Table 9. As expected, the flexible threshold model is uniformly preferred to the zero-threshold model according to the log-likelihood measure. For all three indices, the "optimal" thresholds are found to be negative. With these threshold values, the return and realized volatility are divided into two regimes: regime 1 ($\gamma \leq \gamma^*$) and regime 2 ($\gamma > \gamma^*$). Consequently, there are 287 (20.49%), 407 (29.54%) and 266 (18.95%) observations in regime 1 for S&P 500, DJIA and PX1 respectively. For comparisons, we find that when the threshold is set to be zero, there are 628 (44.83%), 630 (45.75%) and 660 (47.01%) observations in regime 1 for S&P 500, DJIA and PX1 respectively. Clearly, the number of the observations in regime 1 under the flexible threshold model is much less than that under the zero threshold specification.

To visualize the two regimes under the "optimal" threshold, we plot the daily returns and 5-min realized volatilities (across time) in panel (a) and (b) through Figure 2 to 4 for the three indices with regime 1 dotted and regime 2 dashed. Panel (c) presents the evolution path of the optimized log-likelihoods across different threshold values in the domain of returns. In other words, panel (c) is drawn from the results with about 1400 optimizations by imposing different threshold each round. From panel (c), we see that

[9] We also construct the realized volatilities using other frequency-level transaction data. To save space, those summary statistics are not reported in this version of the paper, but they are available upon request.

"arbitrarily" setting zero as the threshold does not yield the maximum log-likelihood values. This could potentially bias the estimation results, which has been demonstrated in the mis-specification Monte Carlo experiments.

From the empirical results in Table 9, some asymmetric behavior are consistently detected. In general, the persistent parameter value (α_0) in regime 1 is higher than that (α_1) in regime 2. In regime 1, generally no significant leverage effect is detected. Interestingly, we do find some positive correlations between the return and the future volatility in regime 1.[10] One possible explanation for this is that when the investors observe the price dropping beyond certain level, they might choose wait-and-see investing strategy rather than trading more. This would lead to smaller volatilities on the market. In other words, the returns and volatilities may exhibit positive correlations when price drops up to certain threshold. In regime 2, we consistently detect the significant leverage effects (negative ρ_1) for all three indices. Note that the majority of the observations (about 70-80%) are in regime 2, overall, the correlation between the return and volatility processes exhibits negative relationship, which is commonly explained in the literature.

Furthermore, as mentioned, the realized volatility can be constructed using different sampling frequencies. In this paper, we also adopt three other popular sampling frequencies (including 10-min, 15-min and 30-min), which are commonly used in the literature, to construct the realized volatility measures. The empirical estimation results for S&P 500 are provided in Table 10.[11]

In general, the empirical results in Table 10 are similar compared to those from 5-min. However, we observe some consistent and interesting findings. As the sampling interval increases (from 5-min to 30-min), the variance estimate of the disturbance from the volatility process (σ_{v_0} or σ_{v_1}) increases regardless of regimes. This is consistent with the findings established in Takahashi, Omori and Watanabe (2008) and Xu and Li (2010). That is, as the sampling frequency decreases, the volatility process might become more noisy. Consequently, as expected, a decreasing persistency in the volatility process (α_0 or α_1) is observed (regardless of regimes) as the sampling interval increases. This implies that the larger the variance of the volatility process is, the less persistent the volatility process is. In general, ρ_0 in regime 1

[10] However, those correlation coefficients (ρ_0) are not statistically significant in general.

[11] The empirical results for DJIA and PX1 are consistent with those reported in Table 10 for S&P 500. To save space, we do not present the results in this paper. The results are available upon request.

is insignificantly positive and ρ_1 in regime 2 is significantly negative across different frequency levels, which captures the overall leverage effect. In other words, the commonly observed leverage effect dominantly comes from regime 2. The most interesting finding in Table 10 is that the "optimal" threshold value stays at the same level (around -0.58). In other words, the "optimal" threshold is found to be pretty stable across different sampling-frequency realized volatility measures. We also provide the plots of the evolution path of the optimized log-likelihoods across different threshold values for each sampling frequency in Figure 5. Panel (a), (b) and (c) in Figure 5 represent the optimized log-likelihood path for 10-min, 15-min and 30-min respectively. We find that the common peak in the three panels is at the similar return level (-0.58).

5 Conclusion

This paper incorporates the realized volatility measures constructing from high frequency transaction data into a threshold stochastic volatility model. Due to availability of the volatility time series, the model parameters' estimation is implemented via the standard MLE. Several groups of mis-specification and sensitivity Monte Carlo experiments are conducted. Our proposed methodology shows good performance according to the Monte Carlo results. In the empirical study, three stock indices are examined under the threshold stochastic volatility structure. Empirical results show that in different regimes, the returns and volatilities exhibit asymmetric behavior. In addition, this paper allows the threshold in the model to be flexible and uses a sequential optimization based on MLE to search for the "optimal" threshold value. We find that the model with a flexible threshold is always preferred to the model with a fixed threshold according to the log-likelihood measure. The "optimal" threshold is also stable across different sampling realized volatility measures.

Finally, we want to point out a few potential issues for our future research directions. In the flexible threshold environment, the threshold variable is not treated as an endogenous parameter in the estimation procedure. Therefore, to achieve the "optimal" threshold, the program needs to search the whole domain of the possible values. In practice, if the sample size is relatively large, the computational cost could be potentially expensive. If the threshold and model parameters can be estimated simultaneously, this would improve the efficiency of the whole procedure. Furthermore, more statistical inferences can be made based on the estimates of the threshold parameter, such as regime specification test by examining the significance of the threshold estimate. In addition, there have been some successes in the ARCH/GARCH and SV modeling with threshold-effects in both returns and volatility processes, see for example Li and Li (1996), So, Li and Lam (2002) and Smith (2009). This concept could also be easily introduced into the realized stochastic volatility framework. It would be also interesting to examine multiple asymmetric threshold-effects under the time-varying volatility structure through a more complicated threshold selection process. We will leave these for future study.

Appendix

Table 1. Monte Carlo Simulation Design # 1.

	λ_0	λ_1	α_0	α_1	ρ_0	ρ_1	σ_{v_0}	σ_{v_1}
MEAN	-0.5075	-1.0023	0.5971	0.8993	-0.1002	-0.3019	0.9970	0.4991
BIAS	-0.0075	-0.0023	-0.0029	-0.0007	-0.0002	-0.0019	-0.0030	-0.0009
RMSE	0.0926	0.0433	0.0270	0.0121	0.0454	0.0403	0.0327	0.0160
K-S	0.0205	0.0209	0.0321	0.0232	0.0229	0.0345	0.0146	0.0207
	(0.7921)	(0.7729)	(0.2497)	(0.6477)	(0.6657)	(0.1810)	(0.9822)	(0.7823)

The data is generated following DGP 1. True model parameter values: $\lambda_0 = -0.5$, $\lambda_1 = -1.0$, $\alpha_0 = 0.6$, $\alpha_1 = 0.9$, $\rho_0 = -0.1$, $\rho_1 = -0.3$, $\sigma_{v_0} = 1.0$, $\sigma_{v_1} = 0.5$. The threshold variable $\gamma = 0$. The numbers in parenthesis are p-values of the Kolmogorov-Smirnov (K-S) statistics. Normality is not rejected at 5% significant level (cut-off K-S value is 0.0428); Normality is not rejected at 1% significant level (cut-off K-S value is 0.0513).

Figure 1. QQ-Plots of the Estimates in the Benchmark Case

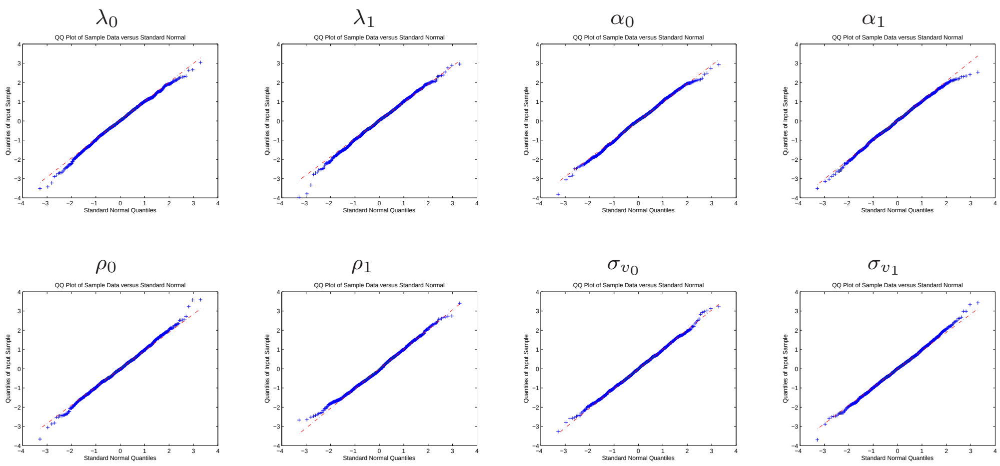


Table 2. Monte Carlo Simulation Design # 2.

	λ_0	λ_1	α_0	α_1	ρ_0	ρ_1	σ_{v_0}	σ_{v_1}
MEAN	-0.5101	-1.0004	0.5978	0.8998	-0.0999	-0.3012	0.9977	0.0998
BIAS	-0.0101	-0.0004	-0.0022	-0.0002	0.0001	-0.0012	-0.0023	-0.0002
RMSE	0.0951	0.0094	0.0275	0.0028	0.0436	0.0393	0.0314	0.0031
K-S	0.0291	0.0311	0.0290	0.0356	0.0238	0.0146	0.0199	0.0186
	(0.3589)	(0.2849)	(0.3665)	(0.1558)	(0.6159)	(0.9825)	(0.8208)	(0.8752)

The data is generated following DGP 1. True model parameter values: $\lambda_0 = -0.5$, $\lambda_1 = -1.0$, $\alpha_0 = 0.6$, $\alpha_1 = 0.9$, $\rho_0 = -0.1$, $\rho_1 = -0.3$, $\sigma_{v_0} = 1.0$, $\sigma_{v_1} =$ **0.1**. The threshold variable $\gamma = 0$. The numbers in parenthesis are p-values of the Kolmogorov-Smirnov (K-S) statistics. Normality is not rejected at 5% significant level (cut-off K-S value is 0.0428); Normality is not rejected at 1% significant level (cut-off K-S value is 0.0513).

Table 3. Monte Carlo Simulation Design # 3.

	λ_0	λ_1	α_0	α_1	ρ_0	ρ_1	σ_{v_0}	σ_{v_1}
MEAN	-0.5090	-1.0002	0.5977	0.9001	-0.0997	-0.9006	0.9987	0.4997
BIAS	-0.0090	-0.0002	-0.0023	0.0001	0.0003	-0.0006	-0.0013	-0.0003
RMSE	0.0839	0.0196	0.0239	0.0050	0.0453	0.0063	0.0314	0.0093
K-S	0.0196	0.0243	0.0288	0.0132	0.0163	0.0289	0.0219	0.0220
	(0.8324)	(0.5924)	(0.3750)	(0.9948)	(0.9523)	(0.3705)	(0.7188)	(0.7152)

The data is generated following DGP 1. True model parameter values: $\lambda_0 = -0.5$, $\lambda_1 = -1.0$, $\alpha_0 = 0.6$, $\alpha_1 = 0.9$, $\rho_0 = -0.1$, $\rho_1 =$**−0.9** , $\sigma_{v_0} = 1.0$, $\sigma_{v_1} = 0.5$. The threshold variable $\gamma = 0$. The numbers in parenthesis are p-values of the Kolmogorov-Smirnov (K-S) statistics. Normality is not rejected at 5% significant level (cut-off K-S value is 0.0428); Normality is not rejected at 1% significant level (cut-off K-S value is 0.0513).

Table 4. Monte Carlo Simulation Design # 4.

	λ_0	λ_1	α_0	α_1	ρ_0	ρ_1	σ_{v_0}	σ_{v_1}
MEAN	-0.5003	-1.0007	0.0988	0.8997	-0.0996	-0.3018	0.9971	0.4995
BIAS	-0.0003	-0.0007	-0.0012	-0.0003	0.0004	-0.0018	-0.0029	-0.0005
RMSE	0.0615	0.0297	0.0286	0.0127	0.0454	0.0405	0.0326	0.0160
K-S	0.0307	0.0264	0.0233	0.0232	0.0214	0.0341	0.0175	0.0192
	(0.2994)	(0.4813)	(0.6445)	(0.6521)	(0.7443)	(0.1909)	(0.9164)	(0.8500)

The data is generated following DGP 1. True model parameter values: $\lambda_0 = -0.5$, $\lambda_1 = -1.0$, $\alpha_0 =$**0.1** , $\alpha_1 = 0.9$, $\rho_0 = -0.1$, $\rho_1 = -0.3$, $\sigma_{v_0} = 1.0$, $\sigma_{v_1} = 0.5$. The threshold variable $\gamma = 0$. The numbers in parenthesis are p-values of the Kolmogorov-Smirnov (K-S) statistics. Normality is not rejected at 5% significant level (cut-off K-S value is 0.0428); Normality is not rejected at 1% significant level (cut-off K-S value is 0.0513).

Table 5. Monte Carlo Simulation Design # 5.

	MEAN		BIAS		RMSE		K-S	
	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$
λ_0	0.4078	-0.5132	0.9078	-0.0132	0.9261	0.1138	0.1114 (0.0000)	0.0250 (0.9103)
λ_1	-1.0068	-1.0071	-0.0068	-0.0071	0.0452	0.0413	0.0373 (0.4826)	0.0353 (0.5542)
α_0	0.9833	0.5975	0.3833	-0.0025	0.3881	0.0295	0.1528 (0.0000)	0.0243 (0.9269)
α_1	0.8985	0.8984	-0.0015	-0.0016	0.0073	0.0062	0.0695 (0.0151)	0.0390 (0.1596)
ρ_0	-0.1070	-0.0979	-0.0070	0.0021	0.0474	0.0594	0.0297 (0.7635)	0.0297 (0.7651)
ρ_1	-0.3037	-0.3024	-0.0037	-0.0024	0.0416	0.0355	0.0229 (0.9530)	0.0209 (0.9796)
σ_{v_0}	1.1236	0.9960	0.1236	-0.0040	0.1482	0.0412	0.0923 (0.0004)	0.0324 (0.6649)
σ_{v_1}	0.4994	0.4998	-0.0006	-0.0002	0.0166	0.0142	0.0380 (0.4597)	0.0238 (0.9365)
γ	–	-0.0209	–	0.0025	–	0.0012	–	–
Log-L	-1579.1	-1453.0	–	–	–	–	–	–

The data is generated following DGP 2. True model parameter values: $\lambda_0 = -0.5$, $\lambda_1 = -1.0$, $\alpha_0 = 0.6$, $\alpha_1 = 0.9$, $\rho_0 = -0.1$, $\rho_1 = -0.3$, $\sigma_{v_0} = 1.0$, $\sigma_{v_1} = 0.5$. The threshold variable $\gamma* =$**–0.02**. The numbers in parenthesis are p-values of the Kolmogorov-Smirnov (K-S) statistics. Normality is not rejected at 5% significant level (cut-off K-S value is 0.0428); Normality is not rejected at 1% significant level (cut-off K-S value is 0.0513).

Table 6. Monte Carlo Simulation Design # 6.

	MEAN		BIAS		RMSE		K-S	
	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$
λ_0	-0.5108	-0.5049	-0.0108	-0.0049	0.0879	0.0851	0.0241 (0.9298)	0.0228 (0.9559)
λ_1	-1.0781	-1.0002	-0.0781	-0.0002	0.1006	0.0466	0.0272 (0.8479)	0.0321 (0.6737)
α_0	0.5968	0.6023	-0.0032	0.0023	0.0277	0.0268	0.0328 (0.6495)	0.0350 (0.5666)
α_1	0.8248	0.8999	-0.0752	-0.0001	0.0784	0.0140	0.0216 (0.9723)	0.0246 (0.9207)
ρ_0	-0.0996	-0.0999	0.0004	0.0001	0.0450	0.0437	0.0247 (0.9170)	0.0242 (0.9278)
ρ_1	-0.2015	-0.2997	0.0985	0.0003	0.1077	0.0421	0.0415 (0.3489)	0.0248 (0.9148)
σ_{v_0}	0.9978	1.0028	-0.0022	0.0028	0.0311	0.0307	0.0245 (0.9228)	0.0190 (0.9932)
σ_{v_1}	0.7214	0.4993	0.2214	-0.0007	0.2254	0.0172	0.0208 (0.9805)	0.0350 (0.5658)
γ	–	0.0213	–	0.0013	–	0.0023	–	–
Log-L	-1267.3	-1107.5	–	–	–	–	–	–

The data is generated following DGP 2. True model parameter values: $\lambda_0 = -0.5$, $\lambda_1 = -1.0$, $\alpha_0 = 0.6$, $\alpha_1 = 0.9$, $\rho_0 = -0.1$, $\rho_1 = -0.3$, $\sigma_{v_0} = 1.0$, $\sigma_{v_1} = 0.5$. The threshold variable $\gamma* =$**0.02**. The numbers in parenthesis are p-values of the Kolmogorov-Smirnov (K-S) statistics. Normality is not rejected at 5% significant level (cut-off K-S value is 0.0428); Normality is not rejected at 1% significant level (cut-off K-S value is 0.0513).

Table 7. Monte Carlo Simulation Design # 7.

	MEAN		BIAS		RMSE		K-S	
	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$
λ_0	-0.1955	-0.1017	-0.0955	-0.0017	0.1017	0.0422	0.0348 (0.5716)	0.0213 (0.9760)
λ_1	-0.3038	-0.3051	-0.0038	-0.0051	0.0368	0.0304	0.0363 (0.5194)	0.0318 (0.6875)
α_0	0.7557	0.7974	-0.0443	-0.0026	0.0531	0.0430	0.0360 (0.5301)	0.0249 (0.9121)
α_1	0.6960	0.6951	-0.0040	-0.0049	0.0318	0.0256	0.0314 (0.7014)	0.0227 (0.9571)
ρ_0	-0.0230	0.1078	-0.1230	0.0078	0.1314	0.0721	0.0266 (0.8670)	0.0209 (0.9804)
ρ_1	-0.1050	-0.1036	-0.0050	-0.0036	0.0459	0.0350	0.0207 (0.9822)	0.0208 (0.9814)
σ_{v_0}	0.4634	0.3962	0.0634	-0.0038	0.0653	0.0194	0.0254 (0.9003)	0.0184 (0.9954)
σ_{v_1}	0.4996	0.4997	-0.0004	-0.0003	0.0166	0.0130	0.0333 (0.6290)	0.0303 (0.7407)
γ	–	-0.4968	–	0.0032	–	0.0150	–	–
Log-L	-1682.4	-1628.6	–	–	–	–	–	–

The data is generated following DGP 3. True model parameter values: $\lambda_0 = -0.1$, $\lambda_1 = -0.3$, $\alpha_0 = 0.8$, $\alpha_1 = 0.7$, $\rho_0 = 0.1$, $\rho_1 = -0.1$, $\sigma_{v_0} = 0.4$, $\sigma_{v_1} = 0.5$. The threshold variable $\gamma* = -0.5$. The numbers in parenthesis are p-values of the Kolmogorov-Smirnov (K-S) statistics. Normality is not rejected at 5% significant level (cut-off K-S value is 0.0428); Normality is not rejected at 1% significant level (cut-off K-S value is 0.0513).

Table 8. Summary Statistics

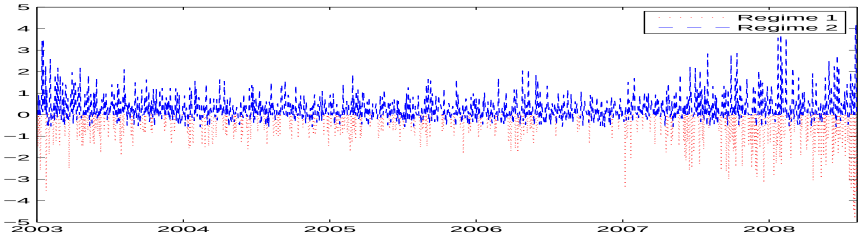
	Mean	Var	Skn	Kurt	Min	Max	J-B Stats
				S&P 500			
DR	0.0248	0.8448	-0.2454	5.9026	-4.8354	4.1780	505.88 (0.00)
RV^{5m}	0.5714	0.6720	7.5754	94.8291	0.0292	14.3848	5.69e5 (0.00)
$\ln(RV^{5m})$	-0.9752	0.7043	0.5402	3.4849	-3.5320	2.6662	81.85 (0.00)
				DJIA			
DR	0.0249	0.7413	-0.2305	5.3446	-4.2258	3.5893	327.60 (0.00)
RV^{5m}	0.5163	0.4707	7.4983	90.5360	0.0313	11.1384	4.52e5 (0.00)
$\ln(RV^{5m})$	-1.0271	0.6302	0.4704	3.6100	-3.4642	2.4104	72.12 (0.00)
				PX1			
DR	0.0284	1.1308	-0.0298	8.0894	-6.6344	8.5064	1.51e3 (0.00)
RV^{5m}	1.1556	3.4715	10.6987	180.4995	0.0570	39.1751	1.86e6 (0.00)
$\ln(RV^{5m})$	-0.3112	0.7888	0.4473	3.2052	-2.8648	3.6680	49.27 (0.00)

Table 9. Empirical Results

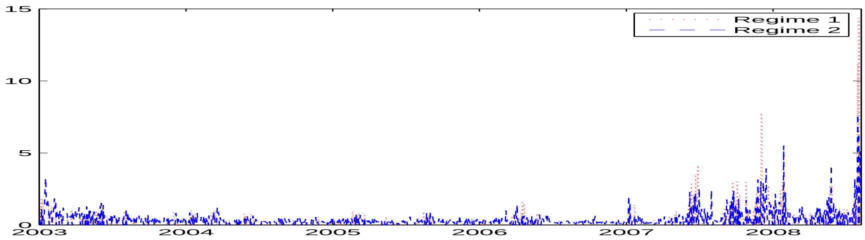
	S&P 500		**DJIA**		**PX1**	
	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$
λ_0	-0.2256	-0.1635	-0.2563	-0.2166	-0.0417	0.0433
	(0.0295)	(0.0321)	(0.0299)	(0.0329)	(0.0242)	(0.0413)
λ_1	-0.2495	-0.2824	-0.3050	-0.3309	-0.1265	-0.1319
	(0.0299)	(0.0260)	(0.0329)	(0.0289)	(0.0228)	(0.0187)
α_0	0.7666	0.7762	0.7492	0.7719	0.7404	0.7686
	(0.0249)	(0.0330)	(0.0250)	(0.0315)	(0.0255)	(0.0390)
α_1	0.7483	0.7277	0.7109	0.6951	0.7109	0.6865
	(0.0215)	(0.0191)	(0.0228)	(0.0201)	(0.0247)	(0.0206)
ρ_0	0.0103	0.1154	0.0221	0.0974	-0.0196	0.1112
	(0.0345)	(0.0536)	(0.0347)	(0.0435)	(0.0427)	(0.0663)
ρ_1	-0.1705	-0.1433	-0.1773	-0.1628	-0.1556	-0.1362
	(0.0289)	(0.0245)	(0.0296)	(0.0265)	(0.0371)	(0.0298)
σ_{v_0}	0.5272	0.4887	0.5261	0.5034	0.6102	0.6265
	(0.0138)	(0.0223)	(0.0113)	(0.0191)	(0.0147)	(0.0221)
σ_{v_1}	0.5420	0.5447	0.5331	0.5389	0.5828	0.5840
	(0.0129)	(0.0104)	(0.0120)	(0.0091)	(0.0136)	(0.0111)
γ	–	-0.5771	–	-0.3004	–	-0.7062
Log-L	-2786.9	-2778.1	-2664.1	-2656.8	-2998.9	-2987.1

The numbers in parenthesis are the standard errors. For each index column, $\gamma = 0$ stands for the model with a fixed threshold value at 0, while $\gamma = \gamma*$ stands for the model with a flexible threshold. The estimates under ($\gamma = \gamma*$) column are taken from the optimization under the "optimal" threshold.

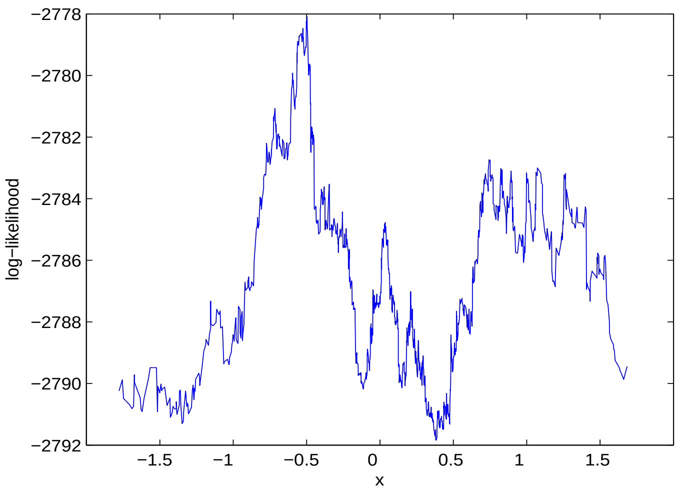
Figure 2. Plots for S&P 500



(a) Daily Return

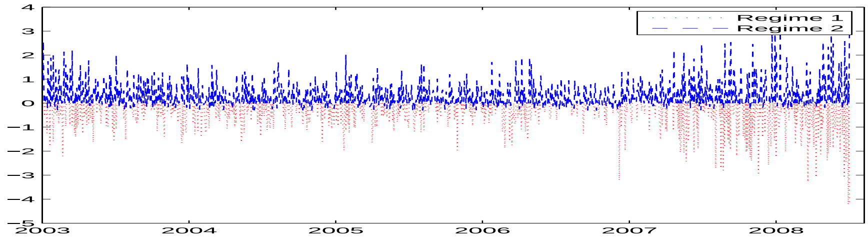


(b) Realized Volatility

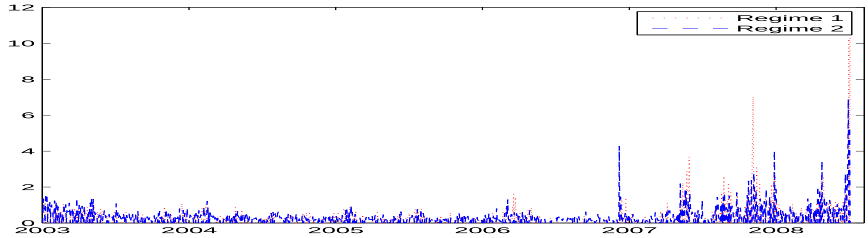


(c) Log-Likelihoods across Different Thresholds

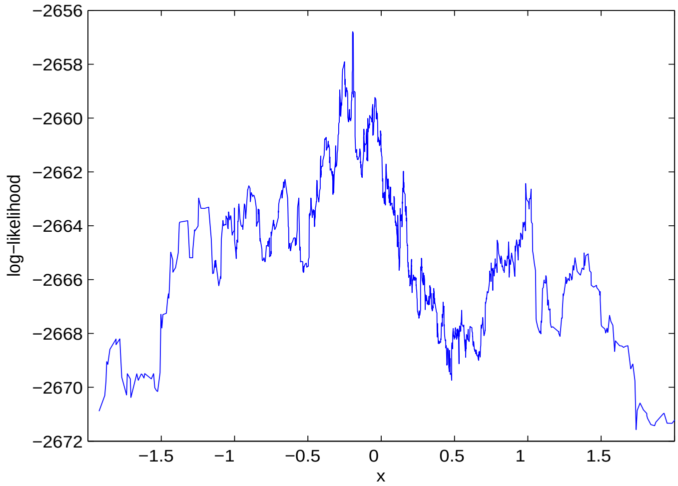
Figure 3. Plots for DJIA



(a) Daily Return

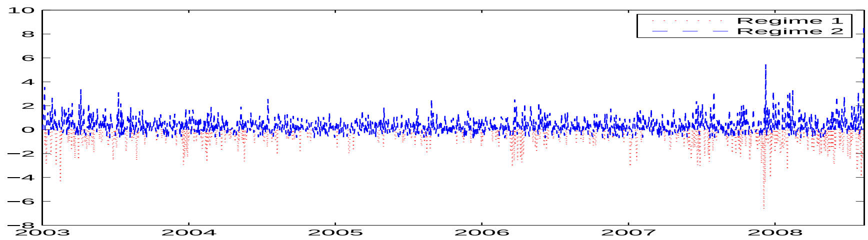


(b) Realized Volatility

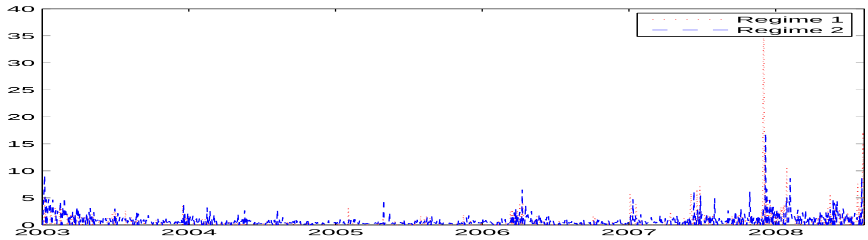


(c) Log-Likelihoods across Different Thresholds

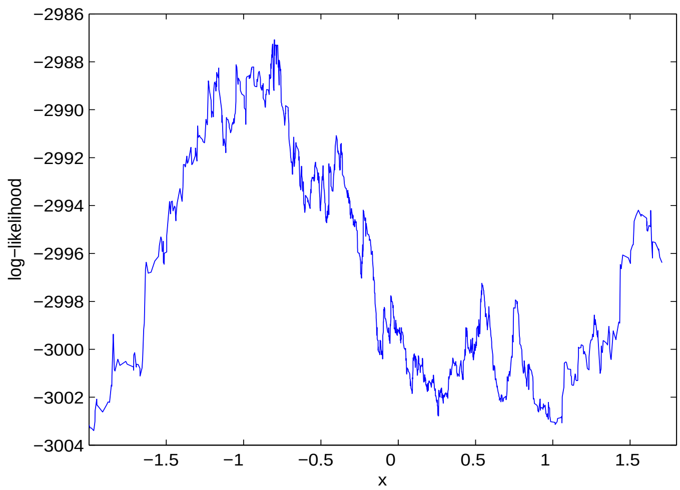
Figure 4. Plots for PX1



(a) Daily Return



(b) Realized Volatility



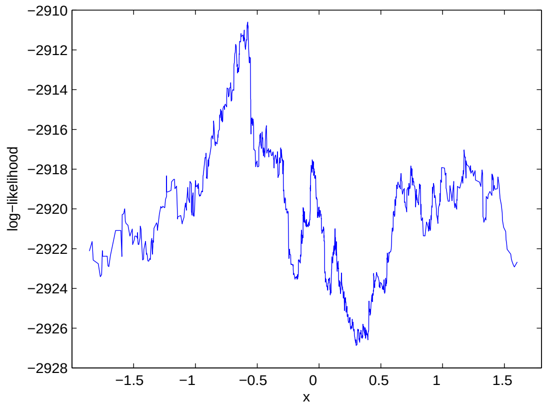
(c) Log-Likelihoods across Different Thresholds

Table 10. Empirical Results for S&P 500 with Different Frequencies

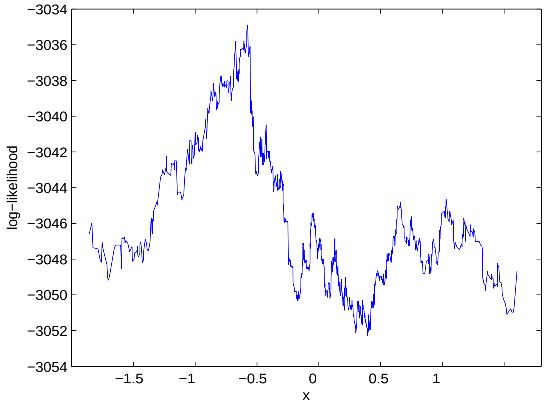
	10-min		**15-min**		**30-min**	
	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$	$\gamma = 0$	$\gamma = \gamma^*$
λ_0	-0.2627	-0.1860	-0.3136	-0.2047	-0.4500	-0.3098
	(0.0320)	(0.0356)	(0.0357)	(0.0389)	(0.0445)	(0.0521)
λ_1	-0.3027	-0.3371	-0.3772	-0.4193	-0.5598	-0.5983
	(0.0340)	(0.0289)	(0.0378)	(0.0321)	(0.0448)	(0.0380)
α_0	0.7278	0.7360	0.6871	0.6900	0.5840	0.5764
	(0.0266)	(0.0368)	(0.0279)	(0.0357)	(0.0313)	(0.0412)
α_1	0.7113	0.6881	0.6578	0.6336	0.5480	0.5288
	(0.0235)	(0.0206)	(0.0257)	(0.0224)	(0.0283)	(0.0246)
ρ_0	0.0108	0.1100	0.0062	0.0765	-0.0091	0.0595
	(0.0323)	(0.0487)	(0.0329)	(0.0504)	(0.0318)	(0.0473)
ρ_1	-0.1710	-0.1431	-0.1652	-0.1350	-0.1324	-0.1176
	(0.0280)	(0.0237)	(0.0270)	(0.0230)	(0.0277)	(0.0231)
σ_{v_0}	0.5766	0.5415	0.6292	0.5736	0.7648	0.7225
	(0.0153)	(0.0234)	(0.0171)	(0.0233)	(0.0212)	(0.0306)
σ_{v_1}	0.5971	0.5963	0.6511	0.6529	0.7877	0.7831
	(0.0143)	(0.0116)	(0.0161)	(0.0134)	(0.0190)	(0.0157)
γ	–	-0.5771	–	-0.5756	–	-0.5756
Log-L	-2919.9	-2910.6	-3047.1	-3034.9	-3358.4	-3344.1

The numbers in parenthesis are the standard errors. For each frequency column, $\gamma = 0$ stands for the model with a fixed threshold value at 0, while $\gamma = \gamma*$ stands for the model with a flexible threshold. The estimates under ($\gamma = \gamma*$) column are taken from the optimization under the "optimal" threshold.

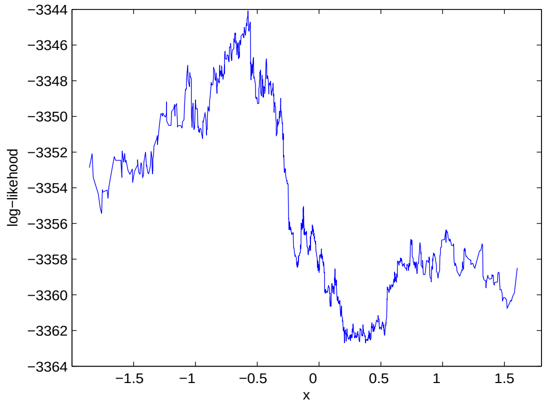
Figure 5. Plots of Log-Likelihoods across Different Thresholds at Different Frequencies for S&P 500



(a) 10-min



(b) 15-min



(c) 30-min

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