Trading Dynamics in Decentralized Markets with Adverse Selection

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Abstract

We study a dynamic, decentralized market environment with asymmetric information and interdependent values between buyers and sellers, and characterize the complete set of equilibria. The model delivers a stark relationship between the severity of the *information frictions* and *market liquidity*. We use this framework to understand how asymmetric information has contributed to the "frozen" credit market at the core of the current financial crisis, and to characterize optimal policy responses to this market failure.

1   Introduction

A central problem in the current financial crisis has been the inability of financial institutions to sell illiquid assets on their balance sheets. More specifically, banks holding large amounts of structured asset-backed securities, such as collateralized debt obligations and credit default swaps, have been mostly unable to find buyers for these assets. This "frozen" market has posed perhaps the greatest risk to the economy as a whole; if financial institutions can not acquire liquid assets (e.g. cash) in exchange for these illiquid assets, they can not make loans. As a result, consumers have more difficulty buying cars and homes, and businesses cannot acquire the financing they need for new investment. This, in turn, can lead to a further decrease in asset prices and a decline in economic growth. Given the danger associated with this downward spiral, the task of identifying the underlying frictions in this market, and understanding the inefficiencies introduced by these frictions, is of crucial importance. Without such an understanding, market participants remain unsure of how this market will behave in the future, and policymakers remain unsure of the optimal form of intervention.

While one could point to a number of potential reasons that trade in this market has broken down, many believe that the primary cause was *asymmetric information*. The story is simple: at the onset of the financial crisis, it became apparent that many assets being held by financial institutions were worth considerably less than had been previously claimed; they were of low quality or, in the language of Akerlof (1970), they were "lemons." Of course, these financial institutions also held assets of higher quality, whose fundamental value (though difficult to discern) was likely at or near pre-crisis evaluations. However, as these assets tend to be relatively complex, it was quite difficult to differentiate high quality from low. Thus the market had many of the basic ingredients of Akerlof's classic "market for lemons": sellers possessed assets that were heterogeneous in quality, and they were more informed about the quality of their assets than potential buyers. The most basic theory would predict that, in this type of environment, trade can break down completely.

However, there are several important features of this particular market that are not consistent with the assumptions typically embedded in existing models of markets with

asymmetric information. For one, the market is *decentralized*; in contrast to the standard competitive paradigm, where the law of one price prevails, buyers and sellers in this market typically negotiate bilaterally. Therefore, a model of this market must allow the exchange of different quality assets to take place at potentially different prices. Moreover, the market is inherently *dynamic* and *non-stationary*; any serious analysis has to consider the manner in which the composition of assets in the market evolves over time, and how this affects both prices and the incentive of market participants to delay trade.

The purpose of this paper is to develop a rigorous economic model that captures the important features of the market discussed above, and to use this model as a laboratory for understanding the effects of informational asymmetries on the patterns of trade in dynamic, decentralized market settings. To be more specific, we consider an discrete time, infinite horizon, one-time entry model with a continuum of buyers and sellers. Sellers each possess a single good of heterogeneous quality (high or low), and this quality is private information. In each period, buyers and sellers are randomly matched, and buyers make a price offer chosen from an exogenously specified set of prices. The parameters are such that there are strictly positive gains from trade in every match. If a seller accepts the buyer's offer, trade ensues and the pair exits; if the seller rejects, the pair remain in the market and are randomly matched again the following period. Finally, we assume that agents are subject to stochastic discount factor shocks in each period, which we interpret as liquidity shocks across agents and over time.[1]

Within this environment, we address a variety of questions that are relevant in the current financial crisis. The first of these questions are positive: *Will this market eventually clear and, if so, how long will it take? How does this length of time depend on the initial composition of high- and low-quality assets? What are the welfare costs associated with buyers being imperfectly informed?* We will study these questions when the degree of informational asymmetry is exogenous, and when it is endogenous; i.e. when financial institutions can choose the quality of the assets they hold, and when potential investors can choose the degree to which they are uninformed about the quality of an asset before engaging in bilateral

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[1]This assumption also allows us to focus on pure strategies.

negotiations.

Then we will turn to normative questions: *Can government intervention increase welfare? If so, what is the optimal policy?* This last point is particularly important in light of the variety of policy responses that have been either proposed or implemented since the financial crisis began. For example, one proposed policy has been for the federal government to buy assets directly from the sellers. An alternative, that has been implemented recently, is to essentially subsidize private-sector buyers to purchase assets from sellers. An important, open question that we intend to address is whether one of these policies implies larger efficiency gains than the other.

1.1 Related Literature

Our work builds on the literature that studies dynamic, decentralized markets with asymmetric information and interdependent values. The primary focus of this literature has been to determine what happens to equilibria in a decentralized environment as market frictions vanish.[2] See Inderst (2005) and Moreno and Wooders (2009) for a steady-state analysis of this issue, and Moreno and Wooders (2002) and Blouin (2003) for analysis of this issue in a one-time entry model. Janssen and Roy (2002) also study a dynamic environment with asymmetric information and interdependent values; however, they assume is assumed takes place in a sequence of Walrasian markets. Though the framework we develop shares certain features in common with several of these papers, the focus will be quite different. We are interested in studying the relationship between information frictions and market liquidity, and the manner in which both market participants and policymakers can respond to overcome these frictions.

Our paper, and those discussed above, are also closely related to the literature that studies sequential bargaining between a single seller and a single buyer in the presence of

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[2]Note that this was an exercise first conducted in a perfect information setting by Rubinstein and Wolinsky (1985) and Gale (1986a, 1986b). A parallel literature has emerged that studies dynamic, decentralized markets with imperfect information and *private values*; see, for example, Satterthwaite and Shneyerov (2007) and the references therein.

asymmetric information.[3] Most relevant to the current project is the work of Vincent (1989), Evans (1989), and Deneckere and Liang (2006), who study the dynamic bargaining game in which a seller has private information about the quality of her good, a buyer makes offers in each period, and the buyer's valuation of the good is correlated with the seller's valuation. Equilibria in this environment tend to have the property that buyers use time to screen the different types of sellers: initially buyers will make low offers that only very low type sellers would accept.[4] If the seller rejects such an offer, the buyer learns that the seller is not a very low type, and updates his posterior accordingly. In the following period, his offer increases, and so on.

This notion of using price dispersion over time to overcome the problem of adverse selection is central to our work, as well as the majority of papers cited above.[5] What is different about the market setting we consider, as opposed to the single buyer/seller setting considered in much of the bargaining literature, is that complementarities can arise in a market setting between e.g. a buyer and other buyers. For example, in our setting multiple equilibria can arise, precisely because a single buyer will have greater incentive to delay trade if other buyers are doing the same. This type of complementarity between agents on the same side of the market is not present in an environment where there is only one agent on each side of the market; this is why there is generally a unique sequential equilibrium in the environments studied by Vincent (1989), Evans (1989), and Deneckere and Liang (2006).

Finally, this paper also adds to the class of models that provide a theory of endogenous market liquidity based on information frictions. In one strand of this literature, which assumes that trade takes place in a centralized competitive market, the pooling equilibrium price decreases as the number of lemons increases, thus decreasing a seller's ability to exchange their good for cash. In this sense, a lemons problem reduces liquidity; see Eisfeldt (2004) for an early contribution in this literature, and Kurlat (2009) for a more recent application. While the assumption of a competitive market tends to buy tractability, there are

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[3]Seminal contributions in this literature include Fudenberg et al. (1985) and Gul et al. (1986), among others.

[4]A "low type seller" is a seller with a good of very low quality.

[5]This basic idea goes back to, at least, Wilson (1980).

several issues that we want to address that cannot be framed within the context of such a model. For example, it would be conceptually difficult to think about a market in which some (but not all) buyers could distinguish high quality from low within the context of a centralized, competitive market. Thus, such a model would not be appropriate for analyzing the incentives of buyers to acquire information about the quality of an asset; here, the assumption of bilateral trade would be more appropriate. More generally, there are a variety of mechanisms that one could allow for in a model of bilateral exchange that could be useful in easing the frictions implied by asymmetric information. Many of these mechanisms have been explored in the search-theoretic literature on the micro-foundations of money; Rocheteau (2009) provides an excellent survey.[6]

2   The Model

Time is discrete, and begins in period $t = 0$. There is an equal measure of infinitely lived buyers and sellers, which we normalize to one. At $t = 0$, each seller possesses a single, indivisible asset, which is either of high (H) quality or low (L) quality. The fraction of sellers with a high quality asset at $t = 0$ is denoted by $q_0 \in (0, 1)$. We describe below the payoffs to a buyer and a seller from each type of asset.

In each period, each agent's discount factor δ is drawn from a continuous and strictly increasing c.d.f. F with support $[0, \overline{\delta}]$, where $\overline{\delta} < 1$. These draws are i.i.d. across both agents and time. This is meant to capture the idea that buyers and sellers have different liquidity needs at different times. At a given time, some sellers may need to sell their asset urgently, while others may be more patient. Likewise, at a given time some buyers may desire consumption urgently, while others may be more patient. Across time, each individual agent may be more or less patient in any given period.[7] The assumption that F is strictly increasing

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[6]A different strand of this literature, pioneered by Glosten and Milgrom (1985) and Kyle (1985), also uses informational asymmetries to generate differences in liquidity by focusing on the problem of a market-maker, and treating the size of the bid-ask spread as a measure of liquidity.

[7]Note that all types of agents draw their discount factors from the same distribution F. Though this is non–essential, we think that it is reasonable. For a deeper look at the use of random discount factors, see Higashi et al. (2009).

rules out mass points in the distribution of discount factors. An asset of quality $j \in \{L, H\}$ yields flow utility y_j to a seller in each period that he holds the asset. It will be convenient to denote the present discounted lifetime value of a type $j \in \{L, H\}$ asset to a seller, computed before the seller draws his discount factor, by c_j, where

$$c_j = \frac{y_j}{1 - \mathbb{E}[\delta]}, \tag{1}$$

with $\mathbb{E}[\delta] = \int \delta dF(\delta) < 1$. We normalize y_L to zero, so that $c_L = 0$. When a buyer purchases the asset, we assume that he receives instantaneous utility u_j.[8] We assume that

$$u_H > y_H + \bar{\delta}c_H > u_L > c_L = 0. \tag{2}$$

The assumptions that $u_H > y_H + \bar{\delta}c_H$ and $u_L > 0$ assure us that there are gains from trade in every match.[9] The assumption that $y_H + \bar{\delta}c_H > u_L$ generates the lemons problem, as the price that buyers are willing to pay for a low quality asset would not be accepted by a sufficiently patient high quality seller.

In every period, after the agents draw their discount factors, buyers and sellers are randomly and anonymously matched in pairs. Discount factors and the quality of the seller's asset are private information. Once matched, the buyer can offer one of two prices, which are fixed exogenously: a high price $p_h \in (y_H + \bar{\delta}c_H, u_H)$ or a low price $p_\ell \in (0, u_L)$.[10] The seller can accept or reject. If a seller accepts, trade ensues and the pair exits the market; there is no entry by additional buyers and sellers. If a seller rejects, no trade occurs and the

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[8]Since buyers exit the market upon trading, it is easiest to model this as an instantaneous payoff. One could define u_j as the expected discounted lifetime value of flow payoffs to a buyer, but this would make the analysis more cumbersome without providing any additional insights. More precisely, if buyers receive flow payoffs $y_j^B > y_j$ and we define $u_j = y_j^B + \delta y_j^B / (1 - \mathbb{E}[\delta])$, since buyers have heterogeneous discount factors, they would be heterogeneous with respect to u_j as well. The current formulation allows for sellers to receive flow payments while they own the asset, without introducing any additional heterogeneity in the buyers' payoffs.

[9]As in Duffie et al. (2005), our preference specification is such that buyers and sellers receive different levels of utility from holding a particular asset. This can arise for a multitude or reasons: agents can have different levels of risk aversion, financing costs, regulatory requirements, or hedging needs. In addition, the correlation of endowments with asset returns may differ across agents. The current formulation is a reduced-form representation of such differences; see Duffie et al. (2007), Vayanos and Weill (2008), and Gârleanu (2009).

[10]Exogenous prices in these types of models have been used extensively; see, for example, Wolinsky (1990) and Blouin and Serrano (2001).

pair remains in the market. This ensures that there is always an equal measure of buyers and sellers. We assume that $u_H - p_h > u_L - p_\ell$, so that a buyer would choose to transact with a type H seller if he could choose. We also assume that

$$y_H + \overline{\delta} p_h \leq p_h. \tag{3}$$

As it turns out, (3) implies sellers accept an offer of p_h regardless of their discount factors.

The history for a buyer is the set of all of his past discount factors and (rejected) price offers. However, a buyer has no reason to condition behavior on his past history: this history is private information, discount factors are i.i.d., and the probability that he meets his current trading partner in the future is zero, as there is a continuum of agents. Moreover, since there is no aggregate uncertainty, the buyer's history of past offers is not helpful in learning any information about the aggregate state. Thus, a pure strategy for a buyer is a sequence $\mathbf{p} = \{p_t\}_{t=0}^{\infty}$, with $p_t : [0, \delta] \rightarrow \{p_\ell, p_h\}$ measurable for all $t \geq 0$, such that $p_t(\delta)$ is the price the buyer offers in period t if he is in the market in this period and his discount factor is δ.

A history for a seller is the set of all of his past discount factors and all price offers that he has rejected. The same argument as above implies that a seller has no reason to condition behavior on his past history. Thus, a pure strategy for a type j seller (i.e. a seller with a type $j \in \{L, H\}$ asset) is a sequence $\mathbf{a}_j = \{a_t^j\}_{t=0}^{\infty}$, with $a_t^j : [0, \overline{\delta}] \times \{p_\ell, p_h\} \rightarrow \{0, 1\}$ measurable for all $t \geq 0$, such that $a_t^j(\delta, p)$ is the seller's acceptance decision in period t as a function of his discount factor and the price offer he receives. We let $a_t^j(\delta, p) = 0$ denote the seller's decision to reject and $a_t^j(\delta, p) = 1$ denote the seller's decision to accept.

We consider symmetric pure–strategy equilibria. A strategy profile can then be described by a list $\sigma = (\mathbf{p}, \mathbf{a}_L, \mathbf{a}_H)$. In order to define equilibria we need to specify what happens when there is a zero measure of agents remaining in the market; more specifically, when all remaining agents trade and exit the market in the current period, we must specify the (expected) payoff to an individual should he choose a strategy that results in *not* trading. In order to avoid imposing ad hoc assumptions, we adopt the following procedure for computing payoffs.

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Suppose that, in every period t, the probability an agent gets the opportunity to trade is $\alpha \in (0, 1]$, and that this probability is independent of his discount factor. Thus, in every period t, a fraction $\alpha \in (0, 1]$ of the buyers and sellers in the market are matched in pairs, and the remainder do not get the opportunity to trade. The definition of strategies when $\alpha < 1$ is the same as when $\alpha = 1$.[11] However, when $\alpha \in (0, 1)$, in every period t there is a positive mass of agents who have not traded.

Let us denote by $V_t^j(\mathbf{a}|\sigma, \alpha)$ the expected payoff to a seller of type $j \in \{L, H\}$ who is in the market in period t following strategy $\mathbf{a}$, given the strategy profile σ for all other agents. The payoff V_t^j is computed before the seller gets the draw for his discount factor and learns whether he can trade or not. For $\alpha \in (0, 1)$, V_t^j is well–defined for all $t \geq 0$, and satisfies the following recursion:

$$
V_t^j(\mathbf{a}|\sigma, \alpha) = (1 - \alpha) \int \left[y_j + \delta V_{t+1}^j(\mathbf{a}|\sigma, \alpha) \right] dF(\delta)
$$
$$
+ \alpha \sum_{i \in \{\ell, h\}} \xi_t(p_i) \int \left\{ a_t^j(\delta, p_i) p_i + \left[1 - a_t^j(\delta, p_i) \right] \left[y_j + \delta V_{t+1}^j(\mathbf{a}|\sigma, \alpha) \right] \right\} dF(\delta), \qquad (4)
$$

where $\xi_t(p)$ is the fraction of buyers who offer $p \in \{p_\ell, p_h\}$ in period t. Note that $\xi_t(p)$ is the probability that a buyer who can trade draws a discount factor δ with $p_t(\delta) = p$. In words, with probability $1 - \alpha$ a seller is not matched in period t, enjoys flow utility y_j, and proceeds to period $t+1$. With probability α the seller is matched, in which case he either accepts the buyer's offer and exits the market, or rejects the offer and stays in the market.

Similarly, we denote by $V_t^B(\mathbf{p}|\sigma, \alpha)$ the expected payoff to a buyer who is in the market in period t following strategy $\mathbf{p}$, given the strategy profile σ for all other agents. The payoff V_t^B is also computed before the buyer gets the draw for his discount factor and learns whether he can trade or not. Again, for $\alpha \in (0, 1)$, V_t^B is well–defined for all $t \geq 0$, and satisfies the

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[11]Now a history for a player also includes the periods in which he was able to trade; for the same reasons given above, a player has no motive to condition his behavior on this information.

following recursion:

$$V_t^B(\mathbf{p}|\sigma, \alpha)$$

$$= (1-\alpha)\delta V_{t+1}^B(\mathbf{p}|\sigma,\alpha) + \alpha\sum_{i\in\{\ell,h\}}\xi_t(p_i)\left\{q_t A_t^H(p_i)[u_H - p_i] + (1-q_t)A_t^L(p_i)[u_L - p_i]\right.$$

$$+ [1 - q_t A_t^H(p_i) - (1-q_t)A_t^L(p_i)]\delta V_{t+1}^B(\mathbf{p}|\sigma,\alpha)\}$$

$$= \delta V_{t+1}^B(\mathbf{p}|\sigma,\alpha) + \alpha\sum_{i\in\{\ell,h\}}\xi_t(p_i)\left\{q_t A_t^H(p_i)\left[u_H - p_i - \delta V_{t+1}^B(\mathbf{p}|\sigma,\alpha)\right]\right.$$

$$+ (1-q_t)A_t^L(p_i)\left[u_L - p_i - \delta V_{t+1}^B(\mathbf{p}|\sigma,\alpha)\right]\},\qquad(5)$$

where q_t is the fraction of H sellers in the market in period t and $A_t^j(p)$ is the likelihood that a seller of type $j\in\{L,H\}$ in the market in period t accepts an offer $p\in\{p_\ell, p_h\}$, i.e.

$$A_t^j(p) = \int a_t^j(\delta, p)dF(\delta).$$

In words, with probability $1-\alpha$ a buyer is not matched in period t, enjoys no utility, and proceeds to period $t+1$. With probability α a buyer is matched, in which case his partner either accepts his offer (and the buyer exits the market) or rejects his offer (and the buyer stays in the market).

Standard dynamic programming arguments show that for each σ, $\mathbf{a}$, $\mathbf{p}$, and $t\geq 0$, the payoffs $V_t^j(\mathbf{a}|\sigma, \alpha)$ and $V_t^B(\mathbf{p}|\sigma, \alpha)$ are continuous functions of α in the interval $(0, 1)$. Hence, the limits of both $V_t^j(\mathbf{a}|\sigma, \alpha)$ and $V_t^B(\mathbf{p}|\sigma, \alpha)$ are well–defined as α converges to one.

Definition 1. *Let σ be the strategy profile under play. The payoff to a buyer who is in the market in period t following the strategy $\mathbf{p}$ is $V_t^B(\mathbf{p}|\sigma) = \lim_{\alpha\to 1}V_t^B(\mathbf{p}|\sigma, \alpha)$. The payoff to a seller of type $j\in\{L,H\}$ who is in the market in period t following the strategy $\mathbf{a}$ is $V_t^j(\mathbf{a}|\sigma) = \lim_{\alpha\to 1}V_t^j(\mathbf{a}|\sigma, \alpha)$.*

We can now define equilibria in our environment.

Definition 2. *The strategy profile $\sigma^* = \left(\mathbf{p}^* = \{p_t^*\}, \mathbf{a}_L^* = \{a_t^{L*}\}, \mathbf{a}_H^* = \{a_t^{H*}\}\right)$ is an equilibrium if for each $t\geq 0$ and $j\in\{L,H\}$, we have that:*
(i) $p_t^*(\delta)$ maximizes

$$q_t A_t^H(p)\left[u_H - p - \delta V_{t+1}^B(\sigma^*)\right] + (1-q_t)A_t^L(p)\left[u_L - p - \delta V_{t+1}^B(\sigma^*)\right]$$

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for all $\delta \in [0, \bar{\delta}]$, where $V_t^B(\sigma^*) = V_t^B(\mathbf{p}^*|\sigma^*)$;

(ii) For each $p \in \{p_\ell, p_h\}$, $a_t^{j*}(\delta, p) = 1$ if, and only if,

$$p \geq y_j + \delta V_{t+1}^j(\sigma^*),\tag{6}$$

where $V_t^j(\sigma^*) = V_t^j(\mathbf{a}_j^*|\sigma^*)$.

Note that (6) implies that in equilibrium a seller accepts any offer that he is indifferent between accepting and rejecting. This is without loss since F has no mass points, and so the probability that a seller is ever indifferent between accepting and rejecting is zero.

3 Properties of Equilibria

For a given strategy profile, we say that the market "clears" in period t if all sellers remaining in the market accept the price offer made by the buyers. In this section we establish that the market clears in finite time in every equilibrium and that, in every period before the market clears, the fraction of type H sellers in the population strictly increases.

We first show that the market clears in period t if, and only if, all buyers in the market offer p_h. From (4) and (6), we have that for any equilibrium σ^*,

$$V_t^j(\sigma^*) = \sum_{i \in \{\ell, h\}} \xi_t(p_i) \int \max\left\{p_i, y_j + \delta V_{t+1}^j(\sigma^*)\right\} dF(\delta)$$

for all $t \geq 0$. Given (3), it should be obvious that $V_t^j(\sigma^*) \leq p_h$ for all $t \geq 0$, so that all sellers in the market always accept an offer of p_h. Thus, the market clears in period t if all buyers offer p_h. Now observe that since a seller has the option of always rejecting any offer he receives, $V_t^j(\sigma^*) \geq c_j$ for all $t \geq 0$. Thus, letting $\underline{\delta} = (u_L - y_H)/c_H$, we have

$$y_H + \underline{\delta} V_{t+1}^H(\sigma^*) \geq y_H + \underline{\delta} c_H = u_L > p_\ell.$$

Therefore, a type H seller with discount factor $\delta \geq \underline{\delta}$ always rejects an offer of p_ℓ. Since $\underline{\delta} < \bar{\delta}$ by (2), there is always a strictly positive mass of such sellers. Hence, the market does not clear in period t if a positive mass of buyers offers p_ℓ.

So, in any equilibrium σ^* the market clears in the first period in which all remaining buyers offer p_h, which we denote by $T = T(\sigma^*)$; we set $T = \infty$ if in every period t a positive mass of buyers offers p_ℓ. For all $t < T$, a positive mass of buyers offer p_ℓ, and the fraction of type j sellers who accept p_ℓ in t is $F\left[(p_\ell - y_j)/V_{t+1}^j(\sigma^*)\right]$. Since all sellers who receive an offer of p_h accept the offer and exit the market, we then have that

$$q_{t+1} = \frac{q_t \left[1 - F\left(\frac{p_\ell - y_H}{V_{t+1}^H(\sigma^*)}\right)\right]}{q_t \left[1 - F\left(\frac{p_\ell - y_H}{V_{t+1}^H(\sigma^*)}\right)\right] + (1 - q_t)\left[1 - F\left(\frac{p_\ell}{V_{t+1}^L(\sigma^*)}\right)\right]}. \tag{7}$$

Now notice that an option for a type H seller is to replicate the behavior of a type L seller. Since $y_H > y_L$, we then have that $V_t^H(\sigma^*) \geq V_t^L(\sigma^*)$ for all $t \geq 0$.[12] Hence, since $F((p_\ell - y_H)/V_{t+1}^H(\sigma^*)) \leq F(\underline{\delta}) < 1$ and F is strictly increasing in its support,

$$F\left(\frac{p_\ell - y_H}{V_{t+1}^H(\sigma^*)}\right) < F\left(\frac{p_\ell}{V_{t+1}^L(\sigma^*)}\right) \tag{8}$$

for all $t \geq 0$; that is, whenever buyers offer p_ℓ, the fraction of type L sellers who accept this offer is larger than the fraction of type H sellers who accept the same offer. This is a fundamental feature of this environment: type H sellers are de facto more patient than type L sellers because their flow payoff from *not* trading is larger. Looking at the law of motion for $\{q_t\}_{t=0}^T$, equation (7), a consequence of this fact is that the fraction of type H sellers in the population increases strictly over time before the market clears.

Proposition 1. *Let $q_0 \in (0, 1)$. In any equilibrium, the market clears in finite time.*

The proof of Proposition 1 is in the Appendix. The intuition for this result is as follows. Suppose, by contradiction, that there is an equilibrium in which, in every period t, the mass of buyers who offer p_ℓ is positive. We know the sequence $\{q_t\}_{t=0}^\infty$ is strictly increasing, and thus convergent (since it is bounded above). If we denote the limit of this sequence by q_∞, it cannot be that $q_\infty = 1$. Indeed, the payoff from offering p_h converges to $u_H - p_h$ as the fraction of type H sellers in the market converges to one. Since the highest payoff possible for

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[12]Indeed, $y_H > y_L$ implies that $V_t^H(\mathbf{a}_L|\sigma^*, \alpha) > V_t^L(\mathbf{a}_L|\sigma^*, \alpha)$ for all $\alpha \in (0, 1)$ and $t \geq 0$. Taking the limit as α converges to one implies the desired result.

a buyer is $u_H - p_h$, and buyers discount the future ($\bar{\delta} < 1$), there is a fraction $q^* < 1$ of type H sellers in the market above which all buyers find it optimal to offer p_h. However, $q_\infty < 1$ implies that eventually the fraction of type H sellers who accept an offer of p_ℓ is arbitrarily close to the fraction of type L sellers who accept the same offer; roughly speaking, as $t \to \infty$, all sellers with discount rate $\delta \in [0, \bar{\delta}]$ behave the same independently of the quality of their asset. This, however, is not possible given (8).

In what follows, we will present the case where the lemons problems is most severe by assuming that $p_\ell < y_H$, so that *no* type H seller accepts p_ℓ. Relaxing this assumption does not substantively change any of our results. We also assume that

$$\bar{\delta} \leq (u_L - p_\ell)/(u_H - p_h), \tag{9}$$

which ensures that the buyer would never prefer to simply not make an offer.

4   Characterizing Equilibria

In this section we provide a complete characterization of the set of equilibria. We start with a characterization of the equilibria in which the market clears immediately (i.e. at $t = 0$). We refer to such equilibria as 0–step equilibria. In general, we refer to equilibria in which the market clears in period k, with $k \geq 0$, as k–step equilibria.

0–step equilibria

Denote by $\pi_i^B(q, \delta, v_L, v_H, v_B)$ the payoff to a buyer who offers p_i, with $i \in \{\ell, h\}$, when: (i) the fraction of type H sellers in the market is $q \in (0, 1)$; (ii) the buyer's discount factor is δ; (iii) the continuation payoff to a seller of type j who chooses not to trade is $v_j \geq c_j$; and (iv) the continuation payoff to the buyer should he not trade is $v_B \in (0, u_H - p_h]$.[13] We know that sellers always accept an offer of p_h. So,

$$\pi_h^B(q, \delta, v_L, v_H, v_B) \equiv \pi_h^B(q) = q[u_H - p_h] + (1 - q)[u_L - p_h].$$

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[13]Note that in any equilibrium σ^*, it must be that $V_t^B(\sigma^*) > 0$ for all $t \geq 0$. The reason is that a buyer always has the option to offer p_ℓ as long as there is a positive mass of type L sellers in the market, and the expected payoff from doing so is strictly positive: since $V_t^L(\sigma^*) \leq p_h$, the probability that a type L seller accepts p_ℓ is at least $F(p_\ell/p_h) > 0$, in which case the buyer's payoff is $u_L - p_\ell > 0$.

We also know that a type H seller always rejects an offer of p_ℓ. So,

$$\pi_\ell^B (q, \delta, v_L, v_H, v_B) \equiv \pi_\ell^B (q, \delta, v_L, v_B)$$
$$= (1-q)F\left(\frac{p_\ell}{v_L}\right)[u_L - p_\ell] + \left\{q + (1-q)\left[1 - F\left(\frac{p_\ell}{v_L}\right)\right]\right\}\delta v_B, \quad (10)$$

where $F(p_\ell/v_L)$ is the fraction of type L sellers who accept p_ℓ. Note that $\pi_\ell^B (q, \delta, v_L, v_B)$ is strictly increasing in v_B. Since $v_B > 0$, $\pi_\ell^B (q, \delta, v_L, v_B)$ is also positive and strictly increasing in δ. Moreover, since $v_B \leq u_H - p_h$, (9) implies that $\delta v_B \leq u_L - p_\ell$, and so $\pi_\ell^B (q, \delta, v_L, v_B)$ is non–increasing in v_L.

Consider now the candidate 0–step equilibrium σ^0 in which, in every $t \geq 0$, $p_t(\delta) = p_h$ for all $\delta \in [0, \overline{\delta}]$, and type j sellers accept an offer p if, and only if, $\delta \leq (p - y_j)/p_h$. It is immediate to see that for all $t \geq 0$,

$$V_t^B(\sigma^0) = \pi_h^B(q_0) \equiv v_B^0(q_0) \quad \text{and} \quad V_t^j(\sigma^0) = p_h \equiv v_j^0.$$

Note that we have introduced the following notation: in a 0–step equilibrium, the expected payoff to a buyer given q_0 is $v_B^0(q_0)$ and the expected payoff to a type j seller is v_j^0, which is independent of q_0.[14] In what follows, we will denote by $v_B^k(\cdot)$ and $v_j^k(\cdot)$ the ex–ante expected payoffs to a buyer and type j seller, respectively, in period $t = 0$ of a k–step equilibrium (before agents draw discount factors).[15]

The strategy profile σ^0 is an equilibrium only if $v_B^0(q_0) > 0$ (for otherwise $V_0^B(\sigma^0) \leq 0$, which cannot happen in equilibrium) and in every period t all buyers find it optimal to offer p_h, which is true as long as

$$\pi_h^B(q_0) \geq \pi_\ell^B \left(q_0, \delta, v_L^0, v_B^0(q_0)\right) \quad (11)$$

for all $\delta \in [0, \overline{\delta}]$. Since $v_B^0(q_0) > 0$ implies that $\pi_\ell^B(q_0, \delta, v_L^0, v_B^0(q_0))$ is strictly increasing in δ, a necessary and sufficient condition for (11) is that

$$\pi_h^B(q_0) \geq \pi_\ell^B \left(q_0, \overline{\delta}, v_L^0, v_B^0(q_0)\right). \quad (12)$$

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[14]In general, we will adopt the convention that a numerical subscript refers to a particular time period, while a numerical superscript refers to the number of periods it takes for the market to clear in equilibrium. In addition, we will use lower case v to denote equilibrium payoffs.

[15]Note that $v_B^k(\cdot)$ and $v_j^k(\cdot)$ are also the ex–ante expected payoffs to a buyer and type j seller, respectively, in period $t > 0$ of a $(k+t)$–step equilibrium; what's important is that this payoff is calculated k steps before the market clears, and before the agent draws his period t discount factor.

Condition (12) is slightly more subtle than it may appear. The left side is clearly the payoff to a buyer from offering p_h. The right side is the payoff to a buyer from offering p_ℓ in the current period and p_h in the ensuing period, conditional on all other buyers offering p_h in the current period. There are two things to notice. First, that when all other buyers offer p_h and exit the market, the payoff to a buyer who remains in the market and offers p_h in the next period is $v_B^0(q_0)$. This comes directly from our refinement for computing payoffs when the mass of agents in the market is zero. Indeed, under σ^0, when the fraction of buyers and sellers who are matched in each period is $\alpha < 1$, all buyers who get the opportunity to trade exit the market, and so the fraction of type H sellers among the sellers who remain in the market stays the same. Second, as we show in the proof of Proposition 2 below, (12) is the loosest possible constraint on q_0 that ensures that a buyer finds it optimal to offer p_h at $t = 0$ when he believes that all other buyers in the market offer p_h as well.

Proposition 2. *Let $\underline{q}^0 \in (0,1)$ denote the unique value of q satisfying (12) with equality. There exists a 0–step equilibrium if, and only if, $q_0 \geq \underline{q}^0$.*

The sketch of the proof of Proposition 2 is as follows; the details are in the Appendix. We first show that (12) is satisfied, and so σ^0 is an equilibrium if, and only if, $q \geq \underline{q}^0$. This follows from the fact that the payoff difference $\pi_h^B(q) - \pi_\ell^B\left(q, \overline{\delta}, v_L^0, v_B^0(q)\right)$ is strictly increasing and continuous in q. We then show that there is no 0–step equilibrium if $q_0 < \underline{q}^0$. Consider any strategy profile $\widetilde{\sigma}^0$ that specifies $p_0(\delta) = p_h$ for all $\delta \in [0, \overline{\delta}]$, so that markets clear at $t = 0$.[16] It is possible to show that it must be that $V_1^B(\widetilde{\sigma}^0) \geq v_B^0(q_0)$ if $\widetilde{\sigma}^0$ is to be an equilibrium. Since $V_1^L(\widetilde{\sigma}^0) \leq p_h$ and $q_0 < \underline{q}^0$ implies that $\pi_\ell^B\left(q_0, \overline{\delta}, v_L^0, v_B^0(q_0)\right) > \pi_h^B(q_0)$, we then have

$$\pi_\ell^B\left(q_0, \overline{\delta}, V_1^L(\widetilde{\sigma}^0), V_1^B(\widetilde{\sigma}^0)\right) \geq \pi_\ell^B\left(q_0, \overline{\delta}, v_L^0, v_B^0(q_0)\right) > \pi_h^B(q_0)$$

for all $q_0 < \underline{q}^0$. Thus, there exists $\delta' < \overline{\delta}$ such that it can *not* be optimal for a buyer with discount factor in $(\delta', \overline{\delta}]$ to offer p_h at $t = 0$ if $q_0 < \underline{q}^0$, so that the market clearing immediately cannot be an equilibrium outcome.

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[16]Note, however, that we haven't placed any restrictions on $p_t(\delta)$ for $t > 0$.

Notice that $q_0 u_H + (1 - q_0) u_L \geq p_h > y_H + \bar{\delta} c_H$ for any q_0 in the interval $[\underline{q}^0, 1)$, since a buyer is only willing to offer p_h if his payoff from doing so is non–negative. Hence, p_h corresponds to a market clearing price in a Walrasian equilibrium. Thus, when the lemons problem is relatively small, i.e., when the fraction of type H sellers is sufficiently large, the market behaves as if it were Walrasian.

We will now show that when the lemons problem gets severe, the market outcomes no longer resemble those of a centralized Walrasian market; instead, these markets appear more like standard decentralized search markets, in the sense that it takes time for buyers and sellers to trade, and they do so at potentially different prices. The following convention will be useful. For any strategy profile σ, let σ_+ be the strategy profile such that for all $t \geq 0$, the agents' behavior in period t is given by their behavior in period $t + 1$ under σ.

1–step equilibria

As the next step, we characterize the set of 1–step equilibria. For this, let

$$q^+ (q, v_L) = \frac{q}{q + (1 - q) [1 - F (p_\ell/v_L)]}, \tag{13}$$

with $q \in (0, 1)$. By construction, $q^+(q, v_L)$ is the fraction of type H sellers in the market in the next period if this fraction is q in the current period, a positive mass of buyers offer p_ℓ, and the continuation payoff to a type L seller in case he rejects a price offer is v_L. Since $v_L \leq p_h$, $F(p_\ell/v_L) \geq F(p_\ell/p_h) > 0$, and so $q^+(q, v_L) > q$ for all $q \in (0, 1)$. Also notice that $q^+(q, v_L)$ is strictly increasing in q if $p_\ell/v_L < \bar{\delta}$ and that $q^+(q, v_L) \equiv 1$ if $p_\ell/v_L \geq \bar{\delta}$.

Consider a strategy profile σ^1 such that a positive mass of buyers offer p_ℓ in $t = 0$ and all buyers offer p_h in $t = 1$. In order for σ^1 to be an equilibrium, it must be that (i) σ_+^1 is a 0–step equilibrium when the initial fraction of type H sellers is $q' = q^+(q_0, v_L^0)$, and (ii) a positive mass of buyers find it optimal to offer p_ℓ in $t = 0$ when the market clears in $t = 1$.[17]

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[17]It must also be the case that the type j sellers accept an offer of p in $t = 0$ if, and only if, $\delta \leq (p - y_j)/p_h$. This optimal behavior of sellers will be implicitly assumed throughout the analysis.

16

Formally, the following conditions are necessary and sufficient for σ^1 to be an equilibrium:

$$q^+(q_0, v_L^0) = q' \tag{14}$$

$$q' \geq \underline{q}^0 \tag{15}$$

$$\pi_h^B(q_0) < \pi_\ell^B\left(q_0, \overline{\delta}, v_L^0, v_B^0(q')\right). \tag{16}$$

Condition (14) is simply the law of motion for q_t from period zero to period one; notice that q' is strictly increasing in q_0. Condition (15) follows from Proposition 2. It ensures that the fraction of type H sellers in $t = 1$ is high enough for market clearing in this period to be an equilibrium outcome. Since $q' \geq \underline{q}^0$ implies that $v_B^0(q') > 0$, $\pi_\ell^B\left(q_0, \delta, v_L^0, v_B^0(q')\right)$ is strictly increasing in δ. Thus, the incentive of a buyer to offer p_ℓ in $t = 0$ when a positive mass of the other buyers also offer p_ℓ and the market clears in $t = 1$ increases with the buyer's patience. Condition (16) then ensures that a positive mass of buyers indeed find it optimal to offer p_ℓ in $t = 0$ when the strategy profile under play is σ^1; if it is optimal for the most patient buyer to offer p_h in $t = 0$, then every other buyer prefers to offer p_h as well. The proposition below fully characterizes the set of 1–step equilibria; the proof is in the Appendix.

Proposition 3. *Let $\overline{q}^1$ denote the unique value of q_0 satisfying (16) with equality and define $\underline{q}^1$ to be such that $q^+(\underline{q}^1, v_L^0) = \underline{q}^0$ if $p_\ell/v_L^0 < \overline{\delta}$ and $\underline{q}^1 = 0$ if $p_\ell/v_L^0 \geq \overline{\delta}$. Then $\underline{q}^1 < \underline{q}^0 < \overline{q}^1 < 1$ and there exists a 1–step equilibrium if, and only if, $q_0 \in [\underline{q}^1, \overline{q}^1) \cap (0, 1)$.*

In words, if $q_0 = \overline{q}^1$ then the most patient buyer is exactly indifferent between offering p_ℓ and p_h when a positive mass of the other buyers are offering p_ℓ; for any $q_0 > \overline{q}^1$ the payoff to such a buyer from immediately trading at price p_h is greater than the payoff from offering p_ℓ and not trading with positive probability, in which case the buyer trades at price p_h in the ensuing period (with a higher fraction of type H sellers in the market). When $p_\ell/v_L^0 < \overline{\delta}$, $\underline{q}^1$ is the unique value of q_0 such that if a positive mass of buyers offer p_ℓ, then the fraction of high quality sellers in the next period is $\underline{q}^0$, the minimum value of the fraction of type H sellers needed for market clearing; notice that $\underline{q}^1 > 0$ in this case. If even the most patient type L seller would rather accept an offer of p_ℓ today than wait one period for an offer of p_h, i.e., if $p_\ell/v_L^0 \geq \overline{\delta}$, then we have $\underline{q}^1 = 0$.

The fact that $\underline{q}^0 < \overline{q}^1$ implies that there are multiple equilibria when $q_0 \in [\underline{q}^0, \overline{q}^1)$. In this region, if all other buyers are offering p_h, it is optimal for an individual buyer to offer p_h. However, if a positive mass of other buyers are offering p_ℓ, the market does not clear at $t = 0$ and the payoff to trading at $t = 1$ increases, rendering it optimal for patient buyers to offer p_ℓ and incur a chance that they trade only in the next period.

The payoff to a buyer in a 1–step equilibrium is

$$v_B^1(q_0) = \int \max \left\{ \pi_h^B(q_0), \pi_\ell^B \left(q_0, \delta, v_L^0, v_B^0 \left[q^+ \left(q_0, v_L^0 \right) \right] \right) \right\} dF(\delta) \geq \pi_h^B(q_0).$$

We denote the fraction of buyers that offer p_h at $t = 0$ in a 1–step equilibrium by

$$\xi^1(q_0) = \int \mathbb{I}\{\pi_h^B(q) \geq \pi_\ell^B \left(q, \delta, v_L^0, v_B^0 \left[q^+(q, v_L^0) \right] \right)\} dF(\delta),$$

where $\mathbb{I}$ represents the indicator function. Thus, the payoff to a type L seller is

$$v_L^1(q_0) = \xi^1(q_0)p_h + (1 - \xi^1(q_0)) \int \max \left\{ p_\ell, \delta v_L^0 \right\} dF(\delta) \leq v_L^0. \tag{17}$$

In Lemma 4 in the Appendix we show that $\xi^1(q_0)$ is continuous and increasing in q_0, and it converges to one as q_0 increases to $\overline{q}^1$. Therefore, the average price, $\xi^1(q_0)p_h + [1 - \xi^1(q_0)]p_\ell$, is increasing in q_0 in the region of 1–step equilibria, and converges to p_h as q_0 converges to $\overline{q}^1$. Moreover, since $q^+(q_0, v_L^0)$ is continuous in q_0, it is easy to see that both v_B^1 and v_L^1 are continuous and increasing in q_0, and they converge to $v_B^0(\overline{q}^1)$ and v_L^0, respectively, as q_0 increases to $\overline{q}^1$. In what follows, we write $v_L^1(\overline{q}^1)$ to denote the limit of $v_L^1(q_0)$ as q_0 increases to $\overline{q}^1$.

2–step equilibria

We now provide a complete characterization of 2–step equilibria. As it turns out, the process of characterizing k–step equilibria is nearly identical for all $k \geq 2$. Thus, the methodology developed here will allow for a complete characterization of equilibria in the next part. Since, by Proposition 3, there are only 0–step and 1–step equilibria when $p_\ell/p_h \geq \overline{\delta}$, we assume that $p_\ell/p_h < \overline{\delta}$ in what follows.

Consider a strategy profile σ^2 such that a positive mass of buyers offer p_ℓ in the first two periods, and then all buyers offer p_h. By construction, the fraction of type H sellers in the market in $t = 1$ is $q' = q^+(q_0, V_1^L(\sigma^2)) > q_0$. So, in order for σ^2 to be an equilibrium, it must be that (i) σ_+^2 is a 1–step equilibrium when the initial fraction of type H sellers is q', and (ii) a positive mass of buyers find it optimal to offer p_ℓ in $t = 0$ when behavior from $t = 1$ is given by σ_+^2. Hence, the following three conditions are necessary and sufficient for σ^2 to be an equilibrium:

$$q^+ \left(q_0, v_L^1(q')\right) = q' \tag{18}$$

$$q' \in \left[\underline{q}^1, \overline{q}^1\right) ; \tag{19}$$

$$\pi_h^B(q_0) < \pi_\ell^B \left(q_0, \overline{\delta}, v_L^1(q'), v_B^1(q')\right) . \tag{20}$$

Though conditions (18) to (20) appear very similar to conditions (14) to (16), there are two differences that warrant discussion. First, as we show in the Appendix, (18) and (19) imply (20). Intuitively, we show that the incentive of the most patient buyer to choose p_ℓ in $t = 0$ is even greater than his incentive to choose p_ℓ in $t = 1$, when the fraction of type H sellers in the market is $q' > q_0$. Therefore, if the most patient buyer strictly prefers to choose p_ℓ in $t = 1$, which is guaranteed by (19), then he will also strictly prefer to offer p_ℓ at $t = 0$ and (20) will be satisfied. Hence, (18) and (19) are necessary and sufficient conditions for a 2–step equilibrium. Second, as is clear from (18), q' is no longer defined by a simple function, as in (14), but rather q' is the solution to a fixed point problem: if the type L sellers expect their continuation payoff to be that of a 1–step equilibrium where the initial fraction of type H sellers is q', then the fraction of type L sellers who accept an offer of p_ℓ in $t = 0$ must be such that this conjecture is correct.

Let $\overline{q}^2$ be such that

$$q^+(\overline{q}^2, v_L^1(\overline{q}^1)) = \overline{q}^1. \tag{21}$$

Notice that $\overline{q}^2$ is unique and that $\overline{q}^2 > \overline{q}^1$. Indeed, since $v_L^1(\overline{q}^1) = v_L^0$, $\overline{q}^1 > \underline{q}^0$, and

$$q^+(\overline{q}^2, v_L^0) = \overline{q}^1 > \underline{q}^0 = q^+(\underline{q}^1, v_L^0),$$

19

the result that $\bar{q}^2$ is unique and $\bar{q}^2 > \underline{q}^1$ follows immediately from the fact that $q^+(q_0, v_L)$ is strictly increasing in q_0 for any v_L such that $p_\ell/v_L < \bar{\delta}$. Now let $\underline{q}^2$ be such that

$$q^+(\underline{q}^2, v_L^1(\underline{q}^1)) = \underline{q}^1 \tag{22}$$

if $p_L/v_L^1(\underline{q}^1) < \bar{\delta}$ and $\underline{q}^2 = 0$ if $p_L/v_L^1(\underline{q}^1) \geq \bar{\delta}$; $\underline{q}^2$ is unique for the same reason that $\bar{q}^2$ is. In the Appendix we show that there is $q_0 \in (0, 1)$ for which (18) and (19) are satisfied if, and only if, $q_0 \in [\underline{q}^2, \bar{q}^2) \cap (0, 1)$, and that for each q_0 in this set there is only one $q' \in [\underline{q}^1, \bar{q}^1)$ for which (18) holds. Moreover, we show that the map $Q_+^2 : q_0 \mapsto q'$ defined by (18) is continuous and strictly increasing; in words, $Q_+^2(q_0)$ is the value of q_1 in a 2–step equilibrium, given initial q_0. To summarize, we have established the following result.

Proposition 4. *Suppose that* $\bar{\delta} > p_\ell/p_h$. *There exists a 2–step equilibrium if, and only if,* $q_0 \in [\underline{q}^2, \bar{q}^2) \cap (0, 1)$. *Moreover, for each* $q_0 \in [\underline{q}^2, \bar{q}^2) \cap (0, 1)$, *there is a unique* $q' \in [\underline{q}^1, \bar{q}^1)$ *such that* q' *is the value of* q_1 *in a 2–step equilibrium when the initial fraction of type* H *sellers is* q_0, *and the map* $Q_+^2 : q_0 \mapsto q'$ *is strictly increasing and continuous.*

The payoff to a buyer in a 2–step equilibrium is

$$v_B^2(q_0) = \int \max\left\{\pi_h^B(q_0), \pi_\ell^B\left(q_0, \delta, v_L^1(Q_+^2(q_0)), v_B^1(Q_+^2(q_0))\right)\right\} dF(\delta) \geq \pi_h^B(q_0). \tag{23}$$

Denote the fraction of buyers that offer p_h at $t = 0$ in a 2–step equilibrium by

$$\xi^2(q_0) = \int \mathbb{I}\{\pi_h^B(q) \geq \pi_\ell^B\left(q, \delta, v_L^1(Q_+^2(q_0)), v_B^1(Q_+^2(q_0))\right)\} dF(\delta). \tag{24}$$

The payoff to a type L seller is then given by

$$v_L^2(q_0) = \xi^2(q_0)p_h + (1 - \xi^2(q_0)) \int \max\left\{p_\ell, \delta v_L^1(Q_+^2(q_0))\right\} dF(\delta) \leq v_L^0. \tag{25}$$

In Lemma 2 in the Appendix, we show that v_B^2 is continuous in q_0 and converges to $v_B^1(\bar{q}^2)$ as q_0 increases to $\bar{q}^2$, while v_L^2 is also continuous and increasing in q_0, and it converges to $v_L^1(\bar{q}_2)$ as q_0 increases to $\bar{q}^2$. This completes the characterization of 2–step equilibria.

A Full Characterization

We know from Proposition 4 that if $\overline{\delta} \leq p_\ell/v_L^1(\underline{q}^1)$, then there are only 0–step, 1–step, and 2–step equilibria. Let $\overline{\delta} > p_L/v_L^1(\underline{q}^1)$ and suppose, by induction, that there exist $k \geq 3$ and finite sequences $\{\underline{q}^s\}_{s=0}^{k-1}$ and $\{\overline{q}^s\}_{s=0}^{k-1}$, with $0 < \underline{q}^s < \underline{q}^{s-1} \leq \overline{q}^s$ for all $s \in \{1, \ldots, k-1\}$ and $\overline{q}^0 = 1$, such that (i) a s–step equilibrium exists if, and only if, $q_0 \in [\underline{q}^s, \overline{q}^s)$ and (ii) the payoffs $v_B^s(q_0)$ and $v_L^s(q_0)$ to buyers and type L sellers in a s–step equilibrium are uniquely defined; this is true when $k = 3$ by Propositions 2 to 4.

The same argument as in the previous part shows that the following conditions are necessary and sufficient for a k–step equilibrium to exist:

$$q^+\left(q_0, v_L^{k-1}(q')\right) = q'; \tag{26}$$

$$q' \in \left[\underline{q}^{k-1}, \overline{q}^{k-1}\right); \tag{27}$$

$$\pi_h^B(q_0) < \pi_\ell^B\left(q_0, \overline{\delta}, v_L^{k-1}(q'), v_B^{k-1}(q')\right). \tag{28}$$

Several crucial features of 2–step equilibria are true for all $k \geq 3$. First, (26) and (27) imply (28), so the necessary and sufficient conditions are given by (26) and (27). Second, if we define $\overline{q}^k$ to be such that

$$\overline{q}^k = q^+(\overline{q}^k, v_L^{k-1}(\overline{q}^{k-1}))$$

and $\underline{q}^k$ to be such that

$$\underline{q}^k = \begin{cases} 0 & \text{if} \quad p_\ell/v_L^{k-1}(\underline{q}^{k-1}) \geq \overline{\delta} \\ q^+(\underline{q}^k, v_L^{k-1}(\underline{q}^{k-1})) = \underline{q}^{k-1} & \text{if} \quad p_\ell/v_L^{k-1}(\underline{q}^{k-1}) < \overline{\delta} \end{cases},$$

then $\underline{q}^k < \underline{q}^{k-1} \leq \overline{q}^k$ and there is a k–step equilibrium if, and only if, $q_0 \in [\underline{q}^k, \overline{q}^k) \cap (0, 1)$. Moreover, for each $q_0 \in [\underline{q}^k, \overline{q}^k) \cap (0, 1)$, there is a unique $q' \in [\underline{q}^{k-1}, \overline{q}^{k-1})$ such that q' is the value of q_1 in a k–step equilibrium when the initial fraction of type H sellers is q_0, and the map $Q_+^k : q_0 \mapsto q'$ is continuous and strictly increasing.

The payoffs in a k–step equilibrium are then given by

$$v_B^k(q_0) = \int \max\left\{\pi_h^B(q_0), \pi_\ell^B\left(q_0, \delta, v_L^{k-1}(Q_+^{k-1}(q_0)), v_B^{k-1}(Q_+^{k-1}(q_0))\right)\right\} dF(\delta) \tag{29}$$

$$v_L^k(q_0) = \xi^k(q_0)p_h + (1 - \xi^k(q_0)) \int \max\left\{p_\ell, \delta v_L^{k-1}(Q_+^{k-1}(q_0))\right\} dF(\delta), \tag{30}$$

where

$$\xi^k(q_0) = \int \mathbb{I}\{\pi_h^B(q) \geq \pi_\ell^B\left(q, \delta, v_L^{k-1}(Q_+^{k-1}(q_0)), v_B^{k-1}(Q_+^{k-1}(q_0))\right)\}dF(\delta) \qquad (31)$$

is the fraction of buyers that offer p_h at $t = 0$ in a k–step equilibrium. The last crucial feature of 2–step equilibria that is true for all $k \geq 3$ is that v_L^k is continuous and increasing in q_0, and has the property that $\lim_{q_0 \to \bar{q}^k} v_L^k(q_0) = v_L^{k-1}(\bar{q}^k)$. To following proposition summarizes the induction step; its proof is in the Appendix.

Proposition 5. *A k–step equilibrium exists if, and only if, $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0, 1)$. Moreover, for each $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0, 1)$, there is a unique $q' \in [\underline{q}^{k-1}, \bar{q}^{k-1})$ such that q' is the value of q_1 in a k–step equilibrium when the initial fraction of type H sellers is q_0, and the map $Q_+^k : q_0 \mapsto q'$ is continuous and strictly increasing. Finally, the payoff v_L^k to a type L seller in a k–step equilibrium is continuous and increasing in q_0, and is such that $\lim_{q_0 \to \bar{q}^k} v_L^k(q_0) = v_L^{k-1}(\bar{q}^k)$.*

The inductive process described above continues as long as $p_\ell / v_L^{k-1}(\underline{q}^{k-1}) < \bar{\delta}$, in which case $\underline{q}^k > 0$. We have thus established the following result, which provides a complete characterization of the equilibrium set.

Theorem 1. *There exist $N \geq 1$ and sequences $\{\underline{q}^k\}_{k=0}^N$ and $\{\bar{q}^k\}_{k=0}^N$, with $\underline{q}^N = 0$, $\bar{q}^0 = 1$, and $\underline{q}^k < \underline{q}^{k-1} \leq \bar{q}^k < 1$ for all $k \in \{1, \dots, N\}$, such that a k–step equilibrium exists if, and only if, $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0, 1)$. Moreover, for each $q_0 \in [\underline{q}^k, \bar{q}^k) \cap (0, 1)$, there is a unique $q' \in [\underline{q}^{k-1}, \bar{q}^{k-1})$ such that q' is the value of q_1 in a k–step equilibrium when the initial fraction of type H sellers is q_0, and the map $Q_+^k : q_0 \mapsto q'$ is continuous and strictly increasing.*

The payoffs for buyers and sellers are uniquely defined in every equilibrium and are determined recursively as follows: (i) $v_B^0(q_0) = q_0[u_H - p_h] + (1 - q_0)[u_L - p_h]$ and $v_L^0(q_0) \equiv p_h$; (ii) for all $k \in \{1, \dots, N\}$, v_B^k and v_L^k are given by (29) and (30), respectively.

The cutoffs $\{\underline{q}^k\}_{k=0}^{N-1}$ and $\{\bar{q}^k\}_{k=1}^N$ are defined recursively as follows: (i) $\underline{q}^0$ is the unique q such that $\pi_h^B(q) = \pi_\ell^B\left(q, \bar{\delta}, v_L^0, v_B^0(q)\right)$ and $q^+(\underline{q}^k, v_L^{k-1}(\underline{q}^{k-1})) = \underline{q}^{k-1}$ for $k = 1, \dots, N-1$; (ii) $\bar{q}^1$ is the unique q such that $\pi_h^B(q_0) = \pi_\ell^B\left(q_0, \bar{\delta}, v_L^0, v_B^0[q^+(q_0, v_L^0)]\right)$ and $q^+(\bar{q}^k, v_L^{k-1}(\bar{q}^{k-1})) = \bar{q}^{k-1}$ for $k = 2, \dots, N$.

Theorem 1 characterizes a sequence of cutoffs which partition the parameter spaces in to regions that take at least (or, alternatively, at most) $k \geq 0$ periods for the market to clear,

for $k = 1, 2, \ldots$. Figure 1 below illustrates.

Figure 1: Equilibrium Cutoffs

$$\vdash\!\dashv$$

| 0 | $\cdots$ | $\underline{q}^2$ | $\overline{q}^3$ | $\underline{q}^1$ | $\overline{q}^2$ | $\underline{q}^0$ $\overline{q}^1$ | 1 |
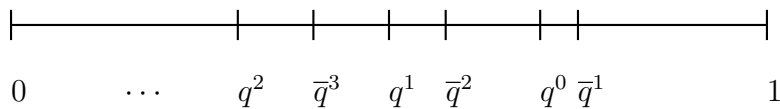
Notice that there is a natural monotonicity to the equilibria: for any $0 < q_0 < \tilde{q}_0 < 1$, if there exists a k–step equilibrium beginning at q_0, then there also exists a $\tilde{k}$–step equilibrium beginning at q_0 such that $\tilde{k} \geq k$.

5 Discussion

The theory developed above can provide insight into two types of issues. First, conditional on the initial composition of high and low quality assets (i.e. q_0), we can study how market prices, trading volume, and average quality evolve over time. Given a full characterization of equilibrium dynamics, we can study how different types of interventions could potentially alter these dynamics, and the subsequent welfare consequences.

Second, we can study how market behavior – and in particular *market liquidity* – differs across values of q_0. A market is typically considered liquid if a good can be sold quickly and at little discount. In many models, trade is instantaneous by construction, and thus the only measure of liquidity is the between the actual price and the frictionless, market-clearing price; time is simply not a margin that can adjust. In the current model, if we focus on high quality goods, the opposite is true: p_h is the only price that can clear the market, and so the appropriate measure of liquidity is the expected amount of time it takes to sell a high quality good. We focus on this measure.

Consider a k–step equilibrium with an initial fraction $q_0 \in \left[\underline{q}^k, \overline{q}^k\right] \cap (0, 1)$ of high quality

assets. Let

$$\eta^s(q,\delta) = \pi_h^B(q) - \pi_\ell^B\left(q, \delta, v_L^{s-1}\left[Q_+^s(q)\right], v_B^{s-1}\left[Q_+^s(q)\right]\right) \tag{32}$$

and define $\delta^s(q)$ such that

$$\delta^s(q) = \begin{cases} 0 & \text{if} \quad \eta^s(q,0) \leq 0 \\ \\ \eta^s(q,\delta^s(q)) = 0 & \text{if} \quad \eta^s(q,0) > 0 \end{cases}$$

for $s = 1, 2, \ldots, k$. In words, when the market is s steps (or periods) away from clearing and the fraction of high quality assets is q, all buyers with $\delta < \delta^s(q)$ offer p_h and all buyers with $\delta \geq \delta^s(q)$ offer p_ℓ. By construction, $\delta^0(q) \equiv \bar{\delta}$, since all buyers offer p_h in the final period of trade. Given q_0, and defining $q_1, q_2, \ldots, q_k$ by the recursion

$$q_{t+1} = Q_+^{k-t}(q_t)$$

for $t = 0, 1, \ldots, k-1$, we can define the expected number of periods it takes to sell an asset of quality H in a k–step equilibrium by

$$E_H^k(q_0) = \sum_{s=0}^{k-1}\left\{\left[\prod_{t=0}^{s}\left\{1 - F\left[\delta^{k-t}(q_t)\right]\right\}\right] F\left[\delta^{k-s-1}(q_{s+1})\right](s+1)\right\}. \tag{33}$$

As we established earlier, for each q_0 there may exist multiple equilibria that take a different number of periods for the market to clear. Let

$$K^*(q_0) = \{k : \exists \text{ a k–step equilibrium given } q_0\}.$$

Below we use a simple numerical example to illustrate the relationship between our notion of liquidity and the initial fraction of high quality assets. The parameter values are chosen such that $\bar{q}^3 = 0$, so that there only exist 2–, 1–, and 0–step equilibria. We plot $E_H^k(q_0)$, letting $k = \max_k K^*(q_0)$ at each point. In general, what we find is that the expected amount of time it takes to sell a high quality asset is decreasing in q_0; this result is independent of our equilibrium selection criterion, so long as we are consistent across values of q_0.

INSERT FIGURE 1 HERE

This finding has important implications for a variety of issues in macroeconomics and finance, where the ability to transact instantaneously at a single, market-clearing price is typically taken as given. This abstraction may seem like a fine approximation to the way that many financial markets behave *most* of the time: in many markets, trades are routinely executed quickly and there is little price dispersion. However, during times of crisis, when the value of many existing assets becomes highly uncertain, the theory developed here suggests that markets will not behave as in the frictionless, Walrasian model. Instead, trade will take time, and the terms of trade will vary across matches. Thus, it will be precisely in those states of the world in which sellers typically need liquidity the most that it will also be most difficult to sell. As a result, those assets that are more vulnerable to adverse selection should, in equilibrium, pay a greater rate of return to compensate for this illiquidity risk. In summary, the theory developed here has important implications for issues relating to asset pricing, the equity premium, and the rate-of-return dominance puzzle.

6 Appendix

6.1 Proof of Proposition 1

Let σ^* be an equilibrium and assume, towards a contradiction, that $T(\sigma^*) = \infty$. First notice that there is $q^* \in (0,1)$ such that

$$q^*[u_H - p_h] + (1 - q^*)[u_L - p_h] = \max\left\{ u_L - p_\ell, \overline{\delta}[u_H - p_h] \right\}.$$

Since the highest payoff possible for a buyer is $u_H - p_h$, the right–hand side of the above equation is the highest payoff possible for a buyer who offers p_ℓ. The definition of q^* then implies that if the fraction of type H sellers in the market is above q^*, then all buyers offer p_h and the market clears. Thus, for all $t \geq 0$, the fraction q_t of type H sellers in the market in period t is bounded above by q^*, and so is the limit q_∞ of the sequence $\{q_t\}$. Now notice that the sequences $\{V_t^L(\sigma^*)\}$ and $\{V_t^H(\sigma^*)\}$ are bounded, and so have convergent subsequences. Dropping subscripts if necessary, we can assume that both sequences converge. Denote their respective limits by V_∞^L and V_∞^H and note that $V_\infty^H \geq V_\infty^L$, given that $V_t^H(\sigma^*) \geq V_t^L(\sigma^*)$ for all t. Since the c.d.f. F is continuous, the law of motion (7) for q_t implies that

$$q_\infty = \frac{q_\infty \left[1 - F\left(\dfrac{p_\ell - y_H}{V_\infty^H} \right) \right]}{q_\infty \left[1 - F\left(\dfrac{p_\ell - y_H}{V_\infty^H} \right) \right] + (1 - q_\infty)\left[1 - F\left(\dfrac{p_\ell}{V_\infty^L} \right) \right]},$$

from which we obtain that

$$q_\infty \left[1 - F\left(\frac{p_\ell - y_H}{V_\infty^H} \right) \right] + (1 - q_\infty)\left[1 - F\left(\frac{p_\ell}{V_\infty^L} \right) \right] = 1 - F\left(\frac{p_\ell - y_H}{V_\infty^H} \right).$$

However, $q_\infty < 1$, and so the last equation implies that $F(p_\ell/V_\infty^L) = F[(p_\ell - y_H)/V_\infty^H]$, a contradiction since $(p_\ell - y_H)/V_\infty^H < p_\ell/V_\infty^L$. Thus, the market must clear in finite time.

6.2 Proof of Proposition 2

Let $\eta^0(q, \delta) = \pi_h^B(q) - \pi_\ell^B(q, \delta, v_L^0, v_B^0(q))$. Straightforward algebra shows that $\eta^0(q, \delta)$ is strictly increasing in q. Since η^0 is continuous in q, $\eta^0(0, \overline{\delta}) < 0$, and $\eta^0(1, \overline{\delta}) > 0$, there is a unique $q \in (0,1)$, that we denote by $\underline{q}^0$, such that $\eta^0(\underline{q}^0, \overline{\delta}) \geq 0$ if, and only if, $q \geq \underline{q}^0$. Since

$v_B^0(q) \leq u_H - p_h$, (9) then implies that $\pi_\ell^B\left(q, \overline{\delta}, v_L^0, v_B^0(q)\right) > 0$, and so $v_B^H(\underline{q}^0) = \pi_h^B(\underline{q}^0) > 0$. This assures that the strategy profile σ^0 is an equilibrium if $q_0 \geq \underline{q}^0$.

As the next step, suppose that $q_0 < \underline{q}^0$ and consider a candidate 0–step equilibrium $\widetilde{\sigma}^0$ with the necessary property that all buyers offer p_h in $t = 0$. One alternative for a buyer is to offer p_h in every period. Let $\widetilde{\mathbf{p}}$ denote this strategy. If $\widetilde{\sigma}^0$ is to be an equilibrium, then it must be that $V_t^B(\widetilde{\sigma}^0) \geq V_t^B(\widetilde{\mathbf{p}}|\widetilde{\sigma}^0)$ for all $t \geq 0$. Now observe that when the probability that an agent can trade in a period is $\alpha \in (0, 1)$,

$$V_t^B(\widetilde{\mathbf{p}}|\widetilde{\sigma}^0, \alpha) = \sum_{\tau=1}^{\infty} \alpha(1-\alpha)^{\tau-1}(\mathbb{E}[\delta])^{\tau-1} v_B^0(q_{t+\tau-1}^\alpha),$$

where $q_{t+\tau-1}^\alpha$ is the fraction of type H sellers in the market in period $t + \tau - 1$. It is easy to see that

$$q_{t+1}^\alpha = \frac{q_t^\alpha[1 - \alpha + \alpha\xi_t(p_\ell)]}{q_t^\alpha[1 - \alpha + \alpha\xi_t(p_\ell)] + (1 - q_t^\alpha)\left\{1 - \alpha + \alpha\xi_t(p_\ell)\left[1 - F\left(p_\ell/V_{t+1}^L(\widetilde{\sigma}^0|\alpha)\right)\right]\right\}},$$

where $\xi_t(p_\ell)$ is the probability that a buyer who gets the opportunity to trade in period t offers p_ℓ. Clearly the sequence $\{q_t^\alpha\}_{t=0}^{\infty}$ is non–decreasing. Hence,

$$V_t^B(\widetilde{\mathbf{p}}|\widetilde{\sigma}^0, \alpha) \geq \sum_{\tau=1}^{\infty} \alpha(1-\alpha)^{\tau-1}(\mathbb{E}[\delta])^{\tau-1} v_B^0(q_0),$$

which implies that $V_t^B(\widetilde{\mathbf{p}}|\widetilde{\sigma}^0) \geq v_B^0(q_0)$. From the main text we know that this last fact, along with the fact that $V_t^L(\widetilde{\sigma}^0) \leq v_L^0$, imply that a positive mass of buyers do not find it optimal to offer p_h in $t = 0$, so that the market clearing immediately cannot be an equilibrium outcome.

6.3 Proof of Proposition 3

Recall that $q^+(q, v_L^0)$ is strictly increasing in q when $p_\ell/v_L^0 < \overline{\delta}$ and that $q^+(q, v_L^0) \equiv 1$ otherwise. From this it is immediate to see that there exists $\underline{q}^1 < \underline{q}^0$ such that $q^+(q_0, v_L^0) \geq \underline{q}^0$ if, and only if, $q_0 \in [\underline{q}^1, 1]$. Note that $\underline{q}^1 = 0$ if $p_\ell/v_L^0 \geq \overline{\delta}$ and $\underline{q}^1$ is such that $q^+(\underline{q}^1, v_L^0) = \underline{q}^0$ otherwise. Now, let $\eta^1(q, \delta) = \pi_h^B(q) - \pi_\ell^B\left(q, \delta, v_L^0, v_B^0\left[q^+(q, v_L^0)\right]\right)$. It is easy to see that

$$\frac{\partial \eta^1}{\partial q}(q, \delta) = F\left(\frac{p_\ell}{p_h}\right)\left\{u_L - p_\ell - \delta v_B^0[q^+(q, p_h)]\right\}$$

$$+ (u_H - u_L)\left\{1 - \delta\left\{q + (1 - q)\left[1 - F\left(\frac{p_\ell}{p_h}\right)\right]\right\}\frac{\partial q^+}{\partial q}\right\}.$$

Thus, from (9) and the fact that

$$\{q + (1-q)\left[1 - F\left(p_\ell/p_h\right)\right]\} \frac{\partial q^+}{\partial q} = 1 - \frac{qF\left(p_\ell/p_h\right)}{q + (1-q)\left[1 - F\left(p_\ell/p_h\right)\right]} < 1,$$

we can conclude that $\partial \eta^1/\partial q > 0$ regardless of the value of p_ℓ/p_h. Since $\eta^1(0, \bar{\delta}) < 0$ and $\eta^1(1, \bar{\delta}) > 0$, there exists $\bar{q}^1 \in (0,1)$ such that $\eta^1(q, \bar{\delta}) < 0$ if, and only if $q_0 \in [0, \bar{q}^1)$. Hence, $\pi_h^B(q_0) < \pi_\ell^B\left(q_0, \bar{\delta}, v_L^0, v_B^0\left[q^+(q_0, v_L^0)\right]\right)$ if, and only if $q_0 \in [0, \bar{q}^1)$. To finish, observe that since $v_B^0[q^+(q, p_h)] > v_B^0(q)$ for all $q \in (0,1)$,

$$\pi_\ell^B\left(\underline{q}^0, \bar{\delta}, v_L^0, v_B^0[q^+(\underline{q}^0, p_h)]\right) > \pi_\ell^B\left(\underline{q}^0, \bar{\delta}, v_L^0, v_B^0(\underline{q}^0)\right) = \pi_h^B(\underline{q}^0).$$

Thus, $\eta^1(\underline{q}^0, \bar{\delta}) < 0$, from which we obtain that $\bar{q}^1 > \underline{q}^0$.

6.4   Lemma 1 and Proof

Lemma 1. *The probability $\xi^1(q_0)$ that a buyer offers p_h in $t = 0$ in a 1–step equilibrium is continuous and increasing in q_0, and it converges to one as q_0 increases to $\bar{q}^1$.*

Since $\pi_\ell^B(q_0, \delta, v_L^0, v_B^0[q^+(q_0, v_L^0)])$ is strictly increasing in δ, the function η^1 in the proof of Proposition 3 is strictly decreasing in δ. Let then $\delta^1(q_0)$, with $q_0 \in [\underline{q}^1, \bar{q}^1) \cap (0, \bar{q}^1)$, be such that

$$\delta^1(q_0) = \begin{cases} 0 & \text{if} \quad \eta^1(q_0, 0) \leq 0 \\ \eta^1(q_0, \delta^1(q_0)) = 0 & \text{if} \quad \eta^1(q_0, 0) > 0 \end{cases}$$

Since $\eta^1(\bar{q}^1, \bar{\delta}) = 0$ and η^1 is strictly increasing in q by the proof of Proposition 3, $\delta^1(q_0)$ is uniquely defined. By construction, δ^1 is the cutoff discount factor below which a buyer finds it optimal to offer p_h in $t = 0$. Hence, the probability $\xi^1(q_0)$ that a buyer offers p_h in $t = 0$ is equal to $F(\delta^1(q_0))$. Since η^1 is jointly continuous, it is easy to see that δ^1 depends continuously on q_0. Moreover, the cutoff $\delta^1(q_0)$ is strictly increasing in q_0 if $\eta^1(q_0, 0) > 0$, as η^1 is strictly increasing in q. The desired result follows from the fact that the c.d.f. F is continuous and strictly increasing and $\lim_{q_0 \to \bar{q}^1} \delta^1(\bar{q}^1) = \bar{\delta}$ (given that $\eta^1(\bar{q}^1, \bar{\delta}) = 0$).

6.5   Proof of Proposition 4

We first show that (18) and (19) imply (20), so that conditions (18) and (19) completely determine the range of initial values of q_0 for which there exists a 2–step equilibrium. Before

we start, notice that

$$\left\{ q + (1 - q) \left[1 - F \left(\frac{p_\ell}{v_L} \right) \right] \right\} q^+(q, v_L) = q$$

for all $q \in (0, 1)$ and $v_L \geq p_\ell$. Hence,

$$\pi_\ell^B \left(q, \delta, v_L, \pi_h^B \left[q^+(q, v_L) \right] \right) = \delta \pi_h^B(q) + (1 - q) F \left(\frac{p_\ell}{v_L} \right) [u_L - p_\ell - \delta(u_L - p_h)] \tag{34}$$

for all $q \in (0, 1)$ and $\delta \in [0, \overline{\delta}]$.

Suppose that $q' \in [\underline{q}^1, \overline{q}^1)$. In order to prove that (20) is satisfied, it is sufficient to show

$$\pi_h^B(q') - \pi_h^B(q_0) \geq \pi_\ell^B \left(q', \delta, v_L^0, v_B^0[q^+(q', v_L^0)] \right) - \pi_\ell^B \left(q_0, \delta, v_L^1(q'), v_B^1(q') \right) \tag{35}$$

for all $\delta \in [0, \overline{\delta}]$. Condition (35) implies that the incentive of a buyer to choose p_ℓ in $t = 0$ is even greater than his incentive to choose p_ℓ in $t = 1$, when the fraction of type H sellers in the market is $q' > q_0$; in particular, this is true for the most patient buyer. First, note that

$$
\begin{aligned}
\pi_\ell^B \left(q', \overline{\delta}, v_L^0, v_B^0 \left[q^+ \left(q', v_L^0 \right) \right] \right) &= \pi_\ell^B \left(q', \overline{\delta}, v_L^0, \pi_h^B \left[q^+ \left(q', v_L^0 \right) \right] \right) \\
&= \delta \pi_h^B(q') + (1 - q') F \left(\frac{p_\ell}{v_L^0} \right) \left[u_L - p_\ell - \overline{\delta}(u_L - p_h) \right].
\end{aligned}
$$

Second, since $v_B^1(q') \geq \pi_h^B(q')$, we have

$$
\begin{aligned}
\pi_\ell^B \left(q_0, \overline{\delta}, v_L^1(q'), v_B^1(q') \right) &\geq \pi_\ell^B \left(q_0, \overline{\delta}, v_L^1(q'), \pi_h^B(q') \right) \\
&= \delta \pi_h^B(q_0) + (1 - q_0) F \left(\frac{p_\ell}{v_L^1(q')} \right) \left[u_L - p_\ell - \overline{\delta}(u_L - p_h) \right];
\end{aligned}
$$

the second equality follows from (18) and (34). Therefore,

$$\pi_\ell^B \left(q', \overline{\delta}, v_L^0, v_B^0 \left[q^+ \left(q', v_L^0 \right) \right] \right) - \pi_\ell^B \left(q_0, \overline{\delta}, v_L^1(q'), v_B^1(q') \right)$$
$$\leq \overline{\delta} \left[\pi_h^B(q') - \pi_h^B(q_0) \right] + \left\{ (1 - q') F \left(\frac{p_\ell}{v_L^0} \right) - (1 - q_0) F \left(\frac{p_\ell}{v_L^1(q')} \right) \right\} \left[u_L - p_\ell - \overline{\delta}(u_L - p_h) \right].$$

Since $v_L^0 > v_L^1(q')$ for all $q' \in [\underline{q}^1, \overline{q}^1)$, $u_L < p_h$, and $q' > q_0$, the second term on the right–hand side of the above inequality is negative, which confirms (35).

We now show that there exists a 2–step equilibrium if, and only if, $q_0 \in [\underline{q}^2, \overline{q}^2) \cap (0, 1)$. Since $0 < \underline{q}^1 < \overline{q}^1 < 1$, (18) and (19) can be satisfied only if the denominator of

$$q^+(q_0, v_L^1(q_0)) = \frac{q_0}{q_0 + (1 - q_0)[1 - F(p_\ell/v_L^1(q'))]}$$

is greater than q_0. In other words, for a given $q' \in [\underline{q}^1, \overline{q}^1)$, there is $q_0 \in (0, 1)$ such that (18) is satisfied only if $p_\ell/v_L^1(q') < \overline{\delta}$. Now observe that if $p_\ell/v_L^1(q') < \overline{\delta}$, then

$$q^-(q') = \frac{q'\left[1 - F\left(p_\ell/v_L^1(q')\right)\right]}{1 - q'F\left(p_\ell/v_L^1(q')\right)}$$

belongs to the interval $(0, 1)$ and is such that $q^+(q^-(q'), v_L^1(q')) = q'$. Thus, (18) is satisfied for $q' \in [\underline{q}^1, \overline{q}^1)$ if, and only if, $p_\ell/v_L^1(q') < \overline{\delta}$. Moreover, it is immediate to see that $q^-(q')$ is the only possible value of q_0 for which (18) and (19) can hold.

Since $v_L^1(q')$ is increasing in q', $p_\ell/v_L^1(\widetilde{q}) < \overline{\delta}$ implies that $p_\ell/v_L^1(q') < \overline{\delta}$ for all $q' > \widetilde{q}$. Let then $\widetilde{q}^1$ be such that $\widetilde{q}^1 = 0$ if $p_\ell/v_L^1(\underline{q}^1) < \overline{\delta}$ and $\widetilde{q}^1 = \sup\{q' \in [\underline{q}^1, \overline{q}^1) : p_\ell/v_L^1(q') = \overline{\delta}\}$ if $p_\ell/v_L^1(\underline{q}^1) \geq \overline{\delta}$; $\widetilde{q}^1$ is well–defined since $p_\ell/v_L^1(\overline{q}^1) = p_\ell/v_L^0 < \overline{\delta}$. By construction, there is $q_0 \in (0, 1)$ such that (18) and (19) are satisfied if, and only if, $q' \in [\underline{q}^1, \overline{q}^1) \cap (\widetilde{q}^1, 1)$, in which case $q_0 = q^-(q')$. Since F and v_L^1 are continuous in δ and q', respectively, it is easy to see that q^- is continuous in q'. Moreover, since v_L^1 is increasing in q', straightforward algebra shows that q^- is strictly increasing in q'. Thus, we have that: (i) when $p_\ell/v_L^1(\underline{q}^1) < \overline{\delta}$, there exists a 2–step equilibrium if, and only if, $q_0 \in [q^-(\underline{q}^1), q^-(\overline{q}^1))$; (ii) when $p_\ell/v_L^1(\underline{q}^1) \geq \overline{\delta}$, there exists a 2–step equilibrium if, and only if, $q_0 \in (q^-(\widetilde{q}^1), q^-(\overline{q}^1))$. It is immediate to see that $q^-(\overline{q}^1) = \overline{q}^2$. We are done if we show that $q^-(\widetilde{q}^1) = 0$ when $p_\ell/v_L^1(\widetilde{q}) \geq \overline{\delta}$. This follows from the fact that $\lim_{q' \to \widetilde{q}^1} F(p_\ell/v_L^1(q')) = 1$.

To finish the proof, notice that since $q^-(q')$ is continuous and strictly increasing in q', it is invertible and its inverse is strictly increasing and continuous. Thus, the map Q_+^2 that takes an initial fraction q_0 of type H sellers into the value of q_1 in a 2–step equilibrium is continuous and strictly increasing.

6.6 Lemma 2 and Proof

Lemma 2. *Suppose that $\overline{\delta} > p_\ell/v_L^0$. Then: (i) v_B^2 is continuous in q_0, and it converges to $v_B^1(\overline{q}^2)$ as q_0 increases to $\overline{q}^2$; (ii) v_L^2 is continuous and increasing in q_0, and it converges to $v_L^1(\overline{q}_2)$ as q_0 increases to $\overline{q}^2$.*

We have shown that Q_+^2 is continuous, strictly increasing, and converges to $\overline{q}^1$ as q_0 increases to $\overline{q}^2$. Given these properties, along with the continuity of v_B^1 and v_L^1, it follows immediately

that v_B^2 is continuous as well. Since $v_L^1(\overline{q}^1) = v_L^0$ and $v_B^1(\overline{q}^1) = v_B^0(\overline{q}^1) = v_B^0[q^+(\overline{q}^2, v_L^0)]$, we then have that

$$\lim_{q_0 \uparrow \overline{q}^2} v_B^2(q_0) = \int_0^{\overline{\delta}} \max \left\{ \pi_h^B(\overline{q}^2), \pi_\ell^B\left(\overline{q}^2, \delta, v_L^0, v_B^0[q^+(\overline{q}^2, v_L^0)]\right) \right\} dF(\delta) = v_B^1(\overline{q}^2).$$

Let $\eta^2(q_0, \delta) = \pi_h^B(q_0) - \pi_\ell^B\left(q_0, \delta, v_L^1(Q_+^2(q_0)), v_B^1(Q_+^2(q_0))\right)$. An argument similar to the one used in the Proof of Proposition 3 shows that for each $q_0 \in [\underline{q}^2, \overline{q}^2)$ there is a unique $\delta^2 = \delta^2(q_0)$ in $[0, \overline{\delta})$, depending continuously on q_0, such that $\eta^2(q_0, \delta) \geq 0$ if, and only if $\delta \leq \delta^2(q_0)$. Thus, $\xi^2(q_0) = F(\delta^2(q_0))$ is continuous and increasing in q_0 (strictly increasing when $\delta^2(q_0) < \overline{\delta}$), from which we obtain that v_L^2 is continuous and increasing in q_0. To finish, notice that

$$\lim_{q_0 \uparrow \overline{q}^2} \eta^2(q_0, \delta) = \pi_h^B(\overline{q}^2) - \pi_\ell^B(\overline{q}^2, \delta, v_L^1(\overline{q}^1), v_B^1(\overline{q}^1))$$
$$= \pi_h^B(\overline{q}^2) - \pi_\ell^B(\overline{q}^2, \delta, v_L^0, v_B^0[q^+(\overline{q}^2, v_L^0)]) = \eta^1(\overline{q}^2, \delta),$$

so that $\lim_{q_0 \uparrow \overline{q}^2} \xi^2(\overline{q}^2) = \xi^1(\overline{q}^2)$, from which we can conclude that

$$\lim_{q_0 \uparrow \overline{q}^2} v_L^2(q_0) = \xi^1(\overline{q}^2) p_h + (1 - \xi^1(\overline{q}^2)) \int_0^{\overline{\delta}} \max \left\{ p_\ell, \delta v_L^0 \right\} dF(\delta) = v_L^1(\overline{q}^1).$$

6.7   Proof of Proposition 5

We omit many of the details, as the proof follows that of Proposition 4 very closely. Suppose, by induction, that the following holds for all $s \in \{1, \ldots, k-1\}$: (i) for all $q_0 \in [\underline{q}^s, \overline{q}^s)$ there is a unique $q' \in [\underline{q}^{s-1}, \overline{q}^{s-1})$ such that $q' = Q_+^s(q_0)$ is the value of q_1 in a s–step equilibrium when the initial fraction of type H sellers is q_0; (ii) if $q' = Q_+^s(q_0)$, then

$$\eta^{s-1}(q', \delta) = \pi_h^B(q') - \pi_\ell^B\left(q', \delta, v_L^{s-1}(Q_+^s(q')), v_B^{s-1}(Q_+^s(q'))\right)$$
$$\geq \eta^s(q_0, \delta) = \pi_h^B(q_0) - \pi_\ell^B\left(q_0, \delta, v_L^s(q'), v_B^s(q')\right) \qquad (36)$$

for all $q_0 \in [\underline{q}^s, \overline{q}^s)$ and $\delta \in [0, \overline{\delta}]$; (iii) v_L^s is continuous and increasing in $[\underline{q}^s, \overline{q}^s)$ and $v_L^s(q_0)$ converges to $v_L^{s-1}(\overline{q}^s)$ as q_0 increases to $\overline{q}^s$. Conditions (i) to (iii) are true when $k = 3$ by Propositions 3 and 4 and Lemmas 1 and 2.

First notice that $\underline{q}^k < \underline{q}^{k-1} \le \bar{q}^k$. This fact follows from (iii) (and the definition of $\bar{q}^k$) and its proof is identical to the proof that $\underline{q}^2 < \underline{q}^1 \le \bar{q}^2$; simply replace the superscripts "1" and "2" with "$k-1$" and "k", respectively.

We now show that if $q' = q^+(q_0, v_L^{k-1}(q'))$ and $q' \in [\underline{q}^{k-1}, \bar{q}^{k-1})$, then $\eta^{k-1}(q', \delta) \ge \eta^k(q_0, \delta)$ for all $\delta \in [0, \bar{\delta}]$. For this, note that

$$
\pi_\ell^B\left(q_0, \delta, v_L^{k-1}(q'), v_B^{k-1}(q')\right)
$$
$$
= (1-q_0)F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)[u_L - p_\ell] + \delta\left\{q_0 + (1-q_0)\left[1 - F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)\right]\right\}v_B^{k-1}(q')
$$
$$
= (1-q_0)F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)[u_L - p_\ell] + \delta\left\{q_0 + (1-q_0)\left[1 - F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)\right]\right\}\pi_h^B(q')
$$
$$
+ \delta\left\{q_0 + (1-q_0)\left[1 - F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)\right]\right\}[v_B^{k-1}(q') - \pi_h^B(q')]
$$
$$
= \delta\pi_h^B(q_0) + (1-q_0)F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)[u_L - p_\ell - \delta(u_L - p_h)]
$$
$$
+ \delta\left\{q_0 + (1-q_0)\left[1 - F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)\right]\right\}[v_B^{k-1}(q') - \pi_h^B(q')]
$$

where the last equality follows from (34). Similarly, one can show that

$$
\pi_\ell^B\left(q', \delta, v_L^{k-2}(q''), v_B^{k-2}(q'')\right)
$$
$$
= \delta\pi_h^B(q') + (1-q')F\left(\frac{p_\ell}{v_L^{k-2}(q'')}\right)[u_L - p_\ell - \delta(u_L - p_h)]
$$
$$
+ \delta\left\{q' + (1-q')\left[1 - F\left(\frac{p_\ell}{v_L^{k-2}(q'')}\right)\right]\right\}[v_B^{k-2}(q'') - \pi_h^B(q'')],
$$

where $q'' = Q_+^{k-1}(q')$. By (36) and Lemma 3 below, we then have that

$$
v_B^{k-1}(q') - \pi_h^B(q') \ge v_B^{k-2}(q'') - \pi_h^B(q''),
$$

so that

$$
\pi_\ell^B\left(q_0, \delta, v_L^{k-1}(q'), v_B^{k-1}(q')\right) - \pi_\ell^B\left(q', \delta, v_L^{k-2}(q''), v_B^{k-2}(q'')\right) \le \delta\left[\pi_h^B(q_0) - \pi_h^B(q')\right]
$$
$$
+ \left\{u_L - p_\ell - \delta\left(u_L - p_h - [v_B^{k-1}(q') - \pi_h^B(q')]\right)\right\}\left\{(1-q')F\left(\frac{p_\ell}{v_L^{k-2}(q'')}\right)\right.
$$
$$
\left. -(1-q_0)F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right)\right\}. \quad (37)
$$

Since $q' \geq q_0$ and $v_L^{k-1}(q') < v_L^{k-2}(q'')$ by Lemma 4 below, we have that

$$(1 - q')F\left(\frac{p_\ell}{v_L^{k-2}(q'')}\right) - (1 - q_0)F\left(\frac{p_\ell}{v_L^{k-1}(q')}\right) < 0.$$

In addition, $u_L < p_h$ and $u_L - p_\ell \geq \delta(u_H - p_h) > \delta\left[v_B^{k-1}(q') - \pi_h^B(q')\right]$. Hence, (37) implies that

$$\pi_\ell^B\left(q, \delta, v_L^{k-1}(q'), v_B^{k-1}(q')\right) - \pi_\ell^B\left(q', \delta, v_L^{k-2}(q''), v_B^{k-2}(q'')\right) < \pi_h^B(q) - \pi_h^B(q'),$$

which is the desired result. Consequently, (ii) holds when $s = k$. Since $\eta^{k-1}(q', \bar{\delta}) < 0$ for all $q' \in [\underline{q}^{k-1}, \overline{q}^{k-1})$, as the most patient buyer must strictly prefer to offer p_ℓ in $t = 0$ in a $(k-1)$–step equilibrium when $k \geq 3$, we then have that (26) and (27) imply (28).

Now observe that the same argument used in the proof of Proposition 4 shows that there is a k–step equilibrium if, and only if, $q_0 \in [\underline{q}^k, \overline{q}^k) \cap (0, 1)$, that for each $q_0 \in [\underline{q}^k, \overline{q}^k) \cap (0, 1)$, there is a unique $q' \in [\underline{q}^{k-1}, \overline{q}^{k-1})$ such that q' is the value of q_1 in a k–step equilibrium when the initial fraction of type H sellers is q_0, and that the map $Q_+^k : q_0 \mapsto q'$ is continuous and strictly increasing; simply replace the superscripts "1" and "2" with "$k-1$" and "k", respectively. In particular, (i) holds when $s = k$.

To finish, notice that the same argument used in the proof of Lemma 2 shows that (iii) holds when $s = k$; once more just replace the superscripts "1" and "2" with "$k-1$" and "k", respectively.

6.8   Lemma 3 and Proof

Lemma 3. *Given* $q' = q_+\left[q, v_L^{k-1}(q')\right]$, $v_B^k(q) - \pi_h^B(q) \geq v_B^{k-1}(q') - \pi_h^B(q')$.

Let $\widetilde{\eta}^k(q, \delta) = -\eta^k(q, \delta)$. Moreover, let $\delta^k(q)$ denote the maximum of zero and the value of δ such that $\widetilde{\eta}^k(q, \delta) = 0$. Then (36) implies that $\delta^{k-1}(q') \geq \delta^k(q)$. This fact, along with

$\widetilde{\eta}^k(q, \delta) \geq \widetilde{\eta}^{k-1}(q', \delta)$ for all δ, implies

$$\begin{aligned} v_B^k(q) - \pi_h^B(q) &= \int_{\delta^k(q)}^{\overline{\delta}} \widetilde{\eta}^k(q, \delta) dF(\delta) \\ &\geq \int_{\delta^k(q)}^{\overline{\delta}} \widetilde{\eta}^{k-1}(q', \delta) dF(\delta) \\ &\geq \int_{\delta^{k-1}(q')}^{\overline{\delta}} \widetilde{\eta}^{k-1}(q', \delta) dF(\delta) = v_B^{k-1}(q') - \pi_h^B(q'). \end{aligned}$$

6.9   Lemma 4 and Proof

Lemma 4. $v_L^{k-1}(q') < v_L^{k-2}(q'')$.

Proof to be added.

References

G.A. Akerlof. The Market for "Lemons": Quality Uncertainty and the Market Mechanism. *The Quarterly Journal of Economics*, 84(3):488–500, 1970.

M.R. Blouin. Equilibrium in a decentralized market with adverse selection. *Economic Theory*, 22(2):245–262, 2003.

M.R. Blouin and R. Serrano. A decentralized market with common values uncertainty: Non-steady states. *Review of Economic Studies*, pages 323–346, 2001.

R. Deneckere and M.Y. Liang. Bargaining with interdependent values. *Econometrica*, 74(5): 1309–1364, 2006.

D. Duffie, N. Garleanu, and L.H. Pedersen. Over-the-counter markets. *Econometrica*, pages 1815–1847, 2005.

D. Duffie, N. Gârleanu, and L.H. Pedersen. Valuation in over-the-counter markets. *Review of Financial Studies*, 20(6):1865, 2007.

A.L. Eisfeldt. Endogenous liquidity in asset markets. *The Journal of Finance*, 59(1):1–30, 2004.

R. Evans. Sequential bargaining with correlated values. *The Review of Economic Studies*, pages 499–510, 1989.

D. Fudenberg, D. Levine, and J. Tirole. Infinite-horizon models of bargaining with one-sided incomplete information. *Game theoretic models of Bargaining*, pages 73–98, 1985.

D. Gale. Bargaining and Competition Part I: Characterization. *Econometrica*, 54(4):785–806, 1986a.

D. Gale. Bargaining and Competition Part II: Existence. *Econometrica*, 54(4):807–818, 1986b.

N. Gârleanu. Portfolio choice and pricing in illiquid markets. *Journal of Economic Theory*, 144(2):532–564, 2009.

L. Glosten and P. Milgrom. Bid, Ask, and Transaction Prices in a Specialist Market with Heterogeneously Informed Agents. *Journal of Financial Economics*, 14:71–100, 1985.

F. Gul, H. Sonnenschein, and R. Wilson. Foundations of dynamic monopoly and the coase conjecture. *Journal of Economic Theory*, 39(1):155–190, 1986.

Y. Higashi, K. Hyogo, and N. Takeoka. Subjective random discounting and intertemporal choice. *Journal of Economic Theory*, 144(3):1015–1053, 2009.

R. Inderst. Matching markets with adverse selection. *Journal of Economic Theory*, 121(2): 145–166, 2005.

M. Janssen and S. Roy. Dynamic Trading in a Durable Good Market with Asymmetric Information. *International Economic Review*, 43:257–282, 2002.

P. Kurlat. Lemons, Market Shutdowns and Learning. 2009.

A.S. Kyle. Continuous Auctions and Insider Trading. *Econometrica*, 53(6):1315–1335, 1985.

D. Moreno and J. Wooders. Prices, delay, and the dynamics of trade. *Journal of Economic Theory*, 104(2):304–339, 2002.

D. Moreno and J. Wooders. Decentralized trade mitigates the lemons problem. *International Economic Review, forthcoming*, 2009.

G. Rocheteau. A monetary approach to asset liquidity. *Working Paper*, 2009.

A. Rubinstein and A. Wolinsky. Equilibrium in a Market with Sequential Bargaining. *Econometrica*, 53(5):1133–1150, 1985.

M. Satterthwaite and A. Shneyerov. Dynamic Matching, Two-Sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition. *Econometrica*, 75(1):155–200, 2007.

D. Vayanos and P.O. Weill. A Search-Based Theory of the On-the-Run Phenomenon. *The Journal of Finance*, 63(3):1361–1398, 2008.

D.R. Vincent. Bargaining with common values. *Journal of Economic Theory*, 48(1):47–62, 1989.

C. Wilson. The nature of equilibrium in markets with adverse selection. *The Bell Journal of Economics*, 11(1):108–130, 1980.

A. Wolinsky. Information revelation in a market with pairwise meetings. *Econometrica*, pages 1–23, 1990.

Figure 2: Liquidity and Lemons