Auctions in Markets: Common Outside Options and the Continuation Value Effect*

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Abstract

We study auctions with outside options that are determined through actions taken in the external market. Such endogenous outside options have important consequences for auction design. In contrast to the case of exogenous outside options, auctions with less information revelation may yield higher revenues. Effects that favor non-transparent auctions include a small payoff difference between different states, a great value of information in the continuation problem, and imprecise signals of the bidders. The timing of information revelation is important: it is never optimal to reveal information after the auction, while it may be optimal before the auction.

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1 Introduction

In many situations information learned in an auction may be valuable in subsequent market interaction. We study such auctions to obtain new insights into auction design. In particular, we consider a situation where the losing bidders have common outside options provided by an external market, and study whether transparent or opaque auctions yield higher revenues. We introduce a stylized model to highlight the main trade-offs that shape the optimal information policy of the auctioneer. There are two risk neutral bidders who participate in a second price auction with a post-auction market. Both bidders have the same known valuation v for the good that is sold in the auction. After one of two possible states of the market is realized, bidders receive private signals that are correlated with the true state, and submit their bids. The winner obtains the object, while the losing bidder makes a take it or leave it offer p to a seller he is randomly matched with. The distribution of the cost of the seller is determined by the state realized, so the state determines the probability with which the offer of the buyer is accepted. Therefore, the expected payoffs of the buyer depend both on the state of the world, and the offer he made. A more precise belief about market conditions allows the losing bidder to make a price offer reflecting those conditions better, and increases the value of his outside option.

We obtain a new explanation for the prevalence of opaque auctions: Auctioneers may decrease the information flow to losing bidders to reduce the value of their outside options. A transparent auction provides a losing bidder with precise information about market conditions, which improves his outside option. Consequently, the bidders bid less in a transparent auction, an outcome we refer to as the *continuation value effect*. Building on this insight, we show that the auctioneer prefers auctions that reveal as little information as possible to losing bidders *after* the auction. Moreover, the auctioneer prefers not to reveal any information *before* the auction, and prefers a second-price auction over the more transparent ascending auction, if and only if the optimal price charged is not very sensitive to market conditions.

To discuss these results, and understand when opaque auctions may enhance revenues, it is instructive to compare our model with standard models of common value auctions, and review a basic design principle for such auctions. The fact that common outside options introduce common value elements into bidding has been noted before[1]. Our model has two

[1] See for example Milgrom and Weber (1982) who discuss why common outside options in the form of resale introduce common value elements.

features that resemble common value auctions. First, the bidders have a common outside option and a common valuation in the original auction. Second, the value of each bidder's outside option depends on the likelihood of each state, so each bidder's utility depends on the others' signal. In fact, if the optimal price in the two states are equal, that is information has no value in the pricing problem, then the auction corresponds to a pure common value auction as studied in Milgrom and Weber (1982). However, we show that the common value auction model does not capture the case where the optimal action in the aftermarket depends on one's belief about market conditions.

The main insight into auction design for common value auctions is the optimality of "transparent" auctions as implied by the *linkage principle*. The intuition for the linkage principle is that revealing extra information alleviates worries that a bidder only won because all other bidders had low signals and thus winning may not actually be profitable (the well known winner's curse phenomenon). The linkage principle implies that the auctioneer should reveal his information about the object, and also reveal the bids placed. As a corollary, an ascending auction is preferable to a second price auction.

Our results show that the insight of the linkage principle may be overturned by the continuation value effect when the value of the outside option depends on the action taken by the losing bidder. If a piece of information is revealed after the auction (like the winning bid), then the linkage principle effect is absent since the bidders cannot incorporate such information into their bids. On the other hand, such information revelation improves outside options, and thus lowers revenues. If information is revealed *before* or *during* (like in an English auction) the auction, then revenue comparisons are ambiguous. We provide three conditions under which the revenue consequences of information revelation are negative: if *maximal* profits in the two states are similar, if bidders have imprecise signals, or if the optimal price is very sensitive to market conditions. The intuition is the following: in the first two cases the linkage principle effect is weak, since bidders are not subject to winner's curse, while in the third case the continuation value effect is strong, and overcomes the linkage principle effect.

Our result that opaque auctions may be revenue-enhancing has further implications. First, it provides an explanation why intransparent auctions may arise in markets like e-Bay,[2] where the auction designer can make the market more attractive for sellers by

[2] On e-Bay, it is difficult to gather information about aggregate market conditions, such as information about past ending prices of close substitutes. In fact, in Germany, it was for a long time impossible to search for past auctions on e-Bay, see Sailer (2008).

restricting availability of information about market conditions. Second, it has implications for the *information percolation* literature initiated by Duffie et. al. (2007). Our results highlight the incentives of sellers to employ non-transparent mechanisms, and thus markets may operate in a much less transparent way than predicted.

We conclude the Introduction with a brief literature review. Our paper provides an alternative framework to the literature on competing mechanisms[3] to study auction design when the auctioneer competes with other sellers on the market. The literatures on auctions with resale[4] and multi unit auctions[5] are also related, because they take post auction market interaction between bidders seriously. Our companion paper (Lauermann, Merzyn and Virag (2010)) study an infinite horizon general equilibrium model where each losing bidder continues to participate in auctions indefinitely. We analyze the bidding pattern over time, and price and belief dispersion across agents.

2 Model and preliminary analysis

2.1 Setup

The interaction unfolds in three stages. First, the auctioneer and the N bidders receive signals about the state of the world. Second, the auctioneer runs an auction for an indivisible object. Third, each losing bidder chooses a price offer in a bilateral bargaining problem.

Information. There are two possible states of the world, $w \in \{H, L\}$, and the realization is not observed by the bidders. The probability of the high state is ρ_0. The unknown state of the world is interpreted as the unknown aggregate market condition. Assume for simplicity that the auctioneer observes the state. The bidders receive private signals that are correlated with the state, these signals are denoted by $s_1, s_2, ..., s_N$. In state w, the bidders' signals are distributed independently according to G_w, so, the bidders' signals are conditionally i.i.d.. We assume that G_w admits a continuous density function g_w. With a signal s, the Bayesian posterior probability of the high state H is $\rho_0 g_H(s) / ((1-\rho_0) g_L(s) + \rho_0 g_H(s))$. Without loss of generality assume that the posterior is a strictly increasing function of s.

Auction. All bidders participate in the seller's auction where a single indivisible good is for sale. We analyze bidding in standard auction formats, including the first-price, the second-price, and the ascending (English) auction. It is worthwhile to point out that

[3] See for example McAfee (1993) and Peters (1997) among other papers.
[4] See for example Hafalir and Krishna (2008), Cheng and Tan (2009) and Garrat and Troger (2007).
[5] See for example Mezzetti et.al. (2008).

although the valuations do not depend on the signals, but the bids may, because the signals influence the beliefs about the state, which influences the option value from losing as it is described below. Moreover, since bidders generally have different beliefs about their outside option, they submit different bids in the auction.

Preferences and payoffs. The winning bidder receives the object and pays price p, while the losing bidders do not make any payments in the auctions studied. The valuation for the object, v, is the same for all bidders and publicly known. The utility of the winner is equal to $v - p$, while that of the losers' is equal to their continuation payoffs (as defined below). Note, that the state affects the value of the outside option (as described below), but not the value of winning.

Outside Option After the auction each losing bidder proceeds to a market where he makes a take-it-or-leave-it offer x to a seller he is randomly matched with. More precisely, in state L, H a losing bidder meets a seller with probability μ_L, μ_H whose cost c is distributed according to distribution functions F_L, F_H that are continuous and twice differentiable, and have a common support $[\underline{c}, \overline{c}]$. The offer x belongs to X, a closed subset of $[\underline{c}, \overline{c}]$.[6] A seller with cost c accepts an offer x if $c \leq x$ holds. Therefore, the expected utility of a buyer who holds belief ρ and makes an offer x can be written as

$$U(\rho, x) = (v - x)\,[\rho\mu_H F_H(x) + (1 - \rho)\mu_L F_L(x)].$$

Let $a(\rho)$ denote the optimal action correspondence,[7] and denote the maximized payoff, the value function by $V(\rho) = (v - a)[\rho\mu_H F_H(a) + (1 - \rho)\,\mu_L F_L(a)]$ for some $a \in \alpha(\rho)$. It follows from standard arguments from the economics of information that the value function V is convex, and the maximum theorem implies that V is a continuous function.

Discussion of the continuation utility function

While the matching and production cost parameters (μ_H, μ_L and F_H, F_L) have similar effects on continuation values, they have different implications for our analysis. First, when only the matching frictions differ (that is $\mu_H \neq \mu_L$, but for all x it holds that $F_H(x) = F_L(x) = F(x)$), then the optimal action does not depend on the beliefs, because $U(\rho, x) = (\rho\mu_H + (1 - \rho)\,\mu_L)\,(v - x)\,F(x)$. In this case the outside option is exogenous in the sense that it is only influenced by the beliefs directly, but there is no adjustment in the

[6]The requirement that $X \subset [\underline{c}, \overline{c}]$ is without loss of generality, since any offer $x \leq \underline{c}$ yields zero utility, while offering more than $\overline{c}$ is strictly dominated by offering exactly $\overline{c}$.

[7]Existence of an optimal decision follows from Weierstrass's theorem.

action when the beliefs change. In this case V is linear, and (as we will see later) our model becomes a special case of the canonical model of interdependent actions that was introduced by Milgrom and Weber (1982). For this case they have shown that the auctioneer reveals all his information to maximize revenues. The main thrust of our analysis is that when this is not the case (that is when the outside option is endogenous), the auctioneer may want to conceal his information.

Further notation for the continuation problem

It is convenient to use the notation $U_H(x), U_L(x)$ that denotes the utility of a losing bidder in states H, L if he takes action x, that is $U_H(x) = (v - x)\mu_H F_H(x)$ and $U_L(x) = (v-x)\mu_L F_L(x)$. With this notation in hand, one can write $U(\rho, x) = \rho U_H(x) + (1-\rho)U_L(x)$, and $V(\rho) = \rho U_H(x) + (1-\rho)U_L(x)$ for some $x \in \alpha(\rho)$. Let $W_H(\rho) = \max_{x\in\alpha(\rho)} U_H(x)$ and let $W_L(\rho) = \min_{x\in\alpha(\rho)} U_L(x)$, denote the value from following an action that is optimal at belief ρ, and it is the action among such optimal actions that yields the best outcome in the high state, and the worst outcome in the low state.[8] As we show it later, the functions W_H, W_L are well behaved (monotone) functions, and also contain all the payoff relevant information for the continuation value problem (together with the value function V), which makes them very useful for our analysis below.

3 Second Price Auction

To start our analysis of the effects of the information policy on revenues we consider an auction format that lends itself to a very tractable analysis, the sealed-bid second price auction. We characterize equilibrium bidding behavior and compare revenues for three different information policies. First, we consider the case where no information (other than who won) is released, second we consider the case where the auctioneer reveals his (perfect) information about the state, and finally we study the policy which reveals the winning bid instead.

[8] If $x \in \alpha(\rho)$ is such that $U_H(x) = W_H(\rho)$, then it must also hold that $U_L(x) = W_L(\rho)$, that is the optimal action that yields the highest utility in state H, is also one with the lowest utility in state L. If this was not the case then action x would payoff dominate some action $y \in \alpha(\rho)$ in both states, which would contradict the fact that $y \in \alpha(\rho)$.

3.1 Equilibrium without information revelation

To make the analysis tractable, we first concentrate on the case where bidders who attach higher probability to the high state (that is bidders with higher ρ) have a lower continuation value. Formally, this assumption states that V is decreasing in ρ. Under this assumption, we show below that there exists a monotone equilibrium. Intuitively, if V is monotone decreasing, then the higher belief (ρ) one holds, the lower the value of his outside option, and the higher bid he should place in the auction.

To start the analysis, note that since the seller does not reveal either his signal or the bids after the auction, a losing bidder does not learn anything beyond the mere fact of having lost. We introduce further notation for our analysis. Let $\eta(s)$ denote an agent's belief conditional on being tied at the top, $s_1 = s_2 = s > s_j$ for all $j > 2$. One can calculate this value as $\eta(s) = \frac{\rho_0 g_H^2(s) G_H^{N-2}(s)}{\rho_0 g_H^2(s) G_H^{N-2}(s) + (1-\rho_0) g_L^2(s) G_L^{N-2}(s)}$. Let $\nu(s)$ denote the probability of the high state conditional on losing with a signal s, that is $\nu(s) = \frac{\rho_0 g_H(s)(1-G_H^{N-1}(s))}{\rho_0 g_H(s)(1-G_H^{N-1}(s)) + (1-\rho_0) g_L(s)(1-G_L^{N-1}(s))}$.

We analyze an equilibrium in monotone and symmetric strategies that is described by a strictly increasing function b. Since both bidders' (continuation) values are affected by the signal of the other bidder, therefore the bidders are in a second price auction with interdependent valuations. Moreover, it is easy to see that signals are affiliated, and thus our model is a special case of the Milgrom and Weber (1982) setup, except for the fact that beliefs affect optimal actions and thus payoffs in the two states. Formally, this difference is captured in our model by having the payoffs in the two states, W_H and W_L, depend on the belief ρ. In the standard Milgrom and Weber setup (the case where W_H and W_L are constant functions of ρ), it is well known that in the symmetric equilibrium of the second price auction, each bidder bids his valuations assuming that he ties at the top spot. We build on this insight, but need to make an adjustment reflecting the new feature of our model that allows beliefs to play a role in determining continuation values through the actions they induce in the future. We can capture this new term, by noting that the relevant continuation value is the one that is assessed conditional on tieing (that is η), but assuming that the action taken is an optimal action induced by the belief upon losing (ν). Formally, then the bid of a bidder with type s can be written as

$$b(s) = v - [\eta(s) W_H(\nu(s)) + (1 - \eta(s)) W_L(\nu(s))]. \tag{1}$$

The following summarizes our findings, providing our existence and uniqueness result:

Proposition 1 *If V is strictly decreasing, then there exist monotone and symmetric equilibria. In every such equilibrium for almost all s each bidder with signal s bids as in (1). If V is not monotone decreasing, then a monotone equilibrium does not exist.*

The proof of the Proposition in Appendix 1 formalizes the argument above. In the course of the proof we establish that W_H is weakly increasing, while W_L is weakly decreasing in ρ for any continuation value problem (see Lemma 2). This intuitive property states that in the high (low) state it is better to take actions that are optimal when the high (low) state is more likely. Monotonicity of these functions imply that W_H, W_L are almost everywhere continuous. Then using the monotonicity properties of W_H, W_L we establish that at all continuity points of these functions any optimal action provides the same utility in the continuation problem, thus there is a unique optimal action in terms of payoff consequences. (See Lemma 2 in the Appendix.) Using this last observation the first order condition (1) is shown to be necessary for optimal bidding for almost all s (that is for all s such that $\nu(s)$ is a continuity point of W_H, W_L). To check that global sufficiency conditions also hold for the bidders' problem we use basic properties of affiliated random variables, extending the analysis of Milgrom and Weber (1982). The fact that b is strictly monotone under our assumptions can be established using these same properties.[9]

Before we continue with our main analysis it is useful to discuss our formal approach. In what follows we will mostly use the induced functions W_H, W_L, V and do not usually relate it to the primitives of the model, μ_H, μ_L, F_H, F_L. This way we emphasize the generality of our findings in the context of models with two states. Moreover, the analysis shows that the main results only depend on the functions W_H, W_L, but not directly on the primitives of the model. However, we also provide a sufficient condition for a value function V to be generated by our model of bargaining with appropriate parameter values for μ_H, μ_L, F_H, F_L.

Lemma 1 *Take any twice differentiable convex function $V : [0,1] \to \mathbb{R}$, and define functions W_L, W_H as $W_L = V - \rho V'$ and $W_H = V + (1-\rho)V'$, and let $\underline{v} = 1 + W_L(1)$. If $W_L, W_H > 0$, and $\frac{W_L}{\underline{v}-\rho}$ is strictly increasing in ρ, then there exist values $v (> \underline{v})$, μ_L, μ_H and functions F_L, F_H such that the induced value function in the continuation bargaining problem is V.*

[9] To prove that b is a strictly increasing function if V is monotone we use two observations. First, the tieing and the losing posterior are monotone in the signal. Second, the tieing posterior is lower than the losing posterior, which holds if bids are monotone.

The proof is provided in Appendix 1. The key step is to prove that a useful envelope condition and a basic incentive condition for type ρ implies that W_H, W_L are pinned down by V as it is stated in the Lemma. Let us provide the sketch of the argument here. Assuming that V is differentiable at a point ρ let us suppose that there is a unique optimal action in the continuation value problem and denote it by $a^*(\rho)$. Then standard arguments imply that

$$V'(\rho) = \frac{\partial U}{\partial \rho}(\rho, x) \mid_{x=a^*(\rho)} = U_H(a^*(\rho)) - U_L(a^*(\rho)) \tag{2}$$

and then by construction

$$V'(\rho) = W_H(\rho) - W_L(\rho) \tag{3}$$

for all such values of ρ. Using the monotonicity properties of functions W_H, W_L in Appendix 1 we establish that these formulas hold for almost all ρ. Also, by construction for almost all ρ it holds that $\rho W_H(\rho) + (1-\rho)W_L(\rho) = V(\rho)$ must hold. Using this and (3) indeed pins down W_H and W_L if one knows V. Then the rest of the proof shows that from the implied functions W_H, W_L one can define a bargaining problem where the optimal action of a buyer with belief ρ is to make an offer ρ. We only need to make sure that the implied functional forms and parameter values of the bargaining problem (candidates for μ_H, μ_L, F_H, F_L) can be interpreted as probabilities and distribution functions, which is guaranteed by the assumptions of the Lemma. In all the examples provided later, all chosen value functions are such that the sufficient conditions of that Lemma are satisfied, and thus one could relate the value function (in a constructive manner) to the underlying bargaining problem. The class of value functions generated by a bargaining problem is thus rich enough to provide the intuition for all our results.

For our mechanism design analysis the monotonicity result of Lemma 1 is a convenient tool, but the result that hiding information may increase revenues does not depend on having a monotone equilibrium. In particular, when we conduct our revenue comparisons we provide an example where V is non-monotone (Example 3), and show that a non-monotone bidding equilibrium exists. Importantly, hiding information is in fact more likely to be profitable when V is non-monotone as we show it in the analysis of that example.

3.2 Revenue comparison when the state may be revealed

For the rest of the paper, except when it is stated otherwise, we assume that V is monotone, and concentrate on the monotone equilibrium in the game with no information revelation.

For analytical simplicity we assume that V is twice differentiable,[10] for which it is sufficient if the optimal action is a smooth function of the belief. Under this assumption, $V' = W_H - W_L$ is continuous. Since W_H is increasing and W_L is decreasing in ρ, therefore it follows that functions W_H, W_L must be continuous. Using the envelope condition (3) and the incentive condition of type ρ we establish in Appendix 1 (as Lemma 3) that if V is twice differentiable, then for all $\rho \in [0,1]$ it holds that

$$W_H(\rho) = \beta - \int_{\rho}^{1} (1-z)V''(z)dz \tag{4}$$

and

$$W_L(\rho) = 1 - \int_{0}^{\rho} zV''(z)dz. \tag{5}$$

Note, that if there is no significant action to be taken, and thus information has no value, then V is linear and W_H, W_L are constant functions. In this case our setup reduces to the common value setup of Milgrom and Weber (1982). Using the above formulas, the following Corollary provides a necessary and sufficient condition for a monotone equilibrium to exist:

Corollary 1 *If V is twice differentiable, then a monotone and symmetric equilibrium exists if and only if $\beta = W_H(1) \leq W_L(1)$.*

Proof. By Proposition 1 we only need to prove that V is monotone if and only if $\beta \leq 1$. First, note that Lemma 2, where we establish that (3) holds almost everywhere, implies that we only need to prove that for all ρ it holds that $W_H(\rho) \leq W_L(\rho)$. Second, (4) and (5) imply that W_H is increasing, while W_L is decreasing in ρ. Putting these observations together yields that for monotonicity of V we need that $W_H(1) \leq W_L(1)$. **Q. E. D.**

To calculate the expected revenue of the second price auction without state revelation, let $g^{(2)}(s)$ denote the density function of the second largest signal of the N signals from an ex-ante perspective. The bidder with such a type will determine the revenue in the second price auction. Formally,

$$g^{(2)}(s) = \rho_0 N g_H(s)(1 - G_H(s))G_H^{N-1}(s) + (1-\rho_0)\, N g_L(s)(1 - G_L(s))G_L^{N-1}(s)$$

[10] This assumption could be relaxed without changing any of the results, but would complicate exposition. Moreover, we know that V is convex and thus twice differentiable almost everywhere, so this assumption is just to avoid kinks at countably many points.

holds. Then using (4) and (5), the ex-ante expected revenue of the seller can be written as

$$ER_n = \int_0^1 g^{(2)}(s)b(s)ds =$$

$$= v - 1 + (1-\beta)\int_0^1 g^{(2)}\eta ds + \int_0^1 g^{(2)}[\eta \int_\nu^1 (1-z)V''(z)dz + (1-\eta)\int_0^\nu zV''(z)dz]ds.$$

Define

$$\gamma = \int_0^1 g^{(2)}(s)[\eta(s)\int_{\nu(s)}^1 (1-z)V''(z)dz + (1-\eta(s))\int_0^{\nu(s)} zV''(z)dz]ds.$$

With this notation in hand, one can rewrite the expected revenues from not revealing the state as

$$ER_n = v - 1 + (1-\beta)\int_0^1 g^{(2)}(s)\eta(s)ds + \gamma. \tag{6}$$

The variable γ has a simple interpretation: it measures the loss in revenue if the state is revealed *after* the auction compared to the case where it is not revealed ever. To see this, imagine that the state is revealed after the auction is run. In this case the optimal action is taken in both states yielding utilities $W_H(1)$ and $W_L(0)$. Therefore, the continuation value conditional on tieing at the top is given as

$$\eta(s)W_H(1) + (1-\eta(s))W_L(0) = \eta(s)\beta + (1-\eta(s)),$$

and thus the equilibrium bid is $v- (\eta(s)\beta + (1-\eta(s))) = v-1+\eta(s)(1-\beta)$. This implies that the expected revenue in the auction where the state is revealed after the auction can be written as $v-1+(1-\beta)\int_0^1 g^{(2)}(s)\eta(s)ds$. Comparing this revenue formula with (6) implies that γ is indeed equal to the revenue loss that accrues from revealing the state after the auction. The variable γ can be naturally linked to the importance of the post auction decision as well. Suppose that the decision problem is trivial, which corresponds to the case where $V''(z) = 0$ for all z. In this case there is no significant action to be taken, and indeed $\gamma = 0$ holds. In general, the higher V'' is, the more the post auction decisions matter (the higher the value of information is), and the higher γ becomes.

If the seller reveals his signal before the auction is run, the symmetric equilibrium is simple. Given that the seller's signal is perfectly informative, the bidders now know the state. Thus, all bidders bid $v - V(1)$ in the high state, and all bidders bid $v - V(0)$ in the low state. The ex-ante expected revenue can be simply written as

$$ER_y = v - (1-\rho_0)\,V(0) - \rho_0 V(1) = v - 1 + \rho_0\,(1-\beta)\,. \tag{7}$$

Using the revenue formulas from above, we are ready to analyze how revealing information in a second price auction affects revenues. Recall that γ measures the importance of the post auction decisions, and β measures the difference in expected payoffs in the two states as defined above. The revenue comparison is as follows:

Proposition 2 *The sellers' revenue is higher without revealing his informative signal, if and only if the parameters of the decision problem are such that*

$$\gamma > (1-\beta)[\rho_0 - \int_0^1 g^{(2)}(s)\eta(s)ds]. \tag{8}$$

Proof. The result is immediate after comparing formulas (6) and (7). **QED**

To interpret the result, let $\delta = (1-\beta)[\rho_0 - \int_0^1 g^{(2)}(s)\eta(s)ds] > 0$. Then δ is equal to the revenue gain from revealing the state before the auction compared to revealing it after the auction.[11] The Proposition can now be understood as a decomposition of the revenue effect of revealing the state (before the auction) into two parts. The right hand side variable δ measures the revenue enhancing effect of revealing the state *before* the auction and is reminiscent of the linkage principle reasoning by Milgrom and Weber (1982). When the two states provide very different utilities, that is β is low, then the standard winner's curse phenomenon arises and δ is high. In this case it is very important for the bidders not to overpay when winning in the low state, and thus bidders reduce their bids to overcome the winner's curse. This in turn implies that revealing any extra information helps the auctioneer, by the well known *linkage principle effect*. The left hand side variable γ is new compared to Milgrom and Weber (1982). By revealing the state *after* the auction, the auctioneer allows the losing bidders to take better actions in the continuation value problem and thus increases their outside options. This depresses the bids in the original auction, a revenue decreasing effect we call the *continuation value effect*. To summarize: the linkage principle effect (favoring information revelation) is strongest when the two states are fundamentally very different in terms of payoffs, that is when β is much lower

[11] Revealing the state before the auction yields ER_y, while revealing it after yields $v - 1 + (1 - \beta)\int_0^1 g^{(2)}(x)\eta(x)dx$ as we argued above. The fact that $\delta > 0$ follows from the law of iterated expectation that implies that $\int_0^1 g^{(2)}(x)\nu(x)dx = \rho_0$ and the fact that for all x it holds that $\nu(x) > \eta(x)$.

than 1, and the continuation value effect (favoring hiding information) is strongest when future actions are important, that is when γ is high. Therefore, the seller should reveal his information about the state if the resulting improvement in decision quality is not too high compared to the resulting gain from alleviating the winner's curse.

The precision of the bidder signals also plays an important determinant of the optimal information policy. If signals are always completely uninformative for all bidders, then $\rho_0 - \int_0^1 g^{(2)}(s)\eta(s)ds = 0$, and thus hiding information is profitable for any $\beta < 1$. In this case both with and without revealing the state all the bidders share the same beliefs (ρ_0 is the common belief when the state is not revealed, and all bidders know the actual state when the state is revealed). Therefore, they engage in Bertrand-competition to compete away all their rents relative to their continuation values. However, the continuation values are higher in the case with state revelation, and thus bids are lower if the state is revealed. In more technical terms, the linkage principle effect is absent in this case, while the continuation value effect is present to favor hiding information.

It is interesting to revisit our bargaining model that we provided in the Setup section. Assume first, that $\mu_H < \mu_L$, but $F_H(p) = F_L(p)$ for all p. In this case, there is less chance in the high state to continue, but the optimal price is independent of the state. Therefore, the optimal action is the same regardless of the beliefs about the state, and thus function V'' is identically zero, and thus $\gamma = 0$ holds. On the other hand, it is more profitable to be in the high state than in the low state, and thus $\beta < 1$. Therefore, the condition of Proposition 2 fails, and revealing information is revenue enhancing. This result can be interpreted as follows: the continuation value effect is absent, since the optimal decision does not depend on the information learnt, but the linkage principle effect is present and thus, following Milgrom and Weber, revealing information is beneficial for the seller. On the other hand, suppose that $F_H(p) \neq F_L(p)$ for all p. In this case the optimal action depends on the belief and thus the continuation value effect becomes important too, and the revenue comparison depends on the exact functional form assumptions.

We are ready to discuss the strength of the condition in the Proposition above, that is how likely it is in general that the continuation value effect dominates. A monotone equilibrium exists if the two states provide different continuation utilities (see Corollary 1), which is exactly the case when the linkage principle effect is strong. Therefore, it may happen that whenever a monotone equilibrium exists the linkage principle effect always dominates the continuation value effect, and thus revealing information is always beneficial. To study this issue, the following Corollary describes under what conditions on the signal distribution and utility functions can one find values of β such that revealing the state may decrease expected revenues:

Corollary 2 *If* $\gamma > (\rho_0 - \int_0^1 g^{(2)}(s)\eta(s)ds) \int_0^1 zV''(z)dz$*, then there exists a high enough value of* β *such that revealing information decreases the expected revenues. If this condition does not hold, then for all values of* β *such that there exists a monotone equilibrium in the game without state revelation, the expected revenue from revealing the state is higher than the expected revenue from hiding it.*

Proof. By Corollary 1, a monotone equilibrium exists if and only if $W_H(1) \leq W_L(1)$ given that V is twice differentiable and thus W_H, W_L are continuous. This condition can be rewritten as

$$1 - \beta > \int_0^1 zV''(z)dz.$$

So for (8) and this last condition to hold simultaneously it must be that $\gamma > (\rho_0 - \int_0^1 g^{(2)}(s)\eta(s)ds) \int_0^1 zV''(z)dz$. Conversely, if this condition holds, then an appropriate β can be found such that both a monotone equilibrium exists, and the revenue is higher without state revelation. **Q.E.D.**

Let us consider an example for each of the two possibilities.

Example 1: Let $\rho_0 = 1/2$, $N = 2$, $g_L = 2(1-s)$ and $g_H = 2s$, and $G_H(s) = s^2$, $G_L(s) = 2s - s^2$, and $V(\rho) = \alpha\rho^n + A\rho + B$ with $n = 1.1$, $\alpha = 1/n$, $B = 1$, $A = -1$. Then, using (16) and (17):

$$W_H(\rho) = \rho^{n-1} + \alpha\rho^n(1-n),$$

and

$$W_L(\rho) = \alpha\rho^n(1-n) + B.$$

One can show that the conditions for monotonicity hold. Moreover, after making the necessary substitutions we obtain that hiding the state is profitable for the seller with $\gamma = 0.01878$ and $\delta = 0.01515$.

Example 2: Let $\rho_0 = 1/2$, $N = 2$, $g_L = 2(1-s)$ and $g_H = 2s$, and $G_H(s) = s^2$, $G_L(s) = 2s - s^2$, and

$$W_H(\rho) = \beta - \alpha(1-\rho)^2,$$

and

$$W_L(\rho) = 1 - \alpha\rho^2,$$

for some $\alpha, \beta \geq 0$. We assume that $\beta \leq 1 - \alpha$, ensuring that $W_H(1) \leq W_L(1)$ and thus V is decreasing. The conditions of Lemma 1 hold if $\alpha < 1/2$ is satisfied. The condition of Proposition 2 for hiding the state to be revenue enhancing becomes $\frac{\alpha}{1-\beta} > C \approx 1.2$, and the monotonicity condition for V is simply $\frac{\alpha}{1-\beta} \leq 1$. The two conditions are not compatible, and thus if one chooses β such that a monotone equilibrium exists then it always holds that $ER_n < ER_y$. In other words the linkage principle effect dominates the continuation value effect whenever a monotone equilibrium exists.

If one considers the case where a monotone equilibrium does not exist in Example 2, then it can be shown that the continuation value effect can dominate the linkage principle effect. This is pursued in the example below.

Example 3:

Let us reconsider Example 2 with $\beta = 1$. There is no monotone equilibrium if $\alpha > 0$. Since the two states are symmetric, so it is natural to concentrate on a signal symmetric equilibrium where $b(s) = b(1-s)$ for all s. In this case one can recalculate the relevant tieing posterior as

$$\widetilde{\eta}(s) = \Pr(H \mid s_1 = s, s_2 = s \text{ or } s_2 = 1 - s) = s.$$

The relevant losing probability is

$$\widetilde{\nu}(s) = \Pr(H \mid s_1 = s, s_2 \in (s, 1-s)) = s.$$

Then in the case of not revealing the state the equilibrium bid function can be written as

$$b(s) = v - [s(1 - \alpha(1-s)^2) + (1-s)1 - \alpha s^2] = v - 1 + \alpha s(1-s) > v - 1.$$

The bid with information revelation is $b = v - 1$ in both states, and thus the revenue comparison favors not revealing the state. This example shows that when the two states are similar (that is $V(0) = V(1)$), then the linkage principle loses its bite, and although a monotone equilibrium does not exist, it follows immediately that revealing the state decreases revenues. This result generalizes to arbitrary density functions and continuation value functions, as long as the two states are similar.

3.3 Revenue when the winning bid can be revealed

The other important question studied for the optimal information policy in auction design is whether any bid information should be revealed by the seller. To study this question in our context, let ER_b denote the expected revenue when the winning bid is revealed, assuming that V is decreasing (that is the conditions of Corollary (1) hold). The following result is established below:

Proposition 3 *Revealing the bids ex post decreases revenue relative to not revealing anything,* $ER_b < ER_n$.

Proof. Suppose the seller reveals only the winning bid. In this case the bid function becomes

$$b_b(s) = v - [\eta(s)W_H(\eta(s)) + (1 - \eta(s))W_L(\eta(s))].$$

Comparing it with the case of no such bid revelation yields

$$b_b(s) = v - [\eta(s)W_H(\eta(s)) + (1 - \eta(s))W_L(\eta(s))]$$

$$< v - [\eta(s)W_H(\nu(s)) + (1 - \eta(s))W_L(\nu(s))] = b(s),$$

where the inequality follows from the fact that $\rho = \arg\max_{q\in[0,1]} \rho W_H(q) + (1-\rho) W_L(q)$. That is the bid of a type s if the winning bid is revealed is lower than in the benchmark case of no information revelation. Therefore,

$$ER_b = \int_0^1 g^{(2)}(s)b_b(s)ds < \int_0^1 g^{(2)}(s)b(s)ds = ER_n.$$

So revealing the bids after the auction decreases revenues. Q. E. D.

A similar argument implies that revealing *any information* after the auction is harmful for the auctioneer. The intuition for this result is simple: by revealing information *after* the auction, the linkage principle effect is absent, since the bidders cannot use the information when bidding, and thus they continue to depress their bids to avoid the winner's curse. On the other hand, in the continuation problem they can take advantage of the information they learnt, hence the continuation value effect is present.

4 Format comparisons: first-price, second-price and English-auctions

In the literature initiated by Milgrom and Weber (1982), another interpretation of the linkage principle is that the more an auction format links the payments to the types of the other agents, the higher the expected revenue is. For example, in a first price auction the payment of a type conditional on winning, is just the submitted bid, which does not depend on the types of the other bidders. In contrast, in a second price auction the winner pays the second highest bid, and thus the expected payment (conditional on winning) depends positively on the types of the other bidders. Therefore, the linkage principle implies that (in the standard case without endogenous outside options, that is when $\gamma = 0$) the second price auction yields a higher expected revenue than the first price auction. Similarly, an English-auction links payments to others' bids (types) even more, since now all the types except for the two highest are revealed by the bid at which those bidders are dropping out. Therefore, the English-auction links the payment to bids even more, and thus yields a higher expected revenue than even the second-price auction.

We saw in the previous Section that in our model with endogenous outside options the linkage principle effect is counteracted by our continuation value effect, when we analyze whether the auctioneer should reveal his exogenous information about the state of the world. It is natural to ask whether the same is true when one compares the three standard auction formats fixing the information policy. To conduct such a comparison, assume that the auctioneer does not reveal any information and runs a first-price, second-price or ascending auction. First, one can show that the revenue ranking is unchanged between the first two formats, that is the second price auction revenue dominates the first price auction. This can be done by modifying the analysis of Krishna (2008), Section 7, pages 105-108. This result follows, because upon losing the same information is learned in the two auction formats: in both auctions the losers only learn that there was a bidder with a higher signal than theirs. This implies that they take the same actions in the continuation decision problems, and therefore the presence of endogenous outside options does not change the comparison between the two formats as compared to the standard case. The formal argument is provided in Appendix 2.

The important novelty when one compares second-price and English auctions is that the English-auction reveals more information than a second-price auction and thus the optimal decision after the two auctions are different. In particular, the English auction allows the losers to take better decisions, and thus the continuation value effect favors the second price auction over the English auction. Since the continuation value and the linkage principle effects work in opposite directions, one needs to assess whether the second-price or the English-auction raises higher revenues. To present our result, we concentrate on a three player version of Example 1 above, where a monotone equilibrium exists and the sellers' revenue is higher in the second price auction than in the ascending auction.[12]

Example 4:

We assume that $\rho_0 = 1/2$, $N = 3$, $g_L = 2(1-s)$ *and* $g_H = 2s$, *and*

$$W_H(\rho) = \rho^{n-1} + \alpha\rho^n(1-n),$$

and

$$W_L(\rho) = \alpha\rho^n(1-n) + B$$

with $\alpha = 1/n$, $n = 1.1$, $B = 1$.

In this example the continuation value effect is stronger than the linkage principle effect, and revealing information via holding a more open auction decreases revenues. If one considers variations in the parameter values, then the revenue comparison has the same qualitative features as in the case of state revelation: if the two states provide very different utility values (that is B is high), then the linkage principle effect dominates. On the other hand, if α is high and thus the actions are important, then the continuation value effect is more likely to be stronger.

5 Discussion

In this Section we consider two extensions to inspect the robustness of our results. First, we show that the assumption of two states can be relaxed without changing the results. We focus on comparing the revenues from the second price auction with and without the revelation of the winning bid, the question addressed in Section 3.3 for the case of two states. Let $t \in [0,1]$ denote the state of the world, and let $U_t(a)$ denote the continuation value when action a is taken in state t. Let g_t denote the conditional distribution of signals in state t, and let h the density function for the state of the world. Assuming that for all $U_t(a)$ is decreasing in t implies that there is an equilibrium with monotone bidding. Let

[12] The calculations related to this exmple are in Appendix 3.

$\eta_t(s)$ be the density of state t if one ties at the top with signal s, that is

$$\eta_t(s) = \frac{h(t)g_t^2(s)G_t^{N-2}(s)}{\int_0^1 h(z)g_z^2(s)G_z^{N-2}(s)dz}.$$

Let $\nu_t(s)$ be the density of state t if one lost with signal s, that is

$$\nu_t(s) = \frac{h(t)g_t(s)(1-G_t^{N-1}(s))}{\int_0^1 h(z)g_z(s)(1-G_z^{N-1}(s))dz}.$$

Then the optimal action after losing with signal s satisfies

$$a(s) = \arg\max_{x\in X} \int_0^1 \nu_t(s)U_t(x)dt.$$

Without bid revelation the equilibrium bid is

$$b_n(s) = v - \int_0^1 \eta_t(s)U_t(a(s))dt.$$

With bid revelation the tieing loser learns that he in fact tied with the winner and takes an action

$$a^*(s) = \arg\max_{x\in X} \int_0^1 \eta_t(s)U_t(x)dt. \tag{9}$$

Therefore, the equilibrium bid becomes

$$b_y(s) = v - \int_0^1 \eta_t(s)U_t(a^*(s))dt.$$

By (9) it follows that $b_n(s) > b_y(s)$, so the revenue comparison result follows the same way as in the two-state model.

Second, suppose that the winning bidder takes an action too, and his continuation utility functions are W_H^w, W_H^w, which have similar properties to W_H, W_L, the continuation utility of the losers. We keep the assumption that the winner obtains a utility v from the object and that there are two states. Let us now concentrate on the question whether in the second-price format state revelation enhances or reduces revenues; the other questions can be studied similarly. Following similar argument as in the benchmark case, the equilibrium bid function without state revelation is

$$v + [\eta(s)(W^w_H(\nu(s)) - W_H(\nu(s))) + (1-\eta(s))(W^w_L(\nu(s) - W_L(\nu(s))].$$

From the bid function, one can calculate the equilibrium expected revenue when no information is revealed by the auctioneer.[13] When the state is revealed, in the high state all bidders bid $v + V^w(1) - V(1)$, and in the low state all bid $v + V^w(0) - V(0)$. From these observations it is obvious that if the winner's continuation values are not very sensitive to the state of the world (that is $W^w_H - W^w_L$ is uniformly close to zero and thus V^w is close to being a constant function), then the revenue comparison is similar to the benchmark case where the winner's continuation problem was omitted. Therefore, our results are robust as long as the winner's continuation problem is not too sensitive to the state as compared to the losers'. It is worth pointing out that if the winner's continuation problem is more important than the loser's, then the continuation value effect favors information revelation, and thus transparent auctions are revenue enhancing. The reason is that if more information is available after the auction, then the winner can make a better decision, which then makes bidders more aggressive since the winning prize has become more valuable. In more technical terms, both the linkage principle and the continuation value effects favor information revelation in this case.

6 Conclusion

We study auctions with endogenous outside options determined through actions taken in the aftermarket. We show that endogenous outside options have important consequences for auction design. In contrast to the case of exogenous outside options, auctions with less information revelation may yield higher revenues. Opaque auctions decrease the information available to losing bidders, which leads to worse decisions in the aftermarket. This leads to worse outside options, and thus more aggressive bidding in the original auction. Effects that favor non-transparent auctions include a small payoff difference between the two states, a great value of information in the continuation problem, and imprecise signals of the bidders. The timing of information revelation is important: it is never optimal to reveal information after the auction, while it may be to optimal to reveal information before the auction. We also show that a less transparent auction format, the second price auction

[13] The main steps are similar to the benchmark case and are omitted.

can yield higher revenues than an English auction, as it fosters less learning and provides lower continuation values for the bidders. The model is robust to introducing several states, and with respect to the winner's having state dependent continuation functions.

7 Appendix

7.1 Appendix 1

In this Appendix we prove Proposition 1, and several useful results about the continuation problem. We start by establishing a monotonicity result:

Lemma 2 *For all $\rho' > \rho$ and $U_H(a') - U_L(a') > U_H(a) - U_L(a)$ it holds that if type ρ weakly prefers a' over a, then type ρ' strictly prefers a' over a. Therefore, for all $b' \in \alpha(\rho')$ and $b \in \alpha(\rho)$ it holds that $U_H(b') - U_L(b') \geq U_H(b) - U_L(b)$. If for some $\rho' > \rho$ it holds that $e \in \alpha(\rho')$ and $f \in \alpha(\rho)$, then $U_H(e) \geq U_H(f)$ and $U_L(e) \leq U_L(f)$; and for almost all $\rho \in [0,1]$ if $c, d \in \alpha(\rho)$ then $U_H(c) = U_H(d)$ and $U_L(c) = U_L(d)$. Function W_H is monotone increasing, while W_L is monotone decreasing.*

Proof of Lemma 2:

Proof. Suppose that $\rho' > \rho$ and $U_H(a') - U_L(a') > U_H(a) - U_L(a)$, and type ρ weakly prefers a' over a, that is

$$\rho U_H(a') + (1-\rho)U_L(a') \geq \rho U_H(a) + (1-\rho)U_L(a). \tag{10}$$

Then $U_H(a') - U_L(a') \geq U_H(a) - U_L(a)$ implies that

$$\left(\rho' - \rho\right)\left(U_H(a') - U_L(a')\right) > \left(\rho' - \rho\right)\left(U_H(a) - U_L(a)\right).$$

Adding the last inequality to (10) implies that

$$\rho' U_H(a') + (1-\rho')U_L(a') > \rho' U_H(a) + (1-\rho')U_L(a),$$

which establishes the first claim. To prove the second statement, suppose that $U_H(b) - U_L(b) > U_H(b') - U_L(b')$. Then the first statement implies that type ρ' strictly prefers b over b', which contradicts with the assumption that $b' \in \alpha(\rho')$. The second statement implies that $U_H(e) - U_L(e) \geq U_H(f) - U_L(f)$. Then $U_H(e) < U_H(f) \Longrightarrow U_L(e) < U_L(f)$, which implies that e is worse than f for any beliefs, and thus $e \in \alpha(\rho')$ could not hold. This contradiction establishes the third claim. To prove the last claim let $\tau(\rho) = \max_{x \in \alpha(\rho)} U_H(x) - U_L(x)$. The second statement implies that τ is weakly increasing, and thus it is almost everywhere continuous. Moreover, at every continuity point ρ of τ it holds that for all $c, d \in \alpha(\rho)$, $U_H(c) - U_L(c) = U_H(d) - U_L(d)$.[14] Then suppose that $U_H(c) > U_H(d)$. In this case, it would follow that $U_H(c) > U_H(d)$, implying that c dominates d and contradicting $d \in \alpha(\rho)$. This contradiction establishes the last result about functions U_H, U_L. The monotonicity claim about W_H, W_L then follows by construction. **Q.E.D.**

Next we prove that the useful envelope conditions from the main text hold for almost all ρ. In particular they hold at every continuity point of W_H, W_L. Let us formally state our claim first:

For almost every ρ it holds that

$$a \in \alpha(\rho) \Longrightarrow V'(\rho) = U_H(a) - U_L(a) \tag{11}$$

and

$$V'(\rho) = W_H(\rho) - W_L(\rho). \tag{12}$$

Proof. Take any ρ and let $a \in \underset{x \in \alpha(\rho)}{\arg\max} U_H(x)$, $b \in \underset{x \in \alpha(\rho)}{\arg\min} U_H(x)$. Note, that by definition of $\alpha(\rho)$ it must hold that $a \in \underset{x \in \alpha(\rho)}{\arg\min} U_L(x)$, $b \in \underset{x \in \alpha(\rho)}{\arg\max} U_L(x)$ and thus for all $x \in \alpha(\rho)$

$$U_H(a) - U_L(a) \geq U_H(x) - U_L(x) \geq U_H(b) - U_L(b).$$

Next, note that for all $\rho' > p$ it holds that $V(\rho') \geq U(\rho', a)$. Therefore,

$$V(\rho') - V(\rho) \geq (\rho' - \rho)(U_H(a) - U_L(a)).$$

Also, the right hand derivative of a convex function exists everywhere, therefore the right hand derivative at ρ satisfies

$$V'_+(\rho) \geq U_H(a) - U_L(a).$$

[14] Suppose that $c, d \in \alpha(\rho)$, and $U_H(c) - U_L(c) > U_H(d) - U_L(d)$. Then for all $\rho' > \rho$ it holds by the second claim that for any $c' \in \alpha(\rho')$, $U_H(c') - U_L(c') \geq U_H(c) - U_L(c)$, and thus $\tau(\rho') \geq U_H(c) - U_L(c)$. Similarly, for any ρ'' and $d' \in \alpha(\rho'')$, $U_H(d') - U_L(d') \leq U_H(d) - U_L(d)$, and thus $\tau(\rho'') \leq U_H(d) - U_L(d)$. Therefore, the function τ must have a jump at such a ρ.

A similar argument yields that the left hand derivative satisfies

$$V'_{-}(\rho) \leq U_H(b) - U_L(b).$$

Since $V'(\rho)$ exists almost everywhere, therefore for almost every ρ it must hold that $U_H(a) - U_L(a) = U_H(b) - U_L(b)$. Therefore, wherever a derivative exists (which is almost everywhere) it holds that $V'(\rho) = U_H(a) - U_L(a)$ for all $a \in \alpha(\rho)$, which establishes that *(2)* for almost all ρ. To establish that (*(12)* holds for all continuity points of W_H, W_L (which is almost everywhere) note that the argument of Lemma 2 implies that for any such continuity point ρ it holds that if $c, d \in \alpha(\rho)$ then $U_H(c) = U_H(d)$ and $U_L(c) = U_L(d)$. Therefore, the argument establishing (*11*) applies to show that $V'(\rho) = U_H(c) - U_L(c) = W_H(\rho) - W_L(\rho)$, concluding the proof. **Q.E.D.**

We are ready to prove the existence of a monotone equilibrium stated in Proposition 1.

Proof of Proposition 1:

Proof: From Lemma 2, we know that for almost all ρ all the optimal actions induce the same utilities in both states. We first concentrate on such values of ρ and then by construction the induced utilities by the optimal action(s) are equal to $W_H(\rho), W_L(\rho)$. We discuss what happens at other values of ρ at the end of the proof.

First, we show that the above defined bid function constitutes an ex-post equilibrium. Symmetry of b is immediate, while monotonicity follows from the facts that η, ν are increasing, $\eta < \nu$ and that the monotonicity of V implies (via Lemma 1) that $U_H(x) \leq U_L(x)$ for all $x \in A$. To see this, note that

$$\frac{d}{dx}[\eta(x)W_H(\nu(x)) + (1 - \eta(x))W_L(\nu(x))] =$$

$$\eta'(x)(W_H(\nu(x)) - W_L(\nu(x))) + \nu'(x)V''(\nu)(\eta(1 - \nu) - (1 - \eta)\nu) < 0$$

follows from the observations above. But this is equivalent to $b'(x) > 0$.

Next, we show that if it is known that $s_1 = s_2 = x$, then winning with $b(x)$ yields the same utility as losing and acting in the future as if the probability of the high state was $\nu(x)$. Losing yields a continuation utility that is equal to $\eta(x)W_H(\nu(x)) + (1 - \eta(x))W_L(\nu(x))$ by construction, while winning with bid $b(x)$ yields a utility $v - b(x)$, which is equal to the continuation utility upon losing.

It also has to be established that if $s_i = x$ then winning against a type $y > x$ with bid $b(y)$ is unprofitable, while if $y < x$ then winning against type y with bid $b(y)$ is profitable.[15] Let us just inspect the $y > x$ case, the other one is similar. In this case winning, upon tieing, yields a utility of

$$v - b(y) = \eta(y) W_H(\nu(y)) + (1 - \eta(y)) W_L(\nu(y)).$$

To calculate the utility from losing, upon tieing, let us introduce the relevant tieing posterior when one bids $b(y)$ and has type x as follows:

$$h(x, y) = \frac{\rho_0 g_H(x) g_H(y) G_H^{N-2}(y)}{\rho_0 g_H(x) g_H(y) G_H^{N-2}(y) + (1 - \rho_0) g_L(x) g_L(y) G_L^{N-2}(y)}.$$

By the fact that g_H and g_L satisfy the MLRP, it follows that $\eta(y) > h(x, y)$. One can similarly define the relevant losing posterior as

$$n(x, y) = \frac{\rho_0 g_H(x)(1 - G_H^{N-1}(y))}{\rho_0 g_H(x)(1 - G_H^{N-1}(y)) + (1 - \rho_0) g_L(x)(1 - G_L^{N-1}(y))}.$$

Again, the MLRP condition implies that $\nu(y) > n(x, y)$. Then the utility upon losing (and tieing) can be written as

$$h(x, y) W_H(n(x, y)) + (1 - h(x, y)) W_L(n(x, y)).$$

First, we show that

$$h(x, y) W_H(n(x, y)) + (1 - h(x, y)) W_L(n(x, y)) \geq$$

$$\geq h(x, y) W_H(\nu(y))) + (1 - h(x, y)) W_L(\nu(y)). \quad (13)$$

To see this, note that by construction

$$n(x, y) W_H(n(x, y)) + (1 - n(x, y)) W_L(n(x, y)) \geq$$

$$\geq n(x, y) W_H(\nu(y))) + (1 - n(x, y)) W_L(\nu(y)). \quad (14)$$

[15] Again, we need to use the relevant tieing belief $\Pr(H \mid s_1 = x, s_2 = y)$ and the relevant action inducing belief upon losing $\Pr(H \mid s_1 = x, s_2 \geq y)$.

Also, it holds that $\nu(y) > n(x,y) > h(x,y)$ and Lemma 1 and the monotonicity of V implies that $W_H(n(x,y)) - W_L(n(x,y) \leq W_H(\nu(y)) - W_L(\nu(y))$. Thus it follows that

$$(h-n)\left(W_H(n(x,y)) - W_L(n(x,y))\right) \geq (h-n)\left(W_H(\nu(y)) - W_L(\nu(y))\right). \qquad (15)$$

Adding up (14) and (15) implies (13). Then using (13), the utility difference between losing and winning satisfies

$$\triangle = h(x,y)W_H(n(x,y)) + (1-h(x,y))W_L(n(x,y)) - (\eta(y)W_H(\nu(y)) + (1-\eta(y))W_L(\nu(y))) =$$

$$\geq hW_H(\nu(y)) + (1-h)W_L(\nu(y)) - (\eta W_H(\nu(y)) + (1-\eta)W_L(\nu(y))) =$$

$$= (h-\eta)(W_H(\nu(y)) - W_L(\nu(y))) \geq 0,$$

where the last inequality follows, because $\eta > h$ and $W_H(\nu(y)) \leq W_L(\nu(y))$ by monotonicity of V (via Lemma 1). Therefore, it is indeed more profitable to lose against a type y than to win if one's type is $x < y$. This concludes the proof of global optimality for the bidders' problem.

Uniqueness of b as in (1) follows from the above argument as well, since upon tieing indifference has to hold in an ex-post equilibrium which yields exactly (1) after taking it into account that the equilibrium is symmetric and monotone. The only caveat is that the bid function is not determined at the (at most countably many) discontinuity points of W_H, W_L. At such a belief x, the optimal action in the continuation problem is not unique which introduces multiple optimal bids when the belief is x. However, there are at most countably many such jump points, so this multiplicity arises only for a small set of types, and for all other beliefs the equilibrium bid is pinned down by formula (1).[16]

Finally, if V is not monotone, then one can show that the candidate bid function (1) is not increasing, which completes our proof. **Q.E.D.**

Proof of Lemma 1:

Proof: Since V is a twice differentiable value function, therefore $V' = W_H - W_L$ for all ρ. This implies that for all ρ

$$V = \rho\left(W_H - W_L\right) + W_L = \rho V' + W_L.$$

[16] When the value function is smooth such discontinuty of W_H, W_L cannot occur and the equilibrium bid is unique for all x. Moreover, the function b is continuous in this case.

Using these two formulas, one obtains that

$$W_L = V - \rho V', \tag{16}$$

and

$$W_H = V + (1-\rho)V'. \tag{17}$$

Let us take any convex value function V, and let us define W_L and W_H as in (16) and (17). We already know from Section 2.2 that there is a unique optimal action for all beliefs when V is twice differentiable. To construct our sufficient condition, assume that when the belief is ρ the unique optimal action is also equal to ρ.[17]. Then by construction $W_H(\rho) = \mu_H F_H(\rho)(v-\rho)$, and $W_L(\rho) = \mu_L F_L(\rho)(v-\rho)$. Given that $\mu_L, \mu_H, F_L, F_H, v-\rho \geq 0$, therefore it must hold that the functions W_L, W_H as induced by (16) and (17), satisfy $W_L, W_H \geq 0$. Moreover, since F_H and F_L are distribution functions it must hold that $W_L(\rho)/(v-\rho)$ and $W_H(\rho)/(v-\rho)$ are increasing in ρ. As we argued above, $W_H'(\rho) = (1-\rho)V''(\rho)$ and $W_L'(\rho) = -\rho V''(\rho)$. The assumption that V is convex then implies that W_H is increasing in ρ, and thus $W_H(\rho)/(v-\rho)$ is increasing too. Let us inspect $W_L(\rho)/(v-\rho)$. It holds that

$$\left(\frac{W_L}{v-\rho}\right)' = \left(\frac{V-\rho V'}{v-\rho}\right)' =$$

$$= \frac{1}{(v-\rho)^2}[V - \rho V' - \rho(v-\rho)V''].$$

Therefore, we only need to establish that for all ρ it holds that $V - \rho V' - \rho(v-\rho)V'' \geq 0$. It is clear that the lower value of v is, the easier this inequality is to satisfy. Let us denote the lowest possible value as $\underline{v}$, where $\underline{v}$ is set such that for all ρ it holds that $\mu_L F_L(\rho) = \frac{W_L(\rho)}{v-\rho} \leq 1$. Since F_L is increasing, this boils down to $\mu_L F_L(1) \leq 1$. Therefore, v must satisfy $v - 1 \geq W_L(1)$. If v is set at any level higher than $\underline{v} = 1 + W_L(1)$, then one can satisfy the inequality that $\mu_L F_L(\rho) \leq 1$, and thus the variables μ_L and $F_L(\rho)$ may be interpreted as probabilities. **Q.E.D.**

[17] This assumption implies that there are some sellers with negative costs, an assumption just made for expositional simplicity playing no role in the analysis.

Lemma 3 *If V is twice differentiable, then for all $\rho \in [0,1]$ it holds that*

$$W_H(\rho) = \beta - \int_\rho^1 (1-z)V''(z)dz \tag{18}$$

and

$$W_L(\rho) = 1 - \int_0^\rho zV''(z)dz. \tag{19}$$

Proof. Since W_H and W_L are monotone, they are almost everywhere differentiable and thus (from (3)) for almost all ρ

$$V'' = W_H' - W_L'. \tag{20}$$

A necessary condition for optimality can be written as $\rho \in \arg\max_{q\in[0,1]} \rho W_H(q) + (1-\rho) W_L(q)$, since an agent with belief ρ can always pretend to be type q. Since W_H and W_L are almost everywhere differentiable, the appropriate first order condition (that holds for almost all ρ) becomes

$$\rho W_H'(\rho) + (1-\rho)W_L'(\rho) = 0. \tag{21}$$

Given (20) and (21) for almost all ρ it holds that $W_H'(\rho) = (1-\rho)V''(\rho)$ and $W_L'(\rho) = -\rho V''(\rho)$. Normalizing, $W_L(0) = 1$, $W_H(1) = \beta$, and using the continuity of W_H, W_L this implies that for all ρ

$$W_H(\rho) = \beta - \int_\rho^1 (1-z)V''(z)dz \tag{22}$$

and

$$W_L(\rho) = 1 - \int_0^\rho zV''(z)dz. \tag{23}$$

indeed hold. **Q.E.D.**

7.2 Appendix 2

Comparison of first and second price auctions:

To formalize the argument in the main text, let us introduce the notation $t(x,y) = \Pr(H \mid s_1 = x, s_2 = y)$ and $l(x,y) = \Pr(H \mid s_1 = x, , s_2 > y)$, and

$$u(x,y) = v - [t(x,y)W_H(l(x,y)) + (1-t(x,y))W_L(l(x,y))].$$

The interpretation is that a bidder with type x values winning at $u(x,y)$ if the he bid as if he had type y. Again, the bidder conditions his posterior upon tieing, but knows that he will make the decision in the continuation problem as if he lost in the auction. Note, that it holds that $\eta(x) = t(x, x)$, $\nu(x) = l(x, x)$, and thus $u(x, x)$ is equal to the equilibrium bid in the second price auction.

Following the notation of Krishna (2008), let $W^A(z, x)$ the expected price paid by a bidder if he is winning when he receives a signal x, but he bids as if his signal was z. Let $W^I(z, x)$ denote this expected price for the first-price, and $W^{II}(z, x)$ for the second-price auction. Denoting the equilibrium bid functions in the two formats as b^I and b^{II}, one can write $W^I(z, x) = b^I(z)$ and

$$W^{II}(z, x) = E[b^{II}(y) \mid s_1 = x, s_2 < z].$$

In the monotone equilibrium of auction $A = I, II$ each bidder maximizes

$$\int_0^z u(x, y)[xg_H(y) + (1 - x)g_L(y)]dy - [xG_H(z) + (1 - x)G_L(z)]W^A(z, x).$$

Therefore, the first order condition becomes

$$[xg_H(x) + (1 - x)g_L(x)](u(x, x) - W^A(x, x)] - [xG_H(x) + (1 - x)G_L(x)]W_1^A(x, x) = 0,$$

where W_1^A denotes the partial derivative of W^A with respect to its first argument. This can be rearranged so that

$$W_1^A(x, x) = \frac{xg_H(x) + (1 - x)g_L(x)}{xG_H(x) + (1 - x)G_L(x)}u(x, x) - \frac{xg_H(x) + (1 - x)g_L(x)}{xG_H(x) + (1 - x)G_L(x)}W^A(x, x).$$

Therefore,

$$W_1^{II}(x, x) - W_1^I(x, x) = -\frac{xg_H(x) + (1 - x)g_L(x)}{xG_H(x) + (1 - x)G_L(x)}(W^{II}(x, x) - W^I(x, x)).$$

Now define

$$\Delta(x) = W^{II}(x, x) - W^I(x, x),$$

so

$$\Delta'(x) = W_1^{II}(x, x) - W_1^I(x, x) + W_2^{II}(x, x) - W_2^I(x, x).$$

Combining these results yields

$$\Delta'(x) = -\frac{xg_H(x) + (1-x)g_L(x)}{xG_H(x) + (1-x)G_L(x)}\Delta(x) + W_2^{II}(x,x) - W_2^I(x,x).$$

Now, by construction $W_2^{II}(x,x) > 0 = W_2^I(x,x)$. Thus if $\Delta(x) \leq 0$, then $\Delta'(x) \geq 0$. Furthermore by assumption $W^{II}(0,0) = b^{II}(0) = v - U_L(1) = b^I(0) = W^I(x,x)$, so $\Delta(0) = 0$. This then implies that for all $x \geq 0$, it indeed holds that $\Delta(x) \geq 0$, which shows that the second price auction yields a higher expected revenue than the first price auction.

7.3 Appendix 3

Proof of the revenue comparison result for Example 4:

Proof. In the second price auction the middle bidder's bid is the revenue, and the bid function can be written as

$$b^{II} = v - [\widehat{\eta}(x)W_H(\widehat{\nu}(x)) + (1-\widehat{\eta}(x))W_L(\widehat{\nu}(x))],$$

where $\widetilde{\eta}, \widetilde{\nu}$ are the relevant tieing and losing posteriors. These beliefs can be written as

$$\widehat{\eta}(x) = \Pr(H \mid s_1 = s_2 = x > s_3) = \frac{x^2\eta}{x^2\eta + (2x - x^2)(1-\eta)},$$

and

$$\widehat{\nu}(x) = \Pr(H \mid s_1 > s_2 = x > s_3) = \frac{x^2\nu}{x^2\nu + (2x - x^2)(1-\nu)}.$$

Let us now calculate the revenue in the English auction. Let z be lowest of the three types, and x be the medium one. Then the revenue is equal to

$$b^E(x,z) = v - [\widehat{\eta}(x,z)W_H(\widehat{\nu}(x,z)) + (1-\widehat{\eta}(x,z))W_L(\widehat{\nu}(x,z))],$$

where

$$\widehat{\eta}(x,z) = \Pr(H \mid s_1 = s_2 = x > s_3 = z) = \frac{z\eta}{z\eta + (1-z)(1-\eta)},$$

and

$$\widehat{\nu}(x,z) = \Pr(H \mid s_1 > s_2 = x > s_3 = z) = \frac{z\nu}{z\nu + (1-z)(1-\nu)}.$$

To calculate the expected revenues let

$$g_{mid}(x) = 12x - 42x^2 + 60x^3 - 30x^4$$

denote the density of the medium type, and let

$$h(z \mid x) = \frac{2\nu(x)z + 2(1-\nu(x))(1-z)}{\nu(x)x^2 + (1-\nu(x))(2x - x^2)}$$

be the density of the low type given the medium types. Then the expected revenues from the two auctions can be written as

$$ER^{II} = \int_0^1 g_{mid}(x)b^{II}(x)dx$$

and

$$ER^E = \int_0^1 g_{mid}(x) \int_0^x h(z \mid x)b^E(x,z)dzdx.$$

After substituting in the relevant functional form assumptions about W_H, W_L, and the parameter values from Example 4, we can use the above bid functions to show that

$$ER^{II} > ER^E$$

indeed holds. **Q.E.D.**

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