Mismatch as choice*

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Abstract

We characterize a competitive search equilibrium in which firms in some markets create jobs that workers seek even though those jobs do not make the most productive use of workers' skills. We refer to markets in which workers purposefully search for and accept inferior jobs as exhibiting *directed mismatch*. This kind of misallocation is driven by the fact that incomplete information about workers' outside options implies that the value of on-the-job search is higher for workers employed in those inferior jobs. Our theory provides new insights into the returns to education as well as the impact of on-the-job search on labor market mismatch. It also suggests that the declining fortunes of college educated American workers in recent decades, like those of high school graduates, are linked to the automation and offshoring of routine-task based jobs.

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1 Introduction

More than 20 percent of employed college graduates aged 25 to 54 in the U.S. work in jobs that are routine-task based. It is implausible that this observation is primarily the result of efficient sorting in frictionless competitive markets, as this would imply that college educated workers employed in routine jobs are, in fact, well suited to those jobs.[1] Instead, the fact that these workers systematically move into non-routine jobs later suggests that they were mismatched to begin with.[2] It is also implausible that college graduates seeking non-routine jobs find routine jobs by accident. Consequently, mismatch in this case cannot be just an *ex post* phenomenon, attributable to bad luck in a purely random matching process.[3] Our understanding of labor market mismatch is limited because, while efficient-sorting models neglect the possibility of mismatch, random-matching models neglect the possibility that such a mismatch is the consequence of informed choice.

To address this issue, we develop an equilibrium theory according to which firms in some markets create jobs that workers search for and accept, even though they do not make the most productive use of workers' skills. We refer to markets in which workers seek inferior jobs, despite the availability of better jobs, as exhibiting *directed mismatch*.

Our analysis builds on previous work on competitive search equilibria with private information (Guerrieri, Shimer and Wright 2010) to address the interaction between directed mismatch and search on the job.[4] Our specification of search on the job combines elements of directed search that are standard in competitive search models (Menzio and Shi 2011) and elements of bargaining that are standard in random matching models (Postel-Vinay and Robin 2002). Separating equilibria of the resulting model retain the block-recursive structure of competitive search equilibria (Shi 2009) while allowing employers to counter outside offers, which is both the most natural assumption and it renders our framework remarkably tractable by limiting the scope for job quits, thereby eliminating the sorts of wage ladders found in Delacroix and Shi (2006). Our model is sufficiently tractable that we are able to work with the decentralized equilibrium directly, rather than by way of a planning problem. This enables us to characterize pooling equilibria, which are neither block-recursive nor constrained efficient.

The ability of workers to search on the job is at the heart of the theory. This is because the option value of on-the-job search, which is an important component of the value of a job, depends on whether a worker is well matched in a job. College educated workers, who are most productive in jobs comprised primarily of non-routine tasks, can be induced to search for routine jobs if the value of on-the-job search from a routine job is sufficiently greater than from a non-routine job.

Incomplete information about workers' outside options naturally gives rise to a situation where

[1] Acemoglu and Autor (2011) and Beaudry, Green and Sand (2016) offer two insightful efficient-sorting perspectives.

[2] See Cortes (2016). For related work on underemployment of recent college graduates, see Abel and Deitz (2016).

[3] Barlevy (2002) and Gautier, Teulings and van Vuuren (2010) are two interesting examples of *ex post* mismatch in random matching models.

[4] Other related analyses of competitive search equilibria with private information include Inderst and Müller (2002), Faig and Jerez (2005), Guerrieri (2008), Delacroix and Shi (2013), Guerrieri and Shimer (2014), Stacey (2015) and Chang (2017).

the value of on-the-job search in routine jobs exceeds that in non-routine jobs. A worker's willingness to switch jobs depends on her ability to elicit a retention offer from her current employer, which is private information to the parties in the match. The fact that potential poachers do not observe the worker's outside option when they make a job offer creates an adverse selection problem: while workers in inferior matches search on the job in order to transition from a poor match to a better one, well-matched workers have an incentive to mimic their on-the-job search behavior. Since well-matched workers seek retention offers rather than new jobs, the poaching of employed workers creates more surplus when the pool of applicants contains a higher fraction of workers in poor matches.

We formalize this adverse selection problem and show that it creates a coordination problem. First, workers' incentives to search for jobs where they expect to be less productive is greater when the future returns to search on the job are expected to be higher. Second, firms are more likely to attempt to poach workers in markets where they believe the fraction of workers in poor matches to be high, hence the return to search on the job for a worker is higher in such markets. These two effects reinforce each other, making directed mismatch self-enforcing.[5]

We characterize a directed mismatch equilibrium, in which a positive fraction of unemployed, college educated workers apply for routine jobs, despite the fact that they have an absolute disadvantage in those jobs. Routine jobs, while paying lower wages on average, are appealing to unemployed, college educated workers because they are easier to find and because they offer attractive opportunities for on-the-job search. Routine jobs being easier to find is an equilibrium result, and is not due to any assumed advantage in either the cost of creating routine jobs or the ease of matching. Despite the fact that routine jobs are unambiguously worse than non-routine jobs from a purely technological perspective, in equilibrium these jobs are more valuable to unemployed college workers than are non-routine jobs because the value of on-the-job search is higher in routine jobs. Indeed, in the equilibrium with directed mismatch, workers only apply to the non-routine jobs when applying to routine jobs is relatively costly.

Whether or not the equilibrium suffers from adverse selection depends on whether or not wages reveal the workers' productivities in their current matches. We first show that both situations can be supported in equilibrium. Then, we argue that the more general issue is whether the adverse selection problem is sufficiently asymmetric across routine and non-routine jobs.

To understand why non-revealing wages can be supported in equilibrium note the existence of an informational externality, whereby firms in the market for unemployed workers do not take into account the informational value of wages to poachers.[6] This externality implies that employers have no direct incentive to post revealing wages, that is, wages that vary with match productivity. This means that non-revealing wages can be equilibrium wages as long as unemployed workers choose to

[5] See Burdett, Imai and Wright (2004) for an interesting analysis of long-term bilateral relationships when both parties can search while they are matched, which can create a different coordination problem.

[6] This informational externality is different from a second one, also present in our problem, whereby poachers do not internalize their effect on the outside option of workers hired in previous periods. Guerrieri (2008) was the first to show how competitive search fails to internalize this second externality.

search for non-revealing contracts when revealing contracts are feasible. This occurs because the option to search on the job constitutes an important component of the value of a job, but the value of this option depends on the beliefs of both workers and potential poaching firms.[7]

Equilibria with non-revealing wages are inefficient. We show that the constrained-efficient allocation — the one that maximizes the present value of aggregate production net of search costs — can be supported as an equilibrium outcome, but one that fully reveals private information and exhibits positive assortative matching, in the sense that college educated workers only ever search for non-routine jobs. Furthermore, in the constrained-efficient equilibrium the option of searching for routine jobs has no value for college educated workers. In the directed mismatch equilibrium, relative to the constrained-efficient allocation, the fraction of college educated workers who search for routine jobs is inefficiently high and the private returns to education are inefficiently low.

Our perspective on mismatch has strong implications concerning the role of on-the-job search in labor markets. First, our framework makes it clear that flows between unemployment and employment cannot be understood independently of on-the-job search opportunities. Since the option to search on the job is an important component of the value of a job, a firm's decision to create a job and an unemployed worker's decision to search for and accept that job all depend crucially on the value of on-the-job search from that job. The fact that employment-to-employment transitions constitute approximately half of all transitions (Fallick and Fleischman, 2004) shows that this is not merely a side issue, or of secondary importance.

Furthermore, the traditional perspective, which views mismatch as an *ex post* phenomenon, suggests that on-the-job search is efficiency enhancing, as it represents a corrective force that pushes the labor market towards a more efficient allocation.[8] In contrast, when mismatch is seen as an *ex ante* phenomenon originating in the optimal search decisions of workers, it becomes clear that on-the-job search can also be an important *cause* of mismatch, as it gives workers an incentive to search for jobs for which they are otherwise not well suited. Moreover, this suggests that there is no simple relationship between the rate of job-to-job transitions, the extent of labor market mismatch, and the efficiency of labor markets. Not only may a reduction in job-to-job transition rates correspond to a decline in mismatch, but it is also impossible to assess the efficiency implications of such a reduction without knowing its cause.

Our analysis also offers an interesting perspective on the labor market experiences of college educated workers. We argue that recent labor market outcomes of college educated American workers can be understood as a consequence of the reduction in the option value of searching for routine jobs in a directed mismatch equilibrium. Ample statistical evidence suggests that the value of routine jobs, whether cognitive or manual, has fallen in recent decades. For example, it is clear that many jobs in the legal profession are under pressure from legal startups that use

[7] Pooling (non-revealing) equilibria in our model rely on one-sided heterogeneity and pure strategies. Pooling in previous literature has been characterized in the context of markets with two-sided heterogeneity, in which agents use mixed strategies. See, for instance, Shi (2002) and Shimer (2005) in the context of labor markets and Chang (2014) and Guerrieri and Shimer (2014) in the context of asset markets.

[8] See, for example, Barlevy (2002), Robin and Postel-Vinay (2002), Gautier, Teulings and van Vuuren (2010), Menzio and Shi (2011).

machine learning and text recognition software to perform routine tasks previously requiring hours of labor.[9] While it is widely recognized that the automation and offshoring of routine jobs underlies the declining fortunes of non-college workers,[10] our analysis implies that the declining fortunes of college educated workers can be traced to the very same forces.

In an equilibrium with directed mismatch, where the option value of routine jobs is positive, a decline in the value of these jobs reduces the value of labor force participation to college educated workers. Furthermore, as routine jobs decline in value, a greater fraction of unemployed college educated workers direct their search towards non-routine jobs, which causes a decline in their employment rates as these jobs are harder to get. This reduction in the value of labor market participation to college educated workers is consistent with a rising wage premium, both for college workers relative to high school workers and for workers in non-routine jobs relative to routine jobs. Our theory also explains the coexistence of increasing college enrollments and declining participation of college educated workers, which may seem puzzling at first pass. Significantly, our theory can also make sense of the remarkable slowdown in the employment and participation rates of college educated women during the 1990s.

It should be noted that the share of college educated workers employed in routine jobs, relative to all employed college educated workers, has fallen in the U.S. after 1990. From the perspective of our theory, this observation corresponds to a decline in mismatch.[11] However, it would be incorrect to conclude from this observation that mismatch does not matter, or that mismatch is less important than it was in prior decades. Rather, declining mismatch, like the other trends discussed above, is a natural consequence of the automation and offshoring of routine jobs.

Finally, our analysis offers a novel perspective on the return to education, which comprises wages, job finding rates, and the value of on-the-job search. In our framework, the value of labor market participation to college graduates is the sum of the value of searching for non-routine jobs plus the option value of searching for routine jobs, which implies that the option to search for routine jobs represents an important component of the value of a college education. This differs from the conventional wisdom, which views the observation of college workers in routine jobs as a failure of education rather than as an important part of the return to education. Interestingly, from the perspective of the directed mismatch equilibrium, the share of college educated workers is inefficiently small, even though the share of those who end up mismatched is inefficiently large.

Naturally, we are not the first to consider why college educated workers seek routine jobs. Intuitively, this may happen for a number of reasons: (i) the supply of non-routine, cognitive jobs lags the supply of college educated workers, (ii) routine jobs are stepping stone jobs that

[9] In 2010, for example, software of the e-discovery firm Clearwell was used by the law firm DLA Piper to search through a half-million documents under a court-imposed deadline of one week (New York Times, Armies of Expensive Lawyers, Replaced by Cheaper Software, March 4, 2011).

[10] See, for example, Autor, Levy and Murnane (2003), Autor, Katz and Kearney (2006), Acemoglu and Autor (2011), Goos, Manning and Salomons (2014).

[11] Our notion of mismatch is very different from that of geographical or sectoral mismatch, (Shimer 2007, Sahin, Song, Topa and Violante 2014).

facilitate promotion, and (iii) routine jobs are easier to get.[12] These arguments, while plausible on the surface, cannot be easily reconciled with the fact that the vector of job types created by the economy is an endogenous variable, which suggests that the economy could create the sorts of jobs that suit the skills and education levels of its workforce. The directed mismatch equilibrium is not subject to this critique: routine jobs are endogenously both stepping stone jobs and easier to get, and it is easy to understand why non-routine, cognitive employment may lag the supply of college educated workers, as the automation and offshoring of routine jobs tends to lower the value of labor force participation to college educated workers while simultaneously increasing the returns to education (relative to non-college workers).[13]

In Section 2 we present the model. In Section 3 we consider a simplified version of the model, with only one type of job, in order to facilitate a more intuitive presentation of some of the key mechanisms of the model. In Section 4 we characterize a directed mismatch equilibrium. In Section 5 we discuss the main implications of our analysis. Technical proofs are in the Appendix.

2 The Model

2.1 Environment

Time is discrete. All agents are risk neutral and discount the future at a rate $r > 0$. There is a unit measure of workers who are either employed or unemployed. An unemployed worker searches for a job and receives a flow benefit from unemployment equal to $b \geq 0$. An employed worker produces. Subsequently, a separation shock makes an employed worker become unemployed with probability $\delta > 0$. Otherwise, the worker can search for a different job while employed.

There are two observable types of jobs indexed by $i = 1, 2$ and the measure of each type of job is determined endogenously by free entry. Workers are *ex ante* identical and they are relatively more productive in type-1 jobs. A worker-job match produces y_h units of output with probability α_i and y_l units of output with probability $1 - \alpha_i$, for $i = 1, 2$, where $b < y_l < y_h$ and $0 < \alpha_2 < \alpha_1 < 1$. The symmetry in the realizations of labor productivity across jobs simplifies the analysis by helping to limit potential job-to-job transitions. The relevant asymmetry between the two jobs is simply that a worker is less likely to be poorly matched in type-1 jobs.

As a matter of interpretation, we think of workers as college graduates, type-1 jobs as jobs involving cognitive, non-routine tasks and type-2 jobs as jobs involving cognitive, but routine tasks. In Section 5 we consider the case where there is a second type of worker who is relatively more productive in type-2 jobs (e.g., a high school graduate).

Employers incur an entry cost $k > 0$ in order to post a vacancy. We assume that $(r + \delta) k < \alpha_2 (y_h - y_l)$ to allow for positive job-to-job transitions from and to either type of job. Unemployed

[12] See, for example, Sicherman and Galor (1990), Autor (2015).

[13] We do not think that the observation of college-educated workers in routine jobs is just an artifact of incomplete data (either on unobserved ability or on search costs). Relatedly, note that the fact that workers make multiple applications is not sufficient to explain why college-educated workers search for routine jobs. Under complete information, they would have an incentive to exhaust all possible non-routine jobs before applying for a routine job.

workers face a positive opportunity cost of searching for jobs where they are more likely to be poorly matched. Specifically, an unemployed worker incurs a cost c if she searches for type-2 jobs in a given period, where c is the realization of a random variable that is independent across workers and over time. For simplicity, we assume that c is a draw from an exponential distribution: $F(c) = 1 - \exp\{-\theta c\}$, for $c \geq 0$, with $\theta > 0$.

Each period there is a continuum of markets. Each market is associated with a single employment contract x, which is specified below. Each employer can post any feasible contract and each worker can direct her search to any market. Let $Q : X \to \mathbb{R}_+$, where X is the set of feasible contracts and $Q(x)$ denotes the queue length associated with a contract x, which is defined as the ratio of workers searching for x to employers posting x. Matching is bilateral, so each employer meets at most one worker and vice versa. Workers who search in a market where $Q(x) = q$ meet an employer with probability $f(q)$ and employers in the same market meet a worker with probability $qf(q)$. We assume that $f(q)$ is twice differentiable, strictly decreasing and convex, with $f(0) = 1$ and $f(\infty) = 0$. We also assume that $qf(q)$ is strictly increasing and concave, approaching 1 as q converges to ∞. These assumptions ensure that the elasticity of job creation, given by $\eta(q) = -qf'(q)/f(q)$, is such that $0 = \eta(0) < \eta(1) \leq 1$, with $\eta'(q) > 0$. For simplicity, we also assume that $\eta(q)$ is concave, with $\eta(\infty) = 1$.[14]

When a worker and an employer meet, both the worker's labor market status and her wage, if currently employed, are observed by the potential employer. Then, the productivity of the potential match is drawn randomly and observed by both parties. However, if the worker is already employed, the productivity of her current match is not observed by the potential employer. Subsequently, employers decide whether or not to make formal offers. We assume that employers make take-it-or-leave-it offers, can counter outside offers and that wages can only be renegotiated by mutual agreement. Workers then decide whether to accept any offers. New matches start producing in the next period.

Since workers are unable to commit not to search on the job, and employers are unable to commit not to counter outside offers, it will facilitate presentation to specify contracts in terms of fixed entry wages, taking into account that wages can be renegotiated by mutual agreement, rather than including hiring and retention policies as part of the contract. A worker that gets a credible outside offer can choose to terminate the current fixed wage contract and agree to a "new" contract with a different wage, which lasts until a new outside offer arrives. If the outside offer is credible and if a better counteroffer is feasible, the employer then commits to a new fixed wage contract. Retention policies will depend on history only through the worker's current wage, which is a sufficient statistic for the payoff-relevant history of the current contract.

A contract $x = \{j, w_l, w_h\}$ specifies a type of job $j \in \{1, 2\}$ and a pair of wages, where $w_l \in [0, y_l]$ and $w_h \in [0, y_h]$ denote the entry wages to be offered when the realizations of match productivity in the new match are $y' = y_l$ and $y' = y_h$, respectively.

Our assumptions imply that employed and unemployed workers in effect do not compete for the

[14] An example of a matching technology that satisfies these assumptions is $M(u, v) = uv/(u + v)$.

same jobs. Moreover, neither workers nor employers can be forced to participate in a match before observing match productivity. That is, employers cannot commit to make a formal job offer and workers cannot commit to accept such an offer before observing the realized match productivity. In this sense, matches are pure inspection goods, rather than experience goods. These assumptions are made to highlight the role of incomplete information about workers' outside options. In Section 5, we argue that our main results continue to apply more generally, as long as match productivity in type-2 jobs is sufficiently easier to observe upon inspection. We think of type-2 jobs as those involving routine tasks and type-1 jobs as those involving non-routine tasks, so this asymmetry is the natural case.

We focus on the adverse selection problem that arises from the combination of limited commitment and asymmetric information. Since match productivity is unobserved by third parties, a worker's current labor productivity is private information to the worker *vis-a-vis* potential new employers. Consequently, poaching offers cannot discriminate between workers with different outside options, unless (equilibrium) wages reveal match productivity. Since workers are unable to commit not to search on the job and employers are unable to commit not to counter outside offers, workers in high-productivity matches have an incentive to seek outside offers solely to elicit retention offers from their current employers.

Finally, we assume that employers face a small costs of making credible offers, so they will never make offers that they know will be rejected with certainty. This assumption rules out potential equilibria where workers in poor matches are able to elicit retention offers. For simplicity, we assume these costs are negligible and so we are not explicit about them.

2.2 Competitive search equilibrium

Let $s \equiv \{i, w, y\} \in S$ denote a worker's payoff-relevant state, where a worker can be unemployed ($i = 0$), employed in a type-1 job ($i = 1$) or employed in a type-2 job ($i = 2$); $w \in [0, y_h]$ denotes her current wage and $y \in \{y_l, y_h\}$ denotes current match productivity. Unemployed workers are associated with the state $s_u = \{0, b, b\}$ by convention, and the feasible state space is given by $S = \{s_u\} \cup S_e$, where $S_e = \{1, 2\} \times [0, y_h] \times \{y_l, y_h\}$.

Focus on stationary equilibria. A competitive search equilibrium (Moen 1997) specifies a mapping Q from feasible contracts to market queues. Workers direct their search across all feasible contracts, taking as given the market queue length $Q(x)$ for all $x \in X$. Workers' decisions must be optimal at any information set, which includes their own state $s \in S$ and the distribution of workers across states, that is, the aggregate state of the economy $\psi : S \to [0, 1]$. It will become clear that competitive search equilibria need not be *block recursive*. That is, the agents' value functions and therefore equilibrium strategies may be a function of the aggregate state. However, in order to minimize clutter, we are not explicit about the potential dependence of the agents' value functions on the aggregate state ψ.

Let $V(s)$ denote the value function of a worker evaluated in state s. Let $U(s, c, x, Q(x))$ denote the expected surplus to a worker with current state s from searching for x, with associated queue length $Q(x)$, when c is the realized cost of searching for type-2 jobs. The worker meets an employer with probability $f(Q(x))$, in which case a draw y' of match productivity is realized and a wage offer w_o is made. Workers reject any offer w_o from a type-j employer such that $V(s) > V(s_o)$, where $s_o = \{j, w_o, y'\}$. Instead, if $V(s) < V(s_o)$, the decision of a worker with current state $s = \{i, w, y\}$ amounts to choosing whether to accept the offer, in which case her state becomes $s_o = \{j, w_o, y'\}$, or reject the offer, in which case her state remains unchanged, if the worker was unemployed (if $s = s_u$), or it becomes $s_c = \{i, w_c, y\}$ if the worker was employed (if $s \neq s_u$) and she got a wage counteroffer w_c. Employed workers only renegotiate contracts if they have a credible outside option; so $V(s_c) > V(s)$ if and only if $V(s_o) > V(s)$.

Thus, we have

$$V(s) = w + \frac{\delta V(s_u)}{1+r} + (1-\delta)\left\{\frac{V(s)}{1+r} + \mathbb{E}_c\left\{\max_{x\in X} U(s, c, x, Q(x))\,|s\right\}\right\}, \tag{1}$$

for all s, where we have restricted attention to pure search policies, for simplicity, with

$$U(s,c,x,Q(x)) = \begin{cases} (1-j)\,c + f(Q(x))\,\mathbb{E}_{y'}\left\{g_h(i,w,s_o)\max\left\{0, \frac{V(s_o)}{1+r} - \frac{V(s)}{1+r}\right\}\right\} & \text{if} \quad s = s_u \\ f(Q(x))\,\mathbb{E}_{y'}\left\{g_h(i,w,s_o)\max\left\{0, \frac{V(s_o)}{1+r} - \frac{V(s)}{1+r}, \frac{V(s_c)}{1+r} - \frac{V(s)}{1+r}\right\}\right\} & \text{if} \quad s \neq s_u, \end{cases}$$

where $\mathbb{E}_c$ and $\mathbb{E}_{y'}$ denote expectations taken with respect to the exogenous variables c and y', respectively; $s = \{i, w, y\}$, $s_o = \{j, w_o, y'\}$ and $s_c = \{i, w_c, y\}$, where the worker anticipates both the hiring policy $g_h(i, w, s_o)$ and the wage offer w_o of the potential new employer, taking as given that $w_c = g_r(s, j, w_o)$, where g_r is her current employer's retention policy, and where g_h and g_r are specified below.

Since $c \geq 0$ is a random draw from F if $s = s_u$, but $c = 0$ if $s \neq s_u$, we define the search policy of a worker as:

$$g_x(s,c) \in \begin{cases} \arg\max_{x\in X} U(s,c,x,Q(x)) & \text{if} \quad s = s_u \\ \arg\max_{x\in X} U(s,0,x,Q(x)) & \text{if} \quad s \neq s_u. \end{cases} \tag{2}$$

Restricting attention to pure acceptance policies, for simplicity, we let

$$g_a(s, s_o, w_c) \in \begin{cases} \arg\max_{a\in\{0,1\}} \{aV(s_o) + (1-a)\,V(s)\} & \text{if} \quad s = s_u \\ \arg\max_{a\in\{0,1\}} \{aV(s_o) + (1-a)\max\{V(s), V(s_c)\}\} & \text{if} \quad s \neq s_u \end{cases} \tag{3}$$

for all $s \in S$, $s_o \in S_e$ and $w_c \in [0, y_h]$, where $g_a(s, s_o, w_c) = 1$ if a worker in state s accepts an offer to work for a type-j employer at the wage w' when w_c is her current employer's counteroffer, with $s \equiv \{i, w, y\}$, $s_o \equiv \{j, w_o, y'\}$ and $s_c = \{i, w_c, y\}$ We let $w_c = b$ if $s = s_u$, by convention.

We next define the present value of an ongoing match to the employer. Note that employers in an ongoing match need to anticipate the worker's search policy (g_x) and her acceptance policy (g_a), as well as the hiring policy of the worker's potential new employer (g_h), which are common knowledge in equilibrium. Also note that the retention policy g_r is contingent on the worker's state $s = \{i, w, y\}$, which includes her type y, since this becomes contractible once the match is formed. Retention offers cannot be made contingent on the realized match productivity associated with an outside offer (i.e., the worker's potential future type), since this is unobserved by the incumbent employer. Of course, the observation that an offer was made and the observed wage offer may reveal match productivity in equilibrium.

Thus, the present value of an ongoing match to the employer, denoted by $J_f(s)$, solves

$$\begin{aligned}
\frac{J_f(s)}{1+r} &= \frac{y-w}{r+\delta+(1-\delta) f\left(Q\left(g_x(s,0)\right)\right) \mathbb{E}_{y'}\left\{g_h\left(i,w,s_o\right)\right\}} \\
&\quad + \left(\frac{(1-\delta) f\left(Q\left(g_x(s,0)\right)\right)}{r+\delta+(1-\delta) f\left(Q\left(g_x(s,0)\right)\right) \mathbb{E}_{y'}\left\{g_h\left(i,w,s_o\right)\right\}}\right) \\
&\quad \times \mathbb{E}_{y'}\left\{g_h\left(i,w,s_o\right) \mathbb{E}_{s_o}\left\{\max_{w_c}\left\{\left(1-g_a\left(s,s_o,w_c\right)\right) \frac{J_f\left(s_c\right)}{1+r}\right\} | j, w_o\right\}\right\} \\
&\text{subject to } w_c \geq w, \qquad (4)
\end{aligned}$$

for all $s = \{i, w, y\} \neq s_u$, where $s_o = \{j, w_o, y'\}$ and $s_c = \{i, w_c, y\}$. The denominator on the right hand side reflects the three sources of discounting: the discount rate (r), the exogenous probability of job destruction (δ), and the probability that the worker receives an outside offer from a poaching firm. Let $g_r(s, j, w_o)$ denote a solution to problem (4).

Given $Q(x)$, workers searching for x do not need to account for the composition of workers in that market. By contrast, employers posting x need to anticipate not only the likelihood of meeting a worker, given by $Q(x) f(Q(x))$, but also the composition of the pool of workers searching for that contract. We let $\mu(\cdot|x)$ denote a probability distribution on S, for each $x \in X$. An employer posting x incurs a flow cost k and meets a worker with probability $Q(x) f(Q(x))$, in which case the expected surplus to the employer is given by $\mathbb{E}_s\{J(s,x)|x\}$, where $J(s,x)$ is the expected value of the employer's surplus conditional on meeting a state-s applicant and $\mathbb{E}_s\{\cdot|x\}$ is taken with respect to $\mu(\cdot|x)$. Thus, the value of posting x to an employer is given by

$$-k + Q(x) f(Q(x)) \mathbb{E}_s\{J(s,x)|x\},$$

where $\mathbb{E}_s\{J(s,x)|x\} = \mathbb{E}_s\{\mathbb{E}_s\{J(s,x)|i,w,x\}|x\}$.

In order to form the latter expectation, note that employers cannot commit to participate in the match before observing the realization of match productivity y'. Hence the decision to make an offer to a worker is made conditional on the realized match productivity. Note further that potential employers observe the workers' current labor market status and their wages, if currently employed, but not their labor productivity in their current match. This implies that potential employers must form expectations concerning the worker's current state, as given by the inner conditional expectation. This implies that:

$$\mathbb{E}_s\left\{J\left(s,x\right)|i,w,x\right\}=\mathbb{E}_{y'}\left\{\max_{h\in\{0,1\}}\left\{h\mathbb{E}_s\left\{g_a\left(s,s_o,g_r\left(s,j,w_o\right)\right)\frac{J_f\left(s_o\right)}{1+r}\left|i,w,y',x\right.\right\}\right\}\right\}, \tag{5}$$

with $s=\{i,w,y\}$, $s_o=\{j,w_o,y'\}$, where $J_f(s_o)$ satisfies equation (4) and where by convention, we set $g_r(s_u,j,w_o)=b$ for all (j,w_o), with $s_u=\{0,b,b\}$. Let $g_h(i,w,s_o)$ denote a solution to the problem in (5). Equation (5) reflects the fact that poachers anticipate the current acceptance policies of the workers they attract (g_a) and the retention policies of their current employers (g_r). In order to minimize clutter, we do not include these explicitly as arguments in the value function J. Similarly, recall that we have assumed that the cost of making an offer is positive, but negligible.

Definition 1 *A stationary equilibrium consists of a set of posted contracts $X^*\subseteq X$ and a set of workers' states $S^*\subseteq S$, value functions $V:S\to R_+$ and $J:S\times X\to R_+$, policy functions $g_x:S\times R_+\to X\cup\emptyset$, $g_a:S\times S_e\times[0,y_h]\to\{0,1\}$, $g_h:\{0,1,2\}\times[0,y_h]\times S\to\{0,1\}$ and $g_r:S_e\times\{1,2\}\times[0,y_h]\to[0,y_h]$, a function $Q:X\to R_+$, a distribution $\mu:S\times X\to[0,1]$ and a distribution $\psi:S\to[0,1]$, such that:*

(A) Atomistic agents: For all $x\in X$, (1) all agents take $Q(x)$, $\mu(\cdot|x)$ and ψ as given, and (2) $\mu(\cdot|x)$ has support on S^*.

(B) Workers' optimal search and acceptance: V satisfies (1); g_x satisfies (2); g_a satisfies (3).

(C) Optimal contract posting and retention with free entry: g_h, g_r and J solve (4) and (5). Moreover, for any $x\in X$, $Q(x)f(Q(x))\int_S J(s,x)\,d\mu(s|x)\le k$, with equality if $x\in X^*$.

(D) Consistent beliefs: For any $x\in X^*$, $\int_S \mathbb{E}_c\{\mathbb{I}_x(g_x(s,c))|s\}\,d\psi(s)>0$ and

$$\mu\left(s|x\right)=\frac{\psi\left(s\right)\mathbb{E}_c\left\{\mathbb{I}_x\left(g_x\left(s,c\right)\right)|s\right\}}{\int_S \mathbb{E}_c\left\{\mathbb{I}_x\left(g_x\left(s,c\right)\right)|s\right\}d\psi\left(s\right)},$$

for all $s\in S$, where $\mathbb{I}_x(g_x(s,c))=1$ if $g_x(s,c)=x$ and $\mathbb{I}_x(g_x(s,c))=0$ if $g_x(s,c)\neq x$.

(E) Consistent allocations: For all $s\in S$,

$$\int_{S^*}\Pr\left(s_{t+1}=\widetilde{s}\,|s_t=s\right)d\psi\left(\widetilde{s}\right)=\int_{S^*}\Pr\left(s_{t+1}=s\,|s_t=\widetilde{s}\right)d\psi\left(\widetilde{s}\right),$$

where $\Pr(s_{t+1}|s_t)$ is the unique distribution associated with g_x, g_h, g_a and g_r, with $g_x(s,c)\in X^*\cup\emptyset$, for all $s\in S^*$ and $S^*=\{s\in S:\psi(s)>0\}$.

Definition 1 requires that markets are complete in the sense that employers can post any contract in the set of feasible contracts and workers direct their search across all feasible contracts. Moreover, all agents are atomistic in the sense that they cannot influence aggregate variables. As is standard in the literature, we use the language of wage posting (e.g., Guerrieri, Shimer and Wright 2010). However, as is well known, the competitive search equilibrium describes the equilibrium of a market, rather than a game. As this distinction turns out to be crucial in the present context, Part (A) of our equilibrium definition makes it explicit.

There are two important aspects to the assumption of atomistic agents in the present context. First, when workers search for a contract $x \in X$, they take as given the probability that they will be rationed, which is $f(Q(x))$. Similarly, when firms post a contract $x \in X$, they take as given the probability that they will be rationed. For the firms, this probability has two distinct dimensions, namely, the meeting probability $Q(x) f(Q(x))$ and the distribution of workers $\mu(\cdot|x)$ who are expected to search for the contract. Rather than including the rationing probabilities in the description of a contingent commodity, as is normally done in the context of Walrasian markets, we treat them as a description of beliefs that all agents share about the trading process. Furthermore, note that, upon meeting a worker, the employers' assessment of the worker's unobservable type is, with a slight abuse of notation, given by $\mu(y|i, w, x)$, for $y \in \{y_l, y_h\}$, which can be constructed from the equilibrium mapping $\mu(\cdot|x)$.

The second important aspect of atomistic agents in our context arises because the presence of on the job search implies not only that workers are heterogeneous in observable as well as unobservable dimensions, but also that the distribution of workers (i.e., the aggregate state) itself is an equilibrium object. Thus, agents take as given not only $Q(x)$ and $\mu(\cdot|x)$, for all $x \in X$, but also ψ, where $\psi(s)$ is the proportion of state-s workers in the economy. This imposes some natural restrictions on beliefs, both on and off the equilibrium path. On the equilibrium path, beliefs must be consistent in the sense that they satisfy Bayes' rule, as usual. Furthermore, both on and off equilibrium, $\mu(\cdot|x)$ must have full support on S^*, for all $x \in X$. In particular, starting along an equilibrium path, when a firm considers posting an off equilibrium contract it understands how both the market queue and the pool of searchers associated with $x \notin X$ will vary with the contract posted, but it takes as given that its posting of a different contract does not influence the distribution of wages in the economy.

Part (B) of Definition 1 ensures that workers' search and acceptance policies are optimal for all states, taking as given the market queue length for all contracts. Part (C) ensures that employers' posting behavior and their subsequent retention policies are optimal, and employers posting equilibrium contracts make zero profits. Part (D) ensures that employers' beliefs are consistent with the workers' search strategies through Bayes' rule along the equilibrium path. It ensures that any contract that is posted in equilibrium, that is, any $x \in X^*$, attracts a positive mass of workers and that the distribution of workers searching for any equilibrium contract is exactly what the employers posting those contracts expect. Free entry of employers then ensures the correct market clearing queue. Part (E) ensures that employers' and workers' equilibrium strategies generate a stationary distribution of workers and jobs and characterizes the sets of equilibrium contracts X^* and worker states S^*. The latter consists of the unemployed plus the support of the equilibrium wage distribution for workers in each type of job. This condition requires that the aggregate flows in and out of any state in S^* must be equal to each other at all times. A formal statement of the transition probability $\Pr(s_{t+1} = s' | s_t = s)$ is straightforward, but cumbersome. In the Appendix, we provide one in the specific context of the equilibria we characterize.

2.3 Equilibrium refinement

Clearly, the above equilibrium definition allows for more or less arbitrary off-equilibrium beliefs (other than the restriction that $\mu(\cdot\,|x)$ and ψ must have common support, for all $x \in X$) and so it allows for many equilibria, each of which is supported by particular beliefs in the markets where no trade takes place. The issue is that some contracts may not be traded because employers fear they would attract only undesirable types of workers. If workers expect the labor market queue associated with those contracts to be sufficiently high then those contracts would in fact not attract any workers and so the employers' pessimistic beliefs are never contradicted. We propose the following refinement of equilibrium, restricting agents' beliefs about contracts that are not traded in equilibrium.

Definition 2. *A refined equilibrium is a stationary equilibrium such that, for any $x' \notin X^*$ there does not exist any queue $q \in R_+$ and any beliefs $\mu'(\cdot\,|x')$ on S with support on S^* such that $qf(q)\int J(s,x')\,d\mu'(s\,|x') \geq k$, where for any feasible (s,c), $\mu'(s\,|x') > 0$ if and only if $U(s,c,x',q) > U(s,c,g_x(s,c),Q(g_x(s,c)))$.*

Our equilibrium refinement is in the spirit of the Intuitive Criterion proposed in Cho and Kreps (1987). It eliminates equilibria if there is some off-equilibrium contract x' and some pair of labor market queues and beliefs $(q,\mu'(\cdot\,|x'))$ that yield some firm offering the deviating contract non-negative profits and some worker seeking the deviating contract a payoff above her equilibrium payoff as long as the firm does not assign a positive probability to the deviation having been made by any type for whom this action is (weakly) equilibrium dominated.

The refined equilibrium proposed in Definition 2 builds on the concepts proposed in Gale (1992, 1996) and Guerrieri, Shimer and Wright (2010), by requiring that beliefs must be such that, for any $s \in S$ and any $x \notin X^*$, $\mu(s\,|x) = 0$ if $U(s,c,x,Q(x)) < U(s,c,g_x(s,c),Q(g_x(s,c)))$, for all $c \in \mathbb{R}_+$, where U is given by (1). This condition amounts to requiring that employers posting an off–equilibrium contract must believe that the only workers the contract would ever attract must be indifferent between the off-equilibrium contract and their preferred equilibrium contract. For if they strictly preferred the off-equilibrium contract, then condition (B) in the equilibrium definition would be violated. In turn, this requires that firms believe that off-equilibrium contracts will attract only those workers who are willing to endure the highest labor market queue.

Our equilibrium refinement also rules out a continuum of equilibria where employers do not post some contracts because they believe they will not attract workers while workers do not search for those contracts because they believe too many other workers would be searching for them as well. An alternative restriction that eliminates all these equilibria is the following: for any $x \notin X^*$, $Q(x) = 0$ if $\mu(s\,|x) = 0$ for all $s \in S$.

Moreover, our refinement rules out equilibria where off-equilibrium poaching contracts are not posted because potential employers believe they would only attract unemployed workers, while employed workers do not search on the job because they believe that there are no off-equilibrium poaching contracts. This is important in our setting because beliefs need to be specified over observable worker characteristics as well as unobservable worker types and, because of the possibility of search on the job, workers in different equilibrium states $s \in S^*$ are not indifferent between all equilibrium contracts $x \in X^*$.

The above requirement applies only to workers who participate in the labor market in equilibrium, that is, only if $s \in S^*$. Intuitively, if a state s is not in the support of the aggregate state ψ, then employers should assign probability zero to the event that such a worker would ever search for any contract, both on and off the equilibrium path. It will become clear that this feature is crucial to support the unique pooling equilibrium that we characterize below. It will will also become clear that the unique separating equilibrium is not robust to small trembles.

3 Equilibrium with homogeneous jobs

In this section we examine an economy with only one type of job. Such an economy cannot, by definition, exhibit the kind of directed mismatch that we are interested in understanding. However, both the relevant informational externality that affects firm wage setting and the adverse selection problem are present. Addressing this, simplified, version of the model allows us to introduce these key elements without the complexity of the full model. Furthermore, by first presenting a one-type-of-job version of the model, we can highlight an important technical problem that has impeded research in this area to date as well as present our resolution of this problem. Throughout this section we suppose that the single type of job created by the economy is of type 1.

Our assumptions about the nature of counteroffers impose a lot of structure on the problem. First, we assume that employers cannot commit to not counter outside offers. In the absence of commitment, incumbent firms will match any offer up to the worker's current productivity. Second, we assume that poaching firms never make offers that will be rejected with certainty. Collectively, these assumptions imply that workers who are known to be in high productivity matches cannot profit from on-the-job search. The reason is that a worker in a high productivity match who receives an outside offer will elicit a retention offer from her current employer, who will be willing to pay up to y_h to retain the worker. Consequently, there are no gains from trade between workers in high productivity matches and potential poaching firms. This implies, for example, that workers in high productivity matches cannot search in separate markets, as no poaching firm would enter a market populated solely by such job seekers.

In principle, workers known to be in high productivity matches could pool with workers in poor matches, which would crowd out those workers. However, well-matched workers seek outside options only to elicit a retention offer. Since poaching firms do not make offers that will be rejected with certainty, such workers will not get outside offers. This possibility is ruled out by our equilibrium refinement, since there would be alternative contracts that could be posted where employers would make non-negative profits and poorly matched workers searching for them would be strictly better off, without making well-matched workers worse off. Note that this means that workers in high productivity matches can only profit from on-the-job search if their match productivity is not revealed in equilibrium.

We further assume that poaching firms cannot commit to offers before observing match productivity, which implies that the maximum wage a poaching wage can offer a worker with whom it has a low productivity match is y_l. Note that, since incumbent firms are willing to make retention offers up to the total amount of worker productivity, poaching firms never make offers to workers with whom they would form a low productivity match.

This shapes the set of possible job and wage transitions as follows: all job switches occur when a worker in a low productivity match meets a firm with which she has a high productivity match. A worker in a high productivity match (who can search on the job by pooling) who meets a firm with which she would also form a high productivity match will elicit both a job offer from the poaching firm and a retention offer from the incumbent firm. Therefore, all job switches and wage changes reveal that a worker is now employed in a high productivity match. An implication of this is that the job ladder has at most one rung. A worker who moves reveals that she has moved into a high productivity match and will no longer be the target of poaching firms while a worker who accepts a retention offer reveals that she is currently employed in a high-productivity match and will also no longer be the target of poaching firms.

Our model shares the well known property that the allocation supported by a competitive search equilibrium can be characterized as the solution of a corresponding dynamic programming problem. However, in our model there exist two equilibria: in one wages reveal match productivity, while in the other wages do not reveal match productivity. While each equilibrium outcome corresponds to the solution of a distinct dynamic programming problem, we will present these two problems using one set of Bellman equations.

To that end, let $\rho \in \{1-\alpha_i, 1\}$ denote the fraction of poorly matched workers among all on-the-job searchers, where we can restrict attention to two types of situations: one where wages are revealing and, consequently, only poorly matched workers search on the job ($\rho = 1$), and another where wages are non-revealing and, consequently, both well-matched and poorly matched workers in type-i jobs search on the job ($\rho = 1 - \alpha_i$).

We begin with the search problem of an employed worker. Even though in this section we are assuming that $i = j = 1$, it will be convenient to index job types by i and j, and to assume that unemployed workers search for type-i jobs while employed workers search for type-j jobs, so that our analysis in this section will extend readily to the case with multiple types of jobs.

For a given value of $\rho \in \{1-\alpha_i, 1\}$, the value of employment to a worker in state $s = \{i, w, y\} \neq s_u$, for $w \in [0, y_h]$ and $y \in \{y_l, y_h\}$, who searches for type-j jobs is given by:

$$\overline{V}(s,\rho) = w + \frac{\delta \overline{V}(s_u,\rho)}{1+r} + (1-\delta)\left\{\frac{\overline{V}(s,\rho)}{1+r} + U_j(s,\rho)\right\}, \tag{P1}$$

where

$$U_j(s,\rho) = \max_{w',q'} \left\{ f(q')\,\alpha_j \left[\max \left\{ 0, \frac{w'}{r+\delta} + \left(\frac{\delta}{r+\delta}\right) \frac{\overline{V}(s_u,\rho)}{1+r} - \frac{\overline{V}(s,\rho)}{1+r} \right\} \right] \right\}$$

subject to

$$\begin{aligned} k &\leq q' f(q')\,\alpha_j \left(\frac{y_h - w'}{r+\delta}\right)\rho, \\ w' &\geq y_l, \\ U_j(\{i,w,y_h\},1) &= 0. \end{aligned}$$

Denote a solution to Problem (P1) by $\{w_e(s,\rho), q_e(s,\rho)\}$, with $q_e(\{i,w,y_h\},1) = \infty$ and $w_e(\{i,w,y_h\},1) = 0$. This normalization simply captures the fact that workers searching on the job from high productivity matches do not crowd out workers searching from low productivity matches in an equilibrium where entry wages reveal match productivities. To avoid clutter, we are not explicit about the dependence of w_e and q_e on j.

$U_j(s,\rho)$ represents the option value of on-the-job search for type-j jobs to an employed worker. As discussed above, the structure of our counteroffer game implies that an employed worker whose wage reveals her to be in a high productivity match cannot profit from on-the-job search. Workers in low productivity matches and workers who are indistinguishable from them can search on the job, and the option value of this search is given by the constrained optimization problem above.

Workers who can profit from on-the-job search face a relatively straightforward competitive search problem. The first constraint imposes that poaching firms must make non-negative expected profits. In this constraint, $\rho \in \{1-\alpha_i, 1\}$ is used to index the two problems. When $\rho = 1$, the non-negative profit constraint is written as if all poaching offers are accepted by workers. This version of the problem corresponds to the equilibrium where wages reveal match productivity, in which case workers in high productivity matches cannot profit from on-the-job search. When $\rho = 1-\alpha_i$, the non-negative profit constraint is written as if poaching offers are accepted by workers with probability $1-\alpha_i$. This version of the problem corresponds to the equilibrium in which wages do not reveal productivity, high productivity workers search on the job, and a fraction $1-\alpha_i$ of applicants to poaching firms reject job offers in favour of retention offers. The other constraint, $w' \geq y_l$, reflects the assumption that poachers recognize the fact that employed workers can only be recruited if the poaching offer exceeds the worker's current productivity.

One can verify that a solution to Problem (P1) is such that

$$q_e(\{i,w,y_h\},1-\alpha_i) = q_e(\{i,w,y_l\},1-\alpha_i)$$

and

$$w_e(\{i,w,y_h\},1-\alpha_i) = w_e(\{i,w,y_l\},1-\alpha_i),$$

which reflects the fact that workers searching on the job from high and low productivity matches have identical incentives in an equilibrium where entry wages do not reveal match productivities. Both types of workers compete for the same outside offers, where subsequent retention offers elicited by well-matched workers will just match the outside offers that will be accepted by poorly matched workers.

The value of unemployment for a worker who searches for type-i jobs is given by:

$$\overline{V}\left(s_u,\rho\right)=b+V_i\left(\rho\right), \tag{P2}$$

where

$$V_i\left(\rho\right)=\frac{\overline{V}\left(s_u,\rho\right)}{1+r}+\max_{w_l,w_h,q}\left\{f\left(q\right)\left[\alpha_i\frac{\overline{V}\left(\left\{i,w_h,y_h\right\},\rho\right)}{1+r}+\left(1-\alpha_i\right)\frac{\overline{V}\left(\left\{i,w_l,y_l\right\},\rho\right)}{1+r}-\frac{\overline{V}\left(s_u,\rho\right)}{1+r}\right]\right\}$$

subject to

$$\begin{aligned}
k &\leq qf(q)\left[\alpha_i\left(\frac{y_h}{r+\delta}-\frac{w_e(\left\{i,w_h,y_h\right\},\rho)}{r+\delta}+\frac{w_e(\{i,w_h,y_h\},\rho)-w_h}{r+\delta+(1-\delta)\alpha_j f(q_e(\{j,w_h,y_h\},\rho))}\right)\right.\\
&+ \left.\left(1-\alpha_i\right)\left(\frac{y_l-w_l}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q_e\left(\left\{j,w_l,y_l\right\},\rho\right)\right)}\right)\right],\\
w_l &\leq y_l,\ w_h\leq y_h \text{ and } w_h\begin{cases}=w_l & \text{if } \rho=1-\alpha_i\\ \neq w_l & \text{if } \rho=1.\end{cases}
\end{aligned}$$

Denote a solution to Problem (P2) by $\left\{w_u^l(i,\rho),w_u^h(i,\rho),q_u(i,\rho)\right\}$. Once again, to avoid clutter, we are not explicit about the dependence of w_u^l, w_u^h and q_u on j.

The last constraints in Problem (P2) reflect the facts that employers cannot commit to pay wages that exceed the worker's marginal product and that wages reveal a worker's current match productivity if and only if entry wages vary across realizations of match productivity.

The first constraint is the non-negative profits constraint, incorporating all possibilities for on-the-job search allowed under our assumptions about counteroffers. The two terms within the parentheses in the first line reflect the profits an employer enjoys when it forms a high productivity match with an unemployed job seeker. The first term is the expected discounted value of the profits received if the employer were to pay the future retention offer $w_e(\{i,w_h,y_h\},\rho)$. The second term reflects the temporary extra profits due to the fact that the entry wage w_h of a high productivity worker is lower than the retention offer the worker will elicit as soon as she receives an outside offer. The denominator reflects the three sources of discounting: the discount rate (r), the exogenous probability of job destruction (δ), and the probability that such a worker receives an outside offer from a poaching firm ($(1-\delta)\alpha_j f(q_e(\{j,w_h,y_h\},\rho))$), in which case the incumbent firm will match and the worker's wage will change. The term within the parentheses in the second line represents the profits a firm enjoys when if forms a low productivity match with an unemployed job seeker. The structure of our counteroffer game implies that such workers are always able to search on the job, never elicit retention offers, and quit whenever they meet a poaching firm with which they form a high productivity match.

The following proposition implies that any allocation supported by a refined equilibrium with positive quits must solve a version of the above problems.

Proposition 1 *Consider a refined equilibrium with positive quits. If the equilibrium is revealing, the equilibrium allocation solves problems (P1) and (P2) with* $\rho = 1$*. If the equilibrium is non-revealing, the equilibrium allocation solves problems (P1) and (P2) with* $\rho = 1 - \alpha_i$*.*

The two possible types of equilibrium correspond to the cases where entry wages either reveal or do not reveal the relevant match productivity realization. Whether wages do or do not reveal this information is critical because it determines whether workers with a high productivity realization can profit from on-the-job search. We employ the terminology of the traditional rational expectations equilibrium literature to refer to these equilibria as *revealing* and *non-revealing*. Revealing equilibria correspond to typical competitive search equilibria. With respect to this, our contribution is to provide a characterization of a non-revealing equilibrium.

Using problems (P1) and (P2) to characterize equilibrium allocations is non-trivial due to the fact that the objective function in problem (P2) is not generally concave in $\{w_l, w_h, q\}$. The main complication arises because poachers do not take workers' future quit rates as given, but rather they understand that workers' future quit rates are a function of their current wages. To see why, consider how a worker' current wage affects her trade-off between quit rates and future wages. For a given current wage, a worker is willing to quit at a relatively slower rate only in exchange for relatively higher future wages. The higher her current wage, the lower the *ex post* surplus she can obtain from a given wage and thus, the lower the worker's quit rate. Since a given (future) wage represents a smaller proportional share of the wage gain in the worker expected surplus for workers with higher current wages, a worker's quit rate declines with her current wage at a decreasing rate. While this property is as one would expect, it implies that the worker's value function $\overline{V}(\{i, w, y\}, \rho)$ may not be a concave function of w, which is problematic. In general, it is unclear whether or not the properties of $q_e(\{i, w, y\}, \rho)$ ensure that both the worker's surplus and the employer's surplus are well-behaved with respect to w.

The above problem complicates significantly the analysis of competitive search on the job (e.g., Delacroix and Shi 2006). In the appendix, we show that this problem can be addressed by viewing the solution to (P1) as a mapping from the workers' quit rates to their current wages, rather than the reverse. This approach is crucial as it allows us to solve directly for the equilibrium, as opposed to characterizing a constrained efficient outcome that corresponds to the equilibrium allocation. This approach allows us to examine both efficient and inefficient equilibria.

3.1 Revealing equilibrium

It is instructive to begin with the separating equilibrium, in which entry wages reveal match productivity. We refer to this kind of equilibria as (fully) revealing.

Proposition 2 *Assume that* $(y_h - b) / (y_h - y_l) \geq (r + \delta + \alpha_1) / (r + \delta + (1 - \delta)\alpha_1)$. *There is a number* $k_0 > 0$ *such that for all* $k \leq k_0$ *there is a refined equilibrium that is revealing. The corresponding equilibrium allocation is uniquely characterized by equations (6)-(9) and (11) below, and it maximizes the present value of aggregate production net of search costs.*

In a revealing equilibrium the wage distribution has three mass points: one wage for each productivity realization for workers who find jobs out of unemployment, and one wage for workers who find jobs via on-the-job search. Equilibrium transitions are as follows: All job offers made to unemployed workers are accepted. Unemployed workers who meet a firm with which they form a low productivity match conduct on-the-job search. These workers change jobs upon meeting another firm with which they form a high productivity match. Our equilibrium refinement implies that workers in high productivity matches will not crowd out poorly matched workers in a revealing equilibrium. Accordingly, such workers do not profit from search on the job and never change jobs. Jobs are destroyed both exogenously (at rate δ) and, for the case of low productivity matches, endogenously by quits.

In the revealing equilibrium low productivity workers are paid their marginal product:

$$w_u^l(i, 1) = y_l. \tag{6}$$

This is an important feature of separating equilibria and is at the core of the constrained efficiency of the revealing equilibrium. Intuitively, the *ex ante* match surplus is maximized when the employer assigns all of the match surplus to poorly matched workers *ex post*, in which case they quit exactly when it is efficient to do so. Such a surplus division is optimal from the viewpoint of employers, because they are able to maximize surplus extraction when workers are well-matched *ex post*.

Otherwise, the revealing equilibrium satisfies the usual zero profit and matching efficiency conditions of competitive search models. In particular, $\{w_e(s, 1), q_e(s, 1)\}$ is the unique pair $\{w', q'\}$ that solves

$$q' f(q')\, \alpha_j \left(\frac{y_h - w'}{r + \delta}\right) = k \tag{7}$$

and

$$\frac{w' - y_l}{r + \delta + (1 - \delta)\alpha_j f(q')} = \left(\frac{1 - \eta(q')}{\eta(q')}\right)\left(\frac{y_h - w'}{r + \delta}\right). \tag{8}$$

Equation (7) requires that the expected value of a vacancy to potential poachers equals the cost of posting the vacancy. It implies that employers are willing to offer higher wages and suffer reductions in the net present value of their profits only if they expect to fill their vacancies at a faster rate.

Equation (8) is the familiar condition of matching efficiency from standard competitive search equilibrium models. The left side of the equation is the present value of forgone wages while a poorly matched worker searches on the job. It is easy to verify that the Bellman equation in problem (P1) implies that this value is equal to the worker surplus in the new match. Recalling that $\eta(q)$ is the elasticity of job creation and $1-\eta(q)$ is the elasticity of job finding, this matching-efficiency condition implies that the ratio of the worker's surplus to the firm's surplus in new matches equals the ratio of their matching elasticities.

Similarly, $\{w_u^h(i,\rho), q_u(i,\rho)\}$ is the unique pair $\{w,q\}$ that satisfies

$$qf(q)\alpha_i\left(\frac{y_h-w}{r+\delta}\right)=k \tag{9}$$

and

$$\alpha_i\frac{\overline{V}\left(\{i,w,y_h\},1\right)}{1+r}+(1-\alpha_i)\frac{\overline{V}\left(\{i,y_l,y_l\},1\right)}{1+r}-\frac{\overline{V}\left(s_u,1\right)}{1+r}=\left(\frac{1-\eta(q)}{\eta(q)}\right)\alpha_i\left(\frac{y_h-w}{r+\delta}\right) \tag{10}$$

together with the Bellman equation in Problem (P2).

Note that the condition for matching efficiency in the market for unemployed workers (i.e. equation (10)) is completely standard. This is because of the result that, in equilibrium, firms earn no profit from low productivity workers. Consequently, the firm's match surplus is entirely a function of the profits it makes when employing high productivity workers. Since these workers cannot profit from on-the-job search in a revealing equilibrium, employers have no incentive to set wages in order manipulate their quit rates.

One can verify that equations (9) and (10), together with the Bellman equation in Problem (P2) imply that $q_u(i,\rho)$ is the unique value of q that solves

$$\frac{y_h-b}{r+\delta}-\frac{(1-\alpha_i)\,k}{\eta\left(q_e(s,1)\right)q_e(s,1)f\left(q_e(s,1)\right)\alpha_j}=\frac{k}{\eta(q)\,qf(q)}+\left(\frac{1-\eta(q)}{\eta(q)}\right)\frac{k}{(r+\delta)\,q}. \tag{11}$$

This completes the characterization of the unique equilibrium allocation associated with a revealing equilibrium.

The assumption that $(y_h-b)/(y_h-y_l)\geq(r+\delta+\alpha_1)/(r+\delta+(1-\delta)\alpha_1)$ made in Proposition 2 is sufficient to ensure that unemployed workers are willing to accept job offers when match productivity is low. Otherwise, a revealing equilibrium with positive job creation may not exist if k is sufficiently low. The assumption requires that the difference between y_l and b be sufficiently large. The smaller the job destruction rate the less restrictive the assumption is.

On the equilibrium path, beliefs about the composition of the pool of applicants must be correct, both in the market for unemployed workers as well as the separate market for workers searching on the job. Moreover, in the latter employers have the most optimistic beliefs, as they believe that their job offers will be accepted with certainty. Therefore, it is straightforward to support the above equilibrium allocation. Clearly, neither the equilibrium mapping Q nor off-equilibrium beliefs that support the equilibrium allocation are unique.

3.2 Non-revealing equilibrium

We now turn to the pooling equilibrium, in which entry wages do not reveal match productivity. We refer to this kind of equilibria as non-revealing.

Proposition 3 *Assume that* $(1-\alpha_1)(1-\delta) > (r+\delta)$. *There is a number* $k_1 > 0$ *such that for all* $k \leq k_1$ *there is a refined equilibrium that is non-revealing. The allocation supported by such a non-revealing equilibrium is uniquely characterized in the Appendix.*

In a non-revealing equilibrium the wage distribution has two mass points. Since wages do not differ across productivity realizations, all entry jobs pay an identical wage. In principle, there could be two wages in the on-the-job search market, as poorly matched workers accept poaching offers whereas well-matched workers transition to retention wages. However, since both types of workers have the same current wage, their incentives to search on the job are identical, so the equilibrium poaching and retention wages are also identical.

Equilibrium transitions are as follows: All job offers made to unemployed workers are accepted, and all workers employed in jobs found out of unemployment search on the job, with workers who are well-matched *ex post* mimicking the on-the-job search behavior of workers who are poorly matched. As a result of pooling, all workers searching on the job face the same matching probabilities. Workers with low productivity realizations in their first jobs change jobs upon meeting another employer with which they form a high productivity match. Workers with high productivity realizations in their first jobs receive retention offers upon meeting another employer with which they form a high productivity match. Jobs are destroyed both exogenously (at rate δ) and, in the case of workers in low productivity matches, endogenously by quits.

Consider the search problem of a worker who is currently employed in a type-i job earning a wage w and searching for a type-j job. Once again, it will be convenient to distinguish between different types of jobs even though we are assuming that $i = j = 1$ throughout this section. It is easy to verify that an interior solution of Problem (P1) with $\rho = 1$, $\{w', q'\}$, satisfies the familiar matching efficiency condition

$$\frac{w' - w}{r + \delta + (1-\delta)\,\alpha_j f\,(q)} = \left(\frac{1 - \eta\,(q')}{\eta\,(q')}\right)\left(\frac{y_h - w'}{r + \delta}\right),$$

according to which the ratio of the worker's surplus to the employer's surplus equals the ratio of their matching elasticities. Notice that this condition is identical to (8), which is the corresponding matching efficiency condition in the revealing equilibrium, though the entry wage (w) is generally different. A solution of Problem (P1) also satisfies the usual zero-profit condition

$$q' f\left(q'\right)(1-\alpha_i)\,\alpha_j \left(\frac{y_h - w'}{r+\delta}\right) = k. \tag{12}$$

Note that, in a non-revealing equilibrium, potential poachers need to anticipate that a fraction $(1-\alpha_i)$ of their pool of applicants are poorly matched in their current jobs, and so a fraction α_i

will turn down their job offers because they are only searching to elicit a retention offer from their current employer.

Solving Problem (P2) is non-trivial because the objective function is not generally concave in $\{w, q\}$. Fortunately, one can address this problem by viewing the solution to Problem (P1) as a mapping from the workers' quit rates to their entry wages, rather than the reverse, and then treat current and future quit rates as the relevant choice variables in Problem (P2). We follow this approach in the proof of Proposition 2 to characterize the equilibrium allocation in the constrained efficient equilibrium. In the Appendix, we show that this approach can be followed more generally to characterize the allocation in the non-revealing equilibrium and prove Proposition 3.

To understand the properties of the non-revealing equilibrium allocation, it is useful to consider the transformed problem in some detail. To that end, use the above first-order conditions to express the worker's entry wage as a function of (the future) q':

$$\widetilde{W}\left(q'\right) \equiv y_h - \left(\frac{k}{q' f\left(q'\right)\left(1-\alpha_i\right)\alpha_j}\right)\left(r+\delta+\left(\frac{1-\eta\left(q'\right)}{\eta\left(q'\right)}\right)\left(r+\delta+\left(1-\delta\right)\alpha_j f\left(q'\right)\right)\right). \quad (13)$$

Observe that employers understand i) that all workers search on the job, and ii) that job finding probabilities in the on-the-job search market depend on the wages earned by workers in their current jobs. Employers take this effect into account and set current wages, in part, in order to influence future quit rates. Let $\widetilde{V}_0\left(i, q'\right)$ denote the value of a type-i job to an employed worker expressed as a function of q':

$$\begin{aligned}\widetilde{V}_0\left(i, q'\right) &\equiv \alpha_i \overline{V}\left(\left\{i, \widetilde{W}\left(q'\right), y_h\right\}, 1-\alpha_i\right) + \left(1-\alpha_i\right)\overline{V}\left(\left\{i, \widetilde{W}\left(q'\right), y_l\right\}, 1-\alpha_i\right) \\ &= \overline{V}\left(\left\{i, \widetilde{W}\left(q'\right), y_h\right\}, 1-\alpha_i\right) = \overline{V}\left(\left\{i, \widetilde{W}\left(q'\right), y_l\right\}, 1-\alpha_i\right)\end{aligned}$$

and let $\widetilde{M}_0\left(i, q'\right)$ denote the *ex ante* surplus associated with a type-i match:

$$\widetilde{M}_0\left(i, q'\right) \equiv \alpha_i \widetilde{M}\left(\left\{i, \widetilde{W}\left(q'\right), y_h\right\}\right) + \left(1-\alpha_i\right)\widetilde{M}\left(\left\{i, \widetilde{W}\left(q'\right), y_l\right\}\right),$$

where $\widetilde{M}\left(\left\{i, \widetilde{W}\left(q'\right), y_h\right\}\right)$ is the *ex post* surplus associated with a high-productivity match and $\widetilde{M}\left(\left\{i, \widetilde{W}\left(q'\right), y_l\right\}\right)$ is the *ex post* surplus associated with a low-productivity match.

It is useful to understand the connection between the total surplus of a match and its allocation between a worker and her employer. To that end, note first that the surplus in low-productivity matches is given by

$$\begin{aligned}\frac{\widetilde{M}\left(\left\{i, \widetilde{W}\left(q'\right), y_l\right\}\right)}{1+r} &= \frac{\overline{V}\left(\left\{i, \widetilde{W}\left(q'\right), y_l\right\}, 1-\alpha_i\right)}{1+r} - \frac{\overline{V}\left(s_u, 1-\alpha_i\right)}{1+r} \\ &\quad + \left(\frac{r+\delta}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q'\right)}\right)\frac{y_l - \widetilde{W}\left(q'\right)}{r+\delta}\end{aligned}$$

where the first line on the right side is the part of the match surplus that goes to the worker and the second line is the part that goes to the employer, which consists of a flow of profits equal to $y_l - \widetilde{W}(q')$ for as long as the worker stays with the employer (where the term in parentheses is the probability that the worker will find an outside offer), which the employer anticipates she will accept with probability one.

The surplus in high-productivity matches is more interesting. In particular,

$$\begin{aligned}
\frac{\widetilde{M}\left(\left\{i, \widetilde{W}(q'), y_h\right\}\right)}{1+r} &= \frac{\overline{V}\left(\left\{i, \widetilde{W}(q'), y_h\right\}, 1-\alpha_i\right)}{1+r} - \frac{\overline{V}(s_u, 1-\alpha_i)}{1+r} \\
&\quad + \left(1 - \frac{r+\delta}{r+\delta+(1-\delta)\alpha_j f(q')}\right) \frac{y_h - w_e\left(\left\{i, \widetilde{W}(q'), y_h\right\}, 1-\alpha_i\right)}{r+\delta} \\
&\quad + \left(\frac{r+\delta}{r+\delta+(1-\delta)\alpha_j f(q')}\right) \frac{y_h - \widetilde{W}(q')}{r+\delta}.
\end{aligned}$$

The first and the third lines in the right side are the obvious counterparts of those in low-productivity matches. The second line reflects the fact that *ex post* well-matched workers will search for outside offers solely to elicit a retention offer from their current employer.

It is easy to verify that an interior solution for the current and future labor market queues $\{q, q'\}$ must satisfy the following conditions:

$$\frac{\widetilde{V}_0(i, q')}{1+r} - \frac{\overline{V}(s_u, 1-\alpha_i)}{1+r} = \lambda_i q \left(\frac{1-\eta(q)}{\eta(q)}\right) \frac{k}{qf(q)}, \tag{14}$$

$$\lambda_i = \frac{f(q)\, \partial \widetilde{V}_0 / \partial q'}{qf(q)\left(\partial \widetilde{V}_0/\partial q' - \partial \widetilde{M}_0/\partial q'\right)} \tag{15}$$

and

$$qf(q)\left(\frac{\widetilde{M}_0(i, q')}{1+r} - \frac{\widetilde{V}_0(i, q')}{1+r} + \frac{\overline{V}(s_u, 1-\alpha_i)}{1+r}\right) = k, \tag{16}$$

where λ_i is the multiplier associated with the employer's zero-profit constraint, given by equation (16). Equation (14) coincides with the standard matching efficiency condition if and only if the multiplier equals $1/q$. Consider equation (15). The multiplier is the expected value of surplus to the worker associated with a higher labor market queue at the margin ($f(q)\, \partial \widetilde{V}_0/\partial q'$) evaluated in terms of the employer's surplus ($qf(q)\left(\partial \widetilde{V}_0/\partial q' - \partial \widetilde{M}_0/\partial q'\right)$). The expected surplus of a match is maximized at $\partial \widetilde{M}_0/\partial q'$, which implies that $\lambda_i = 1/q$. In the Appendix, we show that this happens exactly at the corner when $\widetilde{W}(q') = y_l$.

In an interior non-revealing equilibrium $\lambda_i \geq 1/q$ and the match surplus is not maximized, except in the special case where the first-order conditions hold at the corner and $\lambda_i = 1/q$. The problem is that while employers can lower the workers' future quit rates by raising the entry wages they offer in the first place, they also have an impact on the outside offers the workers will get, because workers with higher wages have an incentive to elicit higher outside offers. Since they cannot prevent well-matched workers from seeking outside offers, the allocation of surplus at the margin is allocated disproportionately to the worker and so employers do not typically have an incentive to raise entry wages all the way to y_l.

The corner allocation is still inefficient because it induces too little entry of employers in the market for unemployed workers. This is because, relative to a revealing equilibrium, employers are forced to share too much surplus with the worker, as high productivity workers who receive outside offers stay with their current employer, but are able to extract some of the surplus. In the Appendix, we show that the allocation in a non-revealing equilibrium is uniquely characterized, although in general we cannot guarantee that the allocation is interior.

To understand why non-revealing wages can be supported in equilibrium note the existence of an informational externality, whereby firms in the market for unemployed workers do not take into account the informational value of wages to poachers. The consequence of this externality is that employers have no direct incentive to post revealing wages. This means that non-revealing wages can be equilibrium wages as long as unemployed workers choose to search for non-revealing contracts when revealing contracts are feasible. This occurs because the option to search on the job constitutes an important component of the value of a job, but the value of this option depends on the beliefs of both workers and potential poaching firms.

To see this, consider a candidate non-revealing equilibrium. Suppose an unemployed worker considers searching for a contract with revealing wages. The value of on-the-job search for such a job, however, depends on the off equilibrium beliefs of poaching firms about the current match quality of their applicant pool. Our equilibrium refinement has no bite for beliefs about non-equilibrium states, therefore these beliefs are unrestricted. Consequently, if potential poachers are sufficiently pessimistic about the composition of the applicant pool in the on-the-job search market associated with a revealing contract (i.e. they believe it will contain a high percentage of well-matched workers looking for retention offers) then the returns to on-the-job search associated with deviations to a revealing contract are sufficiently low that such deviations are unprofitable to workers.

Formally, if $s \notin S^*$, beliefs are arbitrary and we may assume that employers believe that $\mu(s|x) = 0$ for all $s \notin S^*$. In the Appendix, we show that the assumption that $(1-\alpha_1)(1-\delta) > (r+\delta)$ made in Proposition 3 then ensures that no type-1 employer can profit from offering a deviating contract where wage offers to unemployed workers are made conditional on match productivity. The assumption requires that the probability of mismatch in type-1 jobs is sufficiently high, that the jobs are sufficiently durable and that workers value future payoffs sufficiently. Of course, neither the equilibrium mapping Q nor off-equilibrium beliefs that support the equilibrium allocation are unique.

Observe that wages that do not reveal match productivity create adverse selection in the on-the-job search market since, under non-revealing wages, workers in matches with high productivity cannot be identified and, therefore, have an incentive to search on the job in order to elicit retention offers from their current employers. Non-revealing wages, therefore, increase the value of on-the-job search to workers with a high productivity realization relative to the case where wages reveal match quality, thereby preventing well-matched workers from searching on the job. The overall effect of this adverse selection problem, however, is to depress the returns to poaching firms, which reduces the entry of poachers and therefore, depresses the returns to on-the-job search overall. This reduction is concentrated on poorly matched workers. Note that this adverse selection problem is worse the higher is α_1, because a large value of α_1 implies that many workers are well matched to begin with, and therefore only searching on the job to elicit retention offers.

4 Directed mismatch equilibrium

In this section we turn to the analysis of the economy with both types of jobs. Recall that type-2 jobs are less likely to result in high productivity matches than type 1 jobs while being identical to type-1 jobs in terms of creation costs and the matching function. The main result, of both the section and the paper, is that despite the unambiguous technological inferiority of type-2 jobs, there exist equilibria in which type-2 jobs are created alongside type 1 jobs, and some workers choose to search for type-2 jobs. That is, directed mismatch can be supported as an equilibrium outcome.

It is easy to verify that an equilibrium allocation must solve the obvious analogues of Problems (P1) and (P2). First, note that our assumptions about counteroffers continue to restrict the possible job and wage transitions as explained in Section 3. It is straightforward to verify that any allocation supported by an equilibrium with positive quits must be such that employed workers only ever search for type-1 jobs. Intuitively, workers are expected to be more productive in type-1 jobs and, consequently, type-1 employers always drive type-2 employers out of any market where employed workers search.

Taking this into account, Problem (P1), with $j = 1$, can be used to characterize equilibrium allocations, except that now it ought to be recognized that the value functions and the corresponding policy functions are functions of (ρ_1, ρ_2), rather than simply ρ, where $\rho_i \in \{1 - \alpha_i, 1\}$ denotes the fraction of poorly matched workers currently employed in type-i jobs among all those searching for type-1 jobs, for $i = 1, 2$. With a slight abuse of notation we will continue to denote those functions as before.

One can verify that equilibrium allocations in the two-job economy must satisfy the analogue of Problem (P2), with

$$\overline{V}(s_u, \rho_1, \rho_2) = b + \max\{V_1, V_2\}, \tag{17}$$

where V_i is given by Problem (P2) with $\rho = \rho_i$, for $i = 1, 2$.

In order to characterize an equilibrium allocation, first note that, due to the technological inferiority of type-2 jobs, if wages in type-1 matches reveal productivity, the equilibrium is constrained efficient. In this case, no type-2 jobs are created, the equilibrium allocation solves P1 and P2, with

$\rho = 1$, and is as given by Proposition 2, with $j = 1$. Note that, if wages in both types of jobs are revealing, then neither type of job suffers from the adverse selection problem. In this case, the higher productivity of type-1 jobs makes them more attractive to all searchers. If wages in type-1 jobs are revealing, but wages in type-2 jobs are non-revealing then type-2 jobs, in addition to being less productive, also suffer from the adverse selection problem. Clearly, the allocation in a revealing equilibrium is unique within the class of revealing equilibria, but it can be supported in a continuum of different ways.

Thus, the existence of a directed mismatch equilibrium requires that wages in type-1 jobs are non-revealing, in which case on-the-job search from these jobs suffers from the adverse selection problem created by well-matched workers searching for retention offers. The key to understanding these equilibria is to note that the adverse selection problem is more severe in markets for type-1 jobs, because workers in these jobs are relatively less likely to be poorly matched, and therefore more likely to be searching on the job in order to elicit retention offers than are workers in type-2 jobs. Potential poaching firms understand this and, consequently, are less willing to enter markets where workers in type-1 jobs search on the job. This lowers the value of on-the-job search for workers in type-1 jobs. When this effect is sufficiently strong, type-2 jobs have an equilibrium advantage over type-1 jobs despite being inferior along traditional technical dimensions.

There are only two cases to consider: the case where wages do not reveal productivity in either type of job, and the case where wages reveal productivity in type-2 jobs but not type-1 jobs. In the former case, however, the degree of asymmetry may not be powerful enough to support the creation of both types of jobs. Consequently, in the remainder of this section we examine the case where type-1 entry wages are non-revealing and type-2 entry wages are fully revealing. Proposition 4 provides sufficient conditions for existence of an equilibrium where both types of jobs are created, for this case. Below, we argue that this situation can also be interpreted as an extreme case of labor markets where match productivity is imperfectly observable and it is relatively easier to observe in type-2 jobs, which are consequently subject to a less severe adverse selection problem.

Proposition 4 *Assume that* $(1 - \alpha_1)(1 - \delta) > (r + \delta)$. *There are numbers* $\widehat{\alpha} \in (0, \alpha_1)$ *and* $\widehat{k} > 0$ *such that for all* $\alpha_2 \in (\widehat{\alpha}, \alpha_1)$ *and all* $k \in \left(0, \widehat{k}\right)$ *there is a refined equilibrium with positive job quits where both types of entry jobs are created. Thus, this equilibrium exhibits directed mismatch.*

The assumption in the proposition ensures that no type-1 employer can profit from offering a deviating contract in which wage offers to unemployed workers are made conditional on match productivity. The assumption requires that the probability of a low productivity realization in type-1 jobs is sufficiently high, that the jobs are sufficiently durable and that workers value future payoffs sufficiently. Under this assumption, Proposition 4 says that a directed mismatch equilibrium with both types of entry jobs exists whenever the probability of a low productivity realization in type-2 jobs is sufficiently close to that in type-1 jobs and the employers' search costs are sufficiently low.

It is easy to see that the unemployed workers' optimal search policy is characterized by a cutoff c_0 such that they search for type-2 jobs if and only if their current realization of the search cost c

is smaller than the cutoff. Noting that unemployed workers will search for type-2 jobs if and only if their idiosyncratic search cost c is smaller than the utility gain $V_2 - V_1$, we have that

$$\overline{V}(s_u, 1-\alpha_1, 1) - b = V_1 + F(c_0)\left(V_2 - \mathbb{E}(c\,|c \leq c_0) - V_1\right). \tag{18}$$

That is, the net value of unemployment ($\overline{V}(s_u, 1-\alpha_1, 1) - b$) to a worker is equal to the value of searching for jobs where she is more productive (V_1) plus the option value of searching for jobs where she is less productive, which consists of the expected utility gain $F(c_0)(V_2 - V_1)$ minus the expected search costs $F(c_0)\,\mathbb{E}(c\,|c \leq c_0)$.

Since searching for type-1 jobs is costless, an equilibrium allocation must have $V_2 - V_1 \leq c_0$, with equality if and only if $V_2 \geq V_1$. Thus, either the option value of searching for type-2 jobs is non-negative, or else only type-1 jobs are created in equilibrium. In this sense constructing the directed mismatch equilibrium is non-trivial: it requires that the value of search for jobs where the unemployed is relatively less productive to be relatively higher in equilibrium. That is, it requires that $c_0 > 0$. The assumption that $\alpha_2 \in (\widehat{\alpha}, \alpha_1)$ in the proposition ensures that the two types of jobs are sufficiently similar that unemployed workers would strictly prefer to search for type-2 jobs if it were costless to do so. It is clear that there is a number $\widehat{\alpha} \in (0, \alpha_1)$ such that this is the case.

In the equilibrium with two types of entry jobs, workers in type-1 and type-2 jobs conduct on-the-job search on separate markets, because job type is an observable component of a worker's labor market state. It is then easy to see that the equilibrium allocation is such that V_i is given by Problem (P2) subject to (18), for $i = 1, 2$, with $\rho = 1 - \alpha_1$ for $i = 1$, and $\rho = 1$ for $i = 2$.

The next proposition provides further insight on the properties of the directed mismatch equilibrium.

Proposition 5 *Consider an interior refined equilibrium where wages in type-*1 *jobs are non-revealing and wages in type-*2 *jobs are revealing. (i)* $V_2 > \overline{V}(s_u, \rho_1, \rho_2) - b > V_1$ *and (ii)* $f(q_u(2, \rho_1, \rho_2)) > f(q_u(1, \rho_1, \rho_2))$*, for* $(\rho_1, \rho_2) = (1 - \alpha_1, 1)$*.*

Part (i) of the proposition implies that type-2 jobs have higher value than type-1 jobs. This reflects the fact that adverse selection in the on-the-job search market associated with type-1 jobs reduces the values of those jobs. Since wages in type-2 jobs are revealing, the adverse selection does not affect on-the-job search for workers in type-2 jobs.

Part (ii) says that type-2 jobs are also easier to get. Since adverse selection reduces the value of on-the-job search from type-1 jobs, the wage received in those jobs constitutes a larger portion of the value of the job. In type-2 jobs, in contrast, the current wage is relatively less important because the option value of search is higher. Therefore, it is relatively cheaper for firms to compensate workers with job finding probability, rather than with wages, in the market for type-2 jobs.

Unsurprisingly, given the adverse selection problem, labor market outcomes in the directed mismatch equilibrium are inefficient. In particular, the degree of equilibrium mismatch, defined as the fraction of workers searching for type-2 jobs, is inefficiently high relative to the efficient benchmark, under which all workers search for type-1 jobs. Furthermore, the value of unemployment is inefficiently low, due to the fact that the equilibrium fails to maximize the present value of aggregate production net of search costs.

5 Discussion

Our paper has developed an original theory of labor market mismatch. We begin this section by arguing that the coordination failure we have introduced above is not necessary, and that directed mismatch identical to ours can be generated from informational frictions alone. We go on to show that directed mismatch provides several novel insights about labor markets. We focus on the experiences of college educated American workers and think of the type-1 jobs of the previous sections as corresponding to non-routine jobs, which make full use of a college education, while type-2 jobs correspond to routine jobs, which do not.

5.1 Incomplete information in the labor market

Consider the model from the previous section, except that the productivity of type-i entry matches is observed with some probability $\nu_i \in [0, 1]$, for $i = 1, 2$, and suppose that $\nu_2 \geq \nu_1$. For simplicity, suppose that poachers always observe the realization of match productivity before hiring. Our model is the special case where $\nu_2 = \nu_1 = 1$. If $\nu_1 < 1$, the revealing equilibrium fails to exist and moreover, all equilibria are inefficient. If $\nu_2 = \nu_1 = 0$, there is an equilibrium where wages are non-revealing in both markets. This equilibrium also exists in our model. Moreover, in such an equilibrium the adverse selection problem is necessarily more severe in markets where workers employed in type-1 jobs search, precisely because the proportion of poorly matched workers in those markets is relatively low. So everything else equal, the returns to search on the job are higher for workers currently employed in type-2 jobs. We have not been able to establish analytically that this equilibrium can exhibit directed mismatch and numerical simulations of the model suggest that it does not. It is clear, however, that all that is needed to support directed mismatch is that the adverse selection problem is sufficiently more severe in markets where workers employed in type-1 jobs search. Our analysis can be understood as an extreme version of this problem. Whether directed mismatch results from coordination failure or from the fact that non-routine match productivity is imperfectly observable, the structure of the equilibrium is the same and so are the relevant externalities.

5.2 On-the-Job Search

The conventional view, obtained from viewing mismatch as an *ex post* phenomenon, suggests that on-the-job search is an important channel by which labor market mismatch is corrected, as it enables workers in unsuitable jobs to transition to better matches without experiencing unemployment. According to this view, high job-to-job transitions are interpreted to mean that labor markets are effective at allocating workers to jobs. This view underlies much of the academic literature as well as the analysis of other labor market observers.[15]

An *ex ante* perspective on mismatch, however, calls for a radically different view. In a directed mismatch equilibrium, on-the-job search is also the *cause* of mismatch because it encourages workers to search for jobs for which they are otherwise not well suited. This suggests that there is no simple relationship between the rate of job-to-job transitions, the extent of labor market mismatch, and the efficiency of labor markets.

To illustrate this point, we perform a pair of comparative static exercises. Given the richness of the setting, it is not possible to generate analytical results. In lieu of these, we solve the model numerically.[16] Our numerical results should not be mistaken for a serious calibration of the model. The primary objective of our paper is to introduce a rigorous theory of mismatch. Consequently, we employ some fairly extreme symmetry assumptions (for example, that the productivity of good and bad match realizations is identical across job types) that, while simplifying the analytical proofs of existence of equilibrium, render our model quantitatively unrealistic.

Table 1: comparative statics

	increase in α_1	decline in α_2
welfare (value of unemployment)	rises	falls
employment rate	rises	falls
share of routine jobs	falls	falls
non-routine/routine wage premium	rises	rises
job-to-job transition rate	falls	falls

The first column of Table 1 reports the equilibrium response to an increase in α_1, which is the probability that a college worker is highly productive in a non-routine job. A rise in α_1 induces unemployed workers to reallocate their search towards non-routine jobs, which causes mismatch in the economy to fall. This implies that job-to-job transitions also fall, because well-matched workers in non-routine jobs do not benefit from search on-the-job. Furthermore, when α_1 rises, the gap between expected productivity in a non-routine and a routine job also rises, which is reflected in an increase in the wage premium associated with non-routine work and, therefore, a rise in overall wage inequality within the group of college workers. Since the increase in α_1 also raises the expected productivity of non-routine jobs, firms create more such jobs, both in markets where unemployed workers search, and in markets where workers search on-the-job. The result is an increase in overall employment, average wages, and the value of unemployment.

Our second comparative static concerns the equilibrium response to a fall in α_2, which we report in the second column of Table 1. A fall in α_2 also causes the gap between the expected productivity of non-routine and routine jobs and the wage premium paid to workers in non-routine jobs to rise.

[15] "Businesses adding and losing workers and people quitting and taking other jobs – what economists call "churn"— are generally good measures of economic confidence." Goldman Sachs (2014).

[16] For instance, an interior mismatch equilibrium exists under the following parameterization: $y_l = 1$, $y_h = 1.1$, $\alpha_1 = 0.4$, $\alpha_2 = 0.35$, $k = 0.1$, $\delta = 0.1$, $b = 0.1$, $r = 0.02$, $\theta = 1$.

Also, as before, workers reallocate their search towards non-routine jobs, causing both mismatch and job-to-job transitions to fall. A decline in α_2, however, results in a *fall* in the surplus associated with routine jobs. Consequently, firms are willing to create fewer such jobs in equilibrium. Since routine jobs have value to college workers, their destruction causes unemployment, average wages, and the value of unemployment to all fall.

These results stand in sharp contrast to the conventional wisdom. When α_1 rises, the resulting decline in job-to-job transitions does not indicate increased mismatch or reduced social welfare. Indeed, a decline in job-to-job flows can indicate precisely the opposite. The fall in transitions reflects the fact that workers do not search on the job precisely because they are well matched. Furthermore, labor market outcomes are better in the sense that the value of unemployment is higher. The case where α_2 falls is even more problematic. Even though the decline in job-to-job transitions is associated with a *fall* in equilibrium mismatch, this corresponds to a fall in the welfare of labor market participants, as the routine jobs they would prefer to search for become both less profitable and less available. Clearly, the relationship between job-to-job transitions, mismatch, and labor market efficiency is complex and depends on the nature of the shocks that hit the labor market. This further implies that data on job-to-job transitions alone is insufficient to identify either the underlying shocks or the welfare implications.

5.3 Directed mismatch and non-routine biased technological change

In this section we argue that the trends that have impacted college educated workers in recent decades can be understood as responses to the automation and offshoring of routine jobs in a directed mismatch equilibrium.

We begin by extending our model to allow for men and women, high school and college educated workers, a decision to acquire college education and a labor force participation decision.

It is conceptually simple to incorporate differing educational levels into our framework. We assume that high school educated workers cannot be employed in non-routine jobs. As long as educational status is observable a separating competitive search equilibrium exists, under which the analysis of Section 4 continues to apply to college educated workers. Similarly, we can characterize an equilibrium that separates men and women.

It is also straightforward to extend our model to incorporate a labor force participation decision. Suppose that workers have heterogeneous opportunity costs of participating in the labor force. Formally, there is an exogenous and constant flow benefit l from staying out of the labor force, which is worker-specific and is discontinued when a worker enters the labor force. For simplicity, we assume that l is a one-time draw from a distribution H.

Clearly, labor force participation decisions are determined solely by the exogenous opportunity costs of participation and the returns to labor market search. The latter are independent of participation decisions and so optimal participation decisions for college educated workers are characterized by a single cutoff l_0, where

$$\overline{V}(s_u, \rho_1, \rho_2) = \left(\frac{1+r}{r}\right) l_0,$$

for $(\rho_1, \rho_2) = (1 - \alpha_1, 1)$. College educated workers participate in the labor force if and only if their opportunity cost l is smaller than the cutoff l_0. In this setting, $\overline{V}(s_u, 1 - \alpha_1, 1)$ can be understood as the equilibrium value of labor market participation to college educated workers. The important point is that the participation rate will track the value of participation. The participation decisions of high school workers are characterized by a cutoff l_1, which is calculated in a similar manner.

To determine the populations of high school and college educated workers, suppose that workers enter the economy, at time zero, with a high school education and they decide whether to acquire a college education. For simplicity, suppose the idiosyncratic cost of a college education is an independent random draw from some distribution at time zero. In this case, education acquisition decisions are determined solely by the exogenous opportunity costs of acquiring a college education and the returns to labor market participation. Moreover, the latter are independent of education decisions and so optimal education decisions are characterized by a single cutoff, where workers acquire a college education if and only if the cost is lower than the cutoff, and where the cutoff is equal to the expected return to a college education.

We now examine the labor market outcomes of both college and high school educated American workers from the perspective of our extended model. We begin with the observation that automation and offshoring have resulted in a fall in the productivity of routine jobs over recent decades, which we model as a decline of α_2. The impact of these changes on less educated workers has been widely studied. In the directed mismatch equilibrium, however, it is clear that changes in the productivity of routine jobs will also impact college educated workers.

Our discussion of the results in Table 1 illustrates the mechanism by which a number of well-known trends in the U.S. labor market can be understood as the result of a fall in α_2. These include the rising within-group inequality for the highly educated workers (Altonji, Kahn and Speer 2014) and the decline in job-to-job transition rates (Moscarini and Thompsson 2007, Davis, Faberman, and Haltiwanger 2012).

A striking observation regarding college-educated workers is the recent decline in their employment rates. At the aggregate level this trend is noticeable after 2000. However, as Table 2 makes clear, the aggregate trend reflects the composition of two differing trends for men and women, particularly before 1990. For both, men and women, the slow down in the 1990s and the fall in the 2000s are apparent. Of course, the increase in the employment rate of women in the earlier period reflects a number of forces other than non-routine biased technological change. Similar trends are evident in the data on participation rates.

Table 2: employment rates of college graduates, 25-54 (%)[17]

	1980	1990	2000	2012
men	95.3	94.9	94.3	91.2
women	73.0	81.8	81.2	79.0

The decline in the employment rates of college graduates, is particularly striking in light of the rapid increase of educational attainment in recent decades. It is not obvious why the fraction of people choosing to acquire a university education would rise when declining employment and participation rates suggest that the value of labor force participation to college workers has fallen.

The extended model makes sense of these seemingly disparate observations. Because high school educated workers are more concentrated in routine jobs, the decline in the productivity of these jobs will generally affect high school workers more severely relative to college workers. This raises the return to a college education relative to high school and simultaneously reduces the value of labor force participation to college workers. The same force also explains the rising college/high school wage premium (Autor 2015), as well as why employment in non-routine, cognitive jobs may lag the supply of college educated workers. Furthermore, our theory explains the slowdown of employment and participation rates of educated women during the 1990s as resulting from the same forces that drive the employment and participation trends of college educated men.

Perhaps the most apparently counterfactual implication of our explanation is the suggestion that directed mismatch should have fallen along with the value of routine jobs. The first two rows of Table 3, for example, illustrate the fact that, for both men and women, the ratio of routine jobs filled by college educated workers to routine jobs filled by high school workers has risen significantly over recent decades. This could be interpreted as a rise in mismatch related to college educated workers moving down the occupational ladder and into less skilled intensive jobs, perhaps in response to a fall in the creation of non-routine jobs.

Such an interpretation is problematic, however, as it disregards the rapid growth of the population of college graduates relative to the population of high school workers. In the bottom panel of Table 3 we report the results from a counterfactual exercise, where we fix the share of college workers in routine jobs, relative to all employed college graduates, to its 1980 level, and consequently let their employment in routine jobs be driven solely by the changes in the population of college workers. This exercise illustrates that the increases in the populations of college educated men and women are sufficient to account for the entire change in the ratio of college to high school workers employed in routine jobs in the 1980s, and significantly *more* than the total increase after. We conclude that the observed increase in the proportion of routine jobs filled by college educated

[17] Our data is taken from Beaudry, Green and Sand (2016). We thank Ben Sand for making it available to us. The data covers men and women between the ages of 25 and 54, and is drawn from the CPS, 1980-2013. The division of jobs into cognitive and manual, routine and non-routine is based on the types of tasks predominantly performed within them employing the categorization of Acemoglu and Autor (2011).

workers is a reflection of the fact that these workers represent an increasing fraction of the labor force rather than an increase in mismatch, particularly after 1990.

Table 3: ratio of college to high school workers in routine jobs, 25-54 (%)

		1980	1990	2000	2012
actual	men	30.6	32.9	36.8	46.4
	women	16.9	26.0	36.4	72.4
counterfactual*	men	30.6	32.8	39.6	51.0
	women	16.9	25.7	41.4	84.0

*shares of routine and non-routine jobs for college workers are constant at 1980 levels

Table 4 makes this plain by reporting the share of college workers employed in routine-cognitive jobs relative to college workers employed in all cognitive jobs, which is the most natural empirical counterpart to our notion of mismatch. It is clear that this share fell after 1990 for both men and women, with the share for women falling even faster than it did for men. The data after 2000 may reflect the slowdown in cognitive jobs observed by Beaudry, Green, and Sand (2014, 2016), although there are a number of potentially confounding forces at work (Autor, 2015). Remarkably, however, the composition of cognitive jobs in the 2000s did not change in favor of routine jobs.

Table 4: routine-cognitive share of cognitive jobs for college workers, 25-54 (%)

	1980	1990	2000	2012
men	20.9	21.3	19.6	19.2
women	22.6	23.0	19.7	19.8

Our interpretation of the evidence is that the degree of mismatch remained stable in the 1980s, then declined in the 1990s before stabilizing in the 2000s. Whatever skill-biased technological change took place in the 1980s does not seem to be non-routine biased. The evolution of mismatch after 1990 is likely driven by computerization and outsourcing of routine jobs, which has depressed the productivity of routine jobs while increasing the productivity of non-routine jobs, but the effects of these forces after 2000 appear to be masked by the simultaneous slowdown in the productivity of non-routine cognitive jobs. It is worth noting that the decline in mismatch that we emphasize here is consistent with the conclusions of other observers who have analyzed the link between adverse labor market outcomes and labor market mismatch.[18]

While these observations are natural from the perspective of our theory, we view them as problematic for theories based on efficient sorting. Under the most natural variants of efficient sorting, the most able workers have the strongest incentives to obtain a college education. This implies that when the fraction of workers choosing to obtain a college education rises, the ability of both the average and marginal college educated worker to perform non-routine cognitive tasks declines. In terms of labor market outcomes, this suggests that the fraction of college educated

[18] See, for example, Sahin, Song, Topa and Violante (2014), Krugman (2012), and Altig (2012).

workers who are employed in routine jobs should increase with educational attainment.

5.4 The Returns to Education

Our theory of directed mismatch also has implications for the returns to education. Under the conventional, *ex post* view of mismatch, the observation of college workers employed in routine jobs appears to represent a failure of education. That is, such workers would appear to have ended up in jobs that they could have found without a college education. An, often unstated, implication of this view is that the economy produces too many college educated workers.

In the directed mismatch equilibrium, however, the observation that college educated workers search for and are employed in routine jobs does not imply a failure of college education. In the absence of a college education, workers would have the set of choices that high school educated workers face, which is of substantially lower value than the value of the set of choices facing a college educated worker. Furthermore, the value of routine jobs to college workers differs from the value of such jobs to high school educated workers as it embeds the option of on-the-job search, which is of greater value to college educated workers.

Finally, in the directed mismatch equilibrium, the adverse selection problem associated with on-the-job search from non-routine jobs means that the equilibrium exhibits an inefficiently high degree of equilibrium mismatch and, consequently, an inefficiently low return to education. This further implies the economy, in fact, produces too few college educated workers.

Appendix

Proof of Proposition 1

It is easy to see that an allocation that does not solve problems (P1) and (P2), for a given value of $\rho \in (0, 1]$, cannot be supported by a competitive search equilibrium with positive quits, for the proposed equilibrium would necessarily violate the condition of our equilibrium refinement. The discussion leading to Proposition 1 implies that a revealing equilibrium must have $\rho = 1$ whereas a non-revealing equilibrium must have $\rho = 1 - \alpha_i$, as required. **QED**

Proof of Proposition 2

Throughout this proof we maintain the assumption that $\rho = 1$ and we drop the argument ρ from all functions. We keep track of job types under the assumption that unemployed workers search for type-i jobs and employed workers search for type-j jobs, where it is understood that $i = j = 1$ throughout this proof. We first prove existence and uniqueness of the candidate equilibrium allocation. It will become clear that it can be supported by a revealing equilibrium. Then, we show that the revealing equilibrium is constrained efficient.

We begin by characterizing the solution to Problem (P1) as a function of a worker's current wage.

Lemma 1 *Let $s = \{i, w, y_l\}$. For any $w \in [0, y_l]$, $\{w_e(s), q_e(s)\}$ is given by the unique pair (w', q') with $y_l \leq w' < y_h$ and $0 < q_a \leq q \leq q_b < \infty$ that solves the following conditions:*

$$q' f(q') \alpha_j \left(\frac{y_h - w'}{r + \delta} \right) = k,$$

$$\frac{w' - w}{r + \delta + (1 - \delta) \alpha_j f(q')} \geq \left(\frac{1 - \eta(q')}{\eta(q')} \right) \left(\frac{y_h - w'}{r + \delta} \right)$$

and $q' \geq q_a$ with complementary slackness, where q_a is given by

$$q_a f(q_a) \alpha_j \left(\frac{y_h - y_l}{r + \delta} \right) = k$$

and $q_b > q_a$ is given by

$$\frac{y_h - y_l}{r + \delta} = \left(\frac{k}{q_b f(q_b) \alpha_j} \right) \left(1 + \left(\frac{1 - \eta(q_b)}{\eta(q_b)} \right) \left(\frac{r + \delta + (1 - \delta) \alpha_j f(q_b)}{r + \delta} \right) \right). \tag{19}$$

Proof: The relevant first-order conditions for an interior solution of problem (P1) with $\rho = 1$ are given by:

$$\lambda q' = 1,$$

where λ is the relevant Lagrange multiplier, and

$$\frac{w'}{r+\delta} + \left(\frac{\delta}{r+\delta}\right)\frac{\overline{V}(s_u)}{1+r} - \frac{\overline{V}(\{i,w,y_l\})}{1+r} = \lambda q' \left(\frac{1-\eta(q')}{\eta(q')}\right)\left(\frac{y_h - w'}{r+\delta}\right),$$

together with the zero-profit constraint

$$q' f(q')\,\alpha_j \left(\frac{y_h - w'}{r+\delta}\right) = k.$$

This is the first condition stated in the lemma. The second condition follows from combining the first two first-order conditions above and the fact that the Bellman equation implies that a solution to the problem must be such that

$$\frac{w'}{r+\delta} + \left(\frac{\delta}{r+\delta}\right)\frac{\overline{V}(s_u)}{1+r} - \frac{\overline{V}(\{i,w,y_l\})}{1+r} = \frac{w'-w}{r+\delta+(1-\delta)\,\alpha_j f(q')}.$$

Clearly, $w_e(\{i,w,y_l\}) \geq y_l$ if and only if $q_e(\{i,w,y_l\}) \geq q_a$. Our assumption that $(r+\delta)k < \alpha_2(y_h - y_l)$ ensures that $0 < q_a < \infty$.

Combining the two conditions stated in the proposition implies that an interior solution $q_e(\{i,w,y_l\})$ is the unique value of q' that solves

$$\frac{y_h - w}{r+\delta} = \left(\frac{k}{q' f(q')\,\alpha_j}\right)\left(1 + \left(\frac{1-\eta(q')}{\eta(q')}\right)\left(\frac{r+\delta+(1-\delta)\,\alpha_j f(q')}{r+\delta}\right)\right). \tag{20}$$

It follows that $w \leq y_l$ implies that $q_e(\{i,w,y_l\}) \leq q_b$. Clearly, $\infty > q_b > q_a > 0$. **QED**

Invert (20) to express the worker's current wage as a function of q':

$$W(q') \equiv y_h - \left(\frac{k}{q' f(q')\,\alpha_j}\right)\left(r+\delta + \left(\frac{1-\eta(q')}{\eta(q')}\right)\left(r+\delta+(1-\delta)\,\alpha_j f(q')\right)\right), \tag{21}$$

for all $q' \in [q_a, q_b]$.

Lemma 2 *$W(q)$ and $\overline{V}(\{i, W(q), y_l\})$ are strictly increasing and concave functions of q on $[q_a, q_b]$.*

Proof: It is easy to verify that the Bellman equation for $\overline{V}(\{i,w,y_l\})$ implies that

$$\begin{aligned}\frac{\overline{V}(\{i,W(q),y_l\})}{1+r} &= \left(\frac{\delta}{r+\delta}\right)\frac{\overline{V}(s_u)}{1+r} + \left(\frac{r+\delta}{r+\delta+(1-\delta)\,\alpha_j f(q)}\right)\frac{W(q)}{r+\delta} \\ &\quad + \left(1 - \frac{r+\delta}{r+\delta+(1-\delta)\,\alpha_j f(q)}\right)\frac{w_e(\{i,W(q),y_l\})}{r+\delta}\end{aligned} \tag{22}$$

and, using the first-order conditions stated in Lemma 1, one can write

$$\frac{\overline{V}(\{i,W(q),y_l\})}{1+r} = \frac{y_h}{r+\delta} + \left(\frac{\delta}{r+\delta}\right)\frac{\overline{V}(s_u)}{1+r} - \frac{k}{\eta(q)\,q f(q)\,\alpha_j}. \tag{23}$$

One can verify that

$$\frac{\partial}{\partial q}\left(\frac{\overline{V}\left(\{i, W\left(q\right), y_l\}\right)}{1+r}\right) = \frac{k}{qf\left(q\right)\alpha_j}\left(\frac{\eta'\left(q\right)}{\left(\eta\left(q\right)\right)^2} + \frac{1}{q}\left(\frac{1-\eta\left(q\right)}{\eta\left(q\right)}\right)\right),$$

which is positive *on* $[q_a, q_b]$. A sufficient condition for it to be strictly decreasing on $[q_a, q_b]$ is that $\eta'\left(q\right)/\left(\eta\left(q\right)\right)^2$ is a decreasing function, which follows from the concavity of η. Hence $\overline{V}\left(\{i, W\left(q\right), y_l\}\right)$ is strictly concave *on* $[q_a, q_b]$, as required.

Next, differentiating equation (20) with respect to w and q one can verify that

$$\frac{\partial W\left(q\right)}{\partial q} = \left(r+\delta+\left(1-\delta\right)\alpha_j f\left(q\right)\right)\frac{\partial}{\partial q}\left(\frac{\overline{V}\left(\{i, W\left(q\right), y_l\}\right)}{1+r}\right),$$

which is positive and strictly decreasing on $[q_a, q_b]$, because both f and $\partial\overline{V}/\partial q$ are positive and strictly decreasing on $[q_a, q_b]$. Hence, $W\left(q\right)$ is strictly increasing and concave *on* $[q_a, q_b]$, as required. **QED**

Let $M\left(s\right)$ denote the match surplus as a function of the worker's state and note that

$$\frac{M\left(\{i, w, y_h\}\right)}{1+r} = \frac{\overline{V}\left(\{i, w, y_h\}\right)}{1+r} - \frac{\overline{V}\left(s_u\right)}{1+r} + \frac{y_h - w}{r+\delta} \tag{24}$$

and

$$\frac{M\left(\{i, W\left(q\right), y_l\}\right)}{1+r} = \frac{\overline{V}\left(\{i, W\left(q\right), y_l\}\right)}{1+r} - \frac{\overline{V}\left(s_u\right)}{1+r} + \frac{y_l - W\left(q\right)}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q\right)}. \tag{25}$$

Lemma 3 *$M\left(\{i, w, y_h\}\right)$ is independent of w; $M\left(\{i, W\left(q\right), y_l\}\right)$ is a strictly concave function of q on $[q_a, q_b]$ and it is maximized at $q = q_b$; $M\left(\{i, W\left(q\right), y_l\}\right) - \overline{V}\left(\{i, W\left(q\right), y_l\}\right)$ is a strictly decreasing and convex function of q on $[q_a, q_b]$.*

Proof: Fix $\overline{V}\left(s_u\right)$. Noting that

$$\frac{\overline{V}\left(\{i, w, y_h\}\right)}{1+r} = \frac{w}{r+\delta} + \frac{\delta}{r+\delta}\frac{\overline{V}\left(s_u\right)}{1+r}$$

one can write

$$\frac{M\left(\{i, w, y_h\}\right)}{1+r} = \frac{y_h}{r+\delta} - \frac{r\overline{V}\left(s_u\right)}{1+r},$$

which is independent of q. Using (23), together with (21) and (25), one can write

$$\begin{aligned}\frac{M\left(\{i, W\left(q\right), y_l\}\right)}{1+r} \quad &= \quad \frac{y_h}{r+\delta} - \frac{r\overline{V}\left(s_u\right)}{1+r} - \frac{r+\delta}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q\right)}\left(\frac{y_h - y_l}{r+\delta}\right) \\ &\quad - \left(1 - \frac{r+\delta}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q\right)}\right)\left(\frac{k}{qf\left(q\right)\alpha_j}\right).\end{aligned} \tag{26}$$

where $M(\{i,w,y_h\}) > M(\{i,W(q),y_l\})$ whenever $y_h > y_l$. Differentiating equation (26) one can verify that

$$\begin{aligned}\frac{\partial}{\partial q}\left(\frac{M(s)}{1+r}\right) &= \left(\frac{1-\delta}{q^2\left[r+\delta+(1-\delta)\alpha_j f(q)\right]}\right) \\ &\quad\times\left((1-\eta(q))k-\left(\frac{(r+\delta)\eta(q)}{r+\delta+(1-\delta)\alpha_j f(q)}\right)\left(qf(q)\alpha_j\left(\frac{y_h-y_l}{r+\delta}\right)-k\right)\right).\end{aligned}$$

for $s=\{i,W(q),y_l\}$. The term in the first line is decreasing in q since both $qf(q)$ are strictly increasing on $[q_a,q_b]$. The terms in the second line are also decreasing in q since $f(q)$ is decreasing and $\eta(q)$ and $qf(q)$ are increasing on $[q_a,q_b]$, and $qf(q)\alpha_j(y_h-y_l)\geq(r+\delta)k$ for $q\geq q_a$. Hence $M(\{i,W(q),y_l\})$ is strictly concave on $[q_a,q_b]$. It is now easy to verify that equation (19) is a necessary and sufficient condition for $\partial M(\{i,W(q),y_l\})/\partial q=0$. Hence $M(\{i,W(q),y_l\})$ is maximized at $q=q_b$.

Using equations (23) and (26) one can write

$$\begin{aligned}\frac{M(s)-\left(\overline{V}(s)-\overline{V}(s_u)\right)}{1+r} &= \left(\frac{k}{qf(q)\alpha_j}\right)\left(\frac{r+\delta}{r+\delta+(1-\delta)\alpha_j f(q)}+\frac{1-\eta(q)}{\eta(q)}\right) \\ &\quad-\left(\frac{y_h-y_l}{r+\delta+(1-\delta)\alpha_j f(q)}\right),\end{aligned}$$

for $s=\{i,W(q),y_l\}$, and differentiating this equation one can verify that

$$\begin{aligned}\frac{\partial}{\partial q}\left(\frac{M(s)-\left(\overline{V}(s)-\overline{V}(s_u)\right)}{1+r}\right) &= \left(\frac{(1-\delta)\alpha_1 f'(q)}{\left[r+\delta+(1-\delta)\alpha_j f(q)\right]^2}\right)\left(y_h-y_l-\frac{(r+\delta)k}{qf(q)\alpha_j}\right) \\ &\quad-\left(\frac{k}{qf(q)\alpha_j}\right)\left(\left(\frac{1-\eta(q)}{q}\right)\left(\frac{r+\delta}{r+\delta+(1-\delta)\alpha_j f(q)}+\frac{1-\eta(q)}{\eta(q)}\right)+\frac{\eta'(q)}{(\eta(q))^2}\right),\end{aligned}$$

for $s=\{i,W(q),y_l\}$. The term in the first line of the right side is negative since $f'(q)<0$ and $qf(q)\alpha_j(y_h-y_l)\geq(r+\delta)k$ for $q\geq q_a$. The term subtracted in the second line is positive since $\eta(q)<1$ and $\eta'(q)>0$. Hence $M(s)-\left(\overline{V}(s)-\overline{V}(s_u)\right)$, for $s=\{i,W(q),y_l\}$, is a strictly decreasing function of q on $[q_a,q_b]$. Moreover, the term in the first line of the right side is an increasing function of q, because $f'(q)$ and $qf(q)$ are increasing and $f(q)$ is decreasing. The term subtracted in the second line is a decreasing function of q, since $qf(q)$ and $\eta(q)$ are increasing and $f(q)$ and $\eta'(q)/(\eta(q))^2$ are decreasing. Hence, $M(s)-\left(\overline{V}(s)-\overline{V}(s_u)\right)$, for $s=\{i,W(q),y_l\}$, is a strictly convex function of q. **QED**

Next, note that Problem (P2) can be formulated as

$$\overline{V}(s_u)=b+V_i, \tag{P3}$$

where

$$V_i = \frac{\overline{V}(s_u)}{1+r} + \max_{w,q,q'} \left\{ f(q) \left(\frac{V_0(i,w,q')}{1+r} - \frac{\overline{V}(s_u)}{1+r} \right) \right\}$$

subject to

$$k \leq qf(q) \left(\frac{M_0(i,w,q')}{1+r} - \frac{V_0(i,w,q')}{1+r} + \frac{\overline{V}(s_u)}{1+r} \right),$$
$$q' \in [q_a, q_b], \; w \leq y_h, w \neq W\left(q'\right)$$

where

$$V_0\left(i,w,q'\right) = \alpha_i \overline{V}(\{i,w,y_h\}) + (1-\alpha_i)\overline{V}\left(\left\{i, W\left(q'\right), y_l\right\}\right),$$

and

$$M_0\left(i,w,q'\right) = \alpha_i M(\{i,w,y_h\}) + (1-\alpha_i) M\left(\left\{i, W\left(q'\right), y_l\right\}\right).$$

With a slight abuse of notation, we let $\left\{w_u^h(i), q_u(i), q_e^l(i)\right\}$ denote a solution to Problem (P3) while disregarding the constraint $w \neq W(q')$. Even though the objective is not concave in $\{w,q,q'\}$, we prove below that the solution is unique (and it is such that $w_u^h(i) \neq W\left(q_e^l(i)\right)$). It is then easy to see that $\left\{w_u^h(i), W\left(q_e^l(i)\right), q_u(i)\right\}$ solves problem (P2), since $q_e^l(i) = q_e\left(\left\{i, W\left(q_e^l(i)\right), y_l\right\}\right)$.

One can readily verify that an *interior* solution to Problem (P3) is such that the total surplus of the match is maximized. Specifically, it must be that $\partial M_0(i,w,q')/\partial q' = 0$, which requires that $\partial M(\{i, W(q'), y_l\})/\partial q' - 0$. Hence, Lemma 3 implies that $q_e^l(i) = q_b$, where q_b is given by equation (19). Comparing (19) and (20), it follows that $W\left(q_e^l(i)\right) = y_l$. Hence, $w_u^l(i,1) = y_l$ as indicated in (6).

Next, note that

$$\begin{aligned} V_i &= (1 - f(q_u(i))) \frac{\overline{V}(s_u)}{1+r} + f(q_u(i)) \frac{V_0\left(i, w_u^h(i), q_e^l(i)\right)}{1+r} \\ &= (1 - f(q_u(i))) \frac{\overline{V}(s_u)}{1+r} + f(q_u(i)) \left(\frac{\overline{V}(s_u)}{1+r} + \left(\frac{1 - \eta(q_u(i))}{\eta(q_u(i))} \right) \frac{k}{q_u(i) f(q_u(i))} \right), \end{aligned}$$

where the first equality comes from the Bellman equation in Problem (P3) and the second equality follows from the matching-efficiency condition (10) and the zero-profit condition (9). It follows that

$$V_i = \frac{\overline{V}(s_u)}{1+r} + \left(\frac{1 - \eta(q_u(i))}{\eta(q_u(i))} \right) \frac{k}{q_u(i)},$$

which, together with the fact that $\overline{V}(s_u) - b = V_i$, implies that

$$\frac{r\overline{V}(s_u)}{1+r} = b + \left(\frac{1-\eta(q_u(i))}{\eta(q_u(i))}\right)\frac{k}{q_u(i)}.$$

Using this equation, together with equations (9) and (10) and the fact that

$$\begin{aligned}\frac{V_0\left(i, w_u^h(i), q_e^l(i)\right)}{1+r} - \frac{\overline{V}(s_u)}{1+r} &= \alpha_i \frac{w_u^h(i)}{r+\delta} + (1-\alpha_i)\left(\frac{y_h}{r+\delta} - \frac{k}{\eta\left(q_e^l(i)\right) q_e^l(i) f\left(q_e^l(i)\right)\alpha_j}\right) \\ &\quad - \left(\frac{r}{r+\delta}\right)\frac{\overline{V}(s_u)}{1+r},\end{aligned}$$

it follows that $q_u(i)$ satisfies equation (11) in the text.

The right side of (11) is strictly decreasing in $q_u(i)$, it converges to ∞ as $q_u(i)$ approaches 0 and it converges to k as $q_u(i)$ approaches ∞. Hence, there is a unique solution $q_u(i) \in (0,\infty)$ that solves the equation if and only if

$$\frac{y_h - b}{r+\delta} - \frac{(1-\alpha_i)k}{\eta(q_b) q_b f(q_b)\alpha_j} > k.$$

There is a number $k_a > 0$ such that this inequality holds for all $k \in (0, k_a)$. To prove this, differentiate (19) to verify that

$$\partial q_b / \partial k > 0 \text{ and } \frac{\partial}{\partial k}\left(\frac{k}{\eta(q_b) q_b f(q_b)}\right) > 0,$$

with

$$\lim_{k\to 0} q_b = 0 \text{ and } \lim_{k\to 0}\left\{\frac{k}{\eta(q_b) q_b f(q_b)\alpha_j}\right\} = \frac{y_h - y_l}{r+\delta+(1-\delta)\alpha_j} < \frac{y_h - y_l}{r+\delta} < \frac{y_h - b}{r+\delta}.$$

Next, we verify that $\overline{V}(s_u) \leq \min\left\{\overline{V}\left(\left\{i, w_u^h(i), y_h\right\}\right), \overline{V}(\{i, y_l, y_l\})\right\}$. To that end, note that

$$\begin{aligned}\overline{V}(\{i, y_l, y_l\}) - \overline{V}(s_u) &= V_0\left(i, w_u^h(i), q_e^l(i)\right) - \overline{V}(s_u) \\ &\quad - \alpha_i\left[\overline{V}\left(\left\{i, w_u^h(i), y_h\right\}\right) - \overline{V}(\{i, y_l, y_l\})\right]\end{aligned}$$

and

$$\begin{aligned}\overline{V}\left(\left\{i, w_u^h(i), y_h\right\}\right) - \overline{V}(s_u) &= V_0\left(j, w_u^h(i), q_e^l(i)\right) - \overline{V}(s_u) \\ &\quad + (1-\alpha_i)\left[\overline{V}\left(\left\{i, w_u^h(i), y_h\right\}\right) - \overline{V}(\{i, y_l, y_l\})\right],\end{aligned}$$

where

$$V_0\left(i, w_u^h(i), q_e^l(i)\right) = \alpha_i \overline{V}\left(\left\{i, w_u^h(i), y_h\right\}\right) + (1-\alpha_j)\overline{V}\left(\left\{i, w_l\left(j, q_e^l(i)\right), y_l\right\}\right),$$

and use the fact that

$$\frac{V_0\left(i, w_u^h\left(i\right), q_e^l\left(i\right)\right)}{1+r} - \frac{\overline{V}\left(s_u\right)}{1+r} = \left(\frac{1-\eta\left(q_u\left(i\right)\right)}{\eta\left(q_u\left(i\right)\right)}\right)\frac{k}{q_u\left(i\right) f\left(q_u\left(i\right)\right)}$$

and the fact that

$$\frac{\overline{V}\left(\left\{i, w_u^h\left(i\right), y_h\right\}\right)}{1+r} - \frac{\overline{V}\left(\left\{i, y_l, y_l\right\}\right)}{1+r} = \frac{k}{\eta\left(q_b\right) q_b f\left(q_b\right)\alpha_j} - \left(\frac{k}{\alpha_i q_u\left(i\right) f\left(q_u\left(i\right)\right)}\right)$$

to write

$$\frac{\overline{V}\left(\left\{i, y_l, y_l\right\}\right) - \overline{V}\left(s_u\right)}{1+r} = \frac{k}{\eta\left(q_u\left(i\right)\right) q_u\left(i\right) f\left(q_u\left(i\right)\right)} - \alpha_i\left(\frac{k}{\eta\left(q_b\right) q_b f\left(q_b\right)\alpha_j}\right), \tag{27}$$

where

$$\frac{\overline{V}\left(\left\{i, w_u^h\left(i\right), y_h\right\}\right) - \overline{V}\left(s_u\right)}{1+r} = \left(\frac{1}{\eta\left(q_u\left(i\right)\right)} - \frac{1}{\alpha_i}\right)\frac{k}{q_u\left(i\right) f\left(q_u\left(i\right)\right)} + \left(1-\alpha_i\right)\left(\frac{k}{\eta\left(q_b\right) q_b f\left(q_b\right)\alpha_j}\right).$$

Differentiating equation (11), one can verify that $\partial q_u\left(i\right)/\partial k > 0$, with

$$\lim_{k\to 0} q_u\left(i\right) = \lim_{k\to 0}\left\{\frac{k}{q_u\left(i\right) f\left(q_u\left(i\right)\right)}\right\} = 0$$

and

$$\lim_{k\to 0}\left\{\frac{k}{\eta\left(q_u\left(i\right)\right) q_u\left(i\right) f\left(q_u\left(i\right)\right)}\right\} = \left(\frac{r+\delta}{1+r+\delta}\right)\left(\frac{y_h-b}{r+\delta} - \left(1-\alpha_j\right)\frac{y_h-y_l}{r+\delta+\left(1-\delta\right)\alpha_j}\right).$$

It follows that there is a number $k_b > 0$ such that $\overline{V}\left(\left\{i, w_u^h\left(i\right), y_h\right\}\right) > \overline{V}\left(s_u\right)$ for all $k \in (0, k_b)$. Moreover,

$$\lim_{k\to 0}\left\{\frac{\overline{V}\left(\left\{i, y_l, y_l\right\}\right) - \overline{V}\left(s_u\right)}{1+r}\right\} = \frac{y_h-b}{1+r+\delta} - \left(\frac{\alpha_j+r+\delta}{1+r+\delta}\right)\left(\frac{y_h-y_l}{r+\delta+\left(1-\delta\right)\alpha_j}\right).$$

This limit is positive if and only if $\left(y_h-b\right)/\left(y_h-y_l\right) \geq \left(r+\delta+\alpha_j\right)/\left(r+\delta+\left(1-\delta\right)\alpha_j\right)$, which is ensured by the assumption in Proposition 2. It follows that there is a number $k_c > 0$ such that $\overline{V}\left(\left\{i, y_l, y_l\right\}\right) > \overline{V}\left(s_u\right)$ for all $k \in (0, k_c)$.

Furthermore, note that

$$\lim_{k\to 0}\left\{\frac{\overline{V}\left(\left\{i, w_u^h\left(i\right), y_h\right\}\right)}{1+r} - \frac{\overline{V}\left(\left\{i, y_l, y_l\right\}\right)}{1+r}\right\} = \frac{y_h-y_l}{r+\delta+\left(1-\delta\right)\alpha_j} > 0,$$

which implies that $\lim_{k\to 0} w_u^h\left(i\right) > y_l$.

The above arguments together imply that there is a number $k_0 > 0$ such that $k \in (0, k_0)$ is sufficient for $q_u\left(i\right) \in (0, \infty)$ and $\overline{V}\left(s_u\right) \leq \min\left\{\overline{V}\left(\left\{i, w_u^h\left(i\right), y_h\right\}\right), \overline{V}\left(\left\{i, y_l, y_l\right\}\right)\right\}$, and $w_u^h\left(i\right) >$

y_l. Since $w_u^h(i) \neq y_l$, equilibrium wages reveal the current productivity of employed workers, as required.

It is straightforward to characterize ψ. The unemployment rate is given by

$$\psi(s_u) = \frac{\delta}{\delta + f(q_u^*)},$$

where $q_u^* \equiv q_u(i,1)$. The wage distribution has three mass points: two wages for workers who find jobs out of unemployment — a wage w_h^* for those who are well matched and a wage w_l^* for those who are poorly matched — and a wage w_e^* for those workers who find jobs via search on the job. The mass of workers earning the wage w_l^* is

$$\psi(\{i, w_l^*, y_l\}) = \left(\frac{(1-\alpha_i) f(q_u^*)}{\delta + (1-\delta)\alpha_j f(q_e^*)}\right)\psi(s_u),$$

where $w_l^* \equiv w_u^l(i,1) = y_l$ and $q_e^* \equiv q_e(\{i, w_l^*, y_l\}, 1)$. The mass of workers earning the wage $w_h^* \equiv w_u^h(i,1)$ is

$$\psi(\{i, w_h^*, y_h\}) = \left(\frac{\alpha_i f(q_u^*)}{\delta}\right)\psi(s_u)$$

and the mass of workers earning the wage $w_e^* \equiv w_e(\{i, w_l^*, y_l\}, 1)$ is

$$\psi(\{j, w_e^*, y_h\}) = \left(\frac{(1-\delta)\alpha_j f(q_e^*)}{\delta}\right)\psi(\{i, w_l^*, y_l\}).$$

One can verify that $f(q_u^*)$ is an increasing function of y_l, y_h and α_1, and $f(q_e^*)$ is an increasing function of $(y_h - y_l)$ and α_1 in the revealing equilibrium.

It is straightforward to verify that there are functions Q and μ that support the allocation characterized by equations (6)-(9) and (11). If $s \in S^*$, beliefs must be correct and the construction of Q is standard. If $s \notin S^*$, beliefs are arbitrary and we may assume that employers believe that $\mu(s\,|x) = 0$ for all $s \notin S^*$.

Lemma 4 *The revealing equilibrium allocation maximizes the present value of aggregate production net of search costs.*

Proof: First, note that the state of the economy at the beginning of each period can be summarized by $\{u, m\}$, where $u \in [0,1]$ is the measure of unemployed workers, and $m : \{y_l, y_h\} \to [0,1]$, where $m(y)$ denotes the measure of employed workers with match productivity y. Let $p(y)$ denote the probability with which a match has productivity realization y. Let $x_u(y)$ denote the probability with which a meeting between an unemployed worker and a job is turned into a match given the productivity realization y, and $x_e(y'|y)$ denote the probability with which a meeting between a worker and a job with productivity realization y' is turned into a match given that the worker is currently employed in a job with match productivity y. Finally, let q_u denote the labor market queue where unemployed workers search for jobs, and $q_e(y)$ denote the labor market queue where employed workers search given that they are currently employed in jobs with productivity y.

Aggregate output can be written as:

$$Y(u,m) = bu + \sum_y ym(y) - k\frac{u}{q_u} - (1-\delta)k\sum_y \frac{m(y)}{q_e(y)}. \tag{28}$$

Denote by $\hat{u}$ the measure of unemployed workers one period ahead, and by $\hat{m}(y)$ the measure of employed workers with match productivity y one period ahead. Then,

$$\hat{u} = \left(1 - \sum_y f(q_u)x_u(y)\right)u + \delta\sum_y m(y) \tag{29}$$

and

$$\begin{aligned} \hat{m}(y) &= p(y)f(q_u)x_u(y)u + (1-\delta)m(y)\left[1 - p(y)f(q_e(y))x_e(y'|y)\right] \\ &+ (1-\delta)\sum_{y'} m(y')p(y)f(q_e(y'))x_e(y|y'). \end{aligned} \tag{30}$$

The allocation that maximizes aggregate output net of search costs can be characterized as the solution to the planning problem:

$$J(u,m) = \max_{q_u,x_u,q_e,x_e}\left\{Y(u,m) + \frac{J(\hat{u},\hat{m})}{1+r}\right\}, \tag{31}$$

subject to equations (28)-(30). $J(u,m)$ is the unique solution to the planner's problem and can be written as:

$$J(u,m) = J_u u + \sum_y m(y)J_e(y),$$

where

$$J_u = \max_{q_u,x_u} b - \frac{k}{q_u} + \sum_y p(y)f(q_u)x_u(y)\frac{J_e(y)}{1+r} + \left(1 - \sum_y p(y)f(q_u)x_u(y)\right)\frac{J_u}{1+r} \tag{32}$$

and

$$\begin{aligned} J_e(y) &= \max_{x_e,q_e} y - (1-\delta)\frac{k}{q_e(y)} + \delta\frac{J_u}{1+r} + (1-\delta)\left[1 - \sum_{y'} p(y')f(q_e(y))x_e(y'|y)\right]\frac{J_e(y)}{1+r} \\ &+ (1-\delta)\sum_{y'} p(y')f(q_e(y))x_e(y'|y)\frac{J_e(y')}{1+r}. \end{aligned} \tag{33}$$

It is easy to verify that at the optimum $q_e(y_h) = \infty$. This implies:

$$J_e(y_h) = y_h + \delta \frac{J_u}{1+r} + (1-\delta)\frac{J_e(y_h)}{1+r} > J_e(y_l). \tag{34}$$

It is also easy to verify that $x_e(y_h|y_l) = 1$ and $x_e(y_l|y_l) \in [0,1]$ at the optimum. This means that the planner's problem has multiple solutions, all of which yield the same optimal value. The multiplicity concerns the probability with which the planner instructs workers to accept or reject lateral job moves. We characterize the solution when $x_e(y_l|y_l) = 0$.

The necessary condition of (33) with respect to $q_e(y_l)$ can be written:

$$\left[\frac{J_e(y_h)}{1+r} - \frac{J_e(y_l)}{1+r}\right] = \frac{k}{\alpha_1 q_e(y_l) f(q_e(y_l))\eta(q_e(y_l))} \tag{35}$$

and the Bellman equation for $J_e(y_l)$ gives:

$$\frac{J_e(y_l)}{1+r} = \frac{1}{r+\delta+(1-\delta)f(q_e(y_l))\alpha_1}\left(y_l - y_h - (1-\delta)\frac{k}{q_e(y_l)}\right) + \frac{J_e(y_h)}{1+r}.$$

The above equations, along with the expression for $J_e(y_h)$, yield equation (19), which defines the equilibrium value of $q_e(y_l)$.

Conjecture that $x_u(y) = 1$ for $y = \{y_l, y_h\}$. The necessary condition of (32) with respect to q_u can be written:

$$\frac{r}{r+\delta}\frac{J_u}{1+r} = \frac{y_h}{r+\delta} - \frac{k}{q_u f(q_u)\eta(q_u)} - \frac{(1-\alpha_1)k}{\eta(q_b)q_b f(q_b)\alpha_1}. \tag{36}$$

From the Bellman equation for J_u:

$$\frac{rJ_u}{1+r} = b + \frac{k}{q_u}\left(\frac{1-\eta(q_u)}{\eta(q_u)}\right). \tag{37}$$

Combining these two equations yields equation (11) from the text, where $i = j = 1$, and $q_e(s,1) = q_b$. This defines the equilibrium value of q_u.

To show that $x_u(y) = 1$ for $y = \{y_l, y_h\}$, combine equations (35) and (36) to obtain:

$$\frac{J_e(y_l)}{1+r} - \frac{J_u}{1+r} = \frac{k}{q_u f(q_u)\eta(q_u)} - \alpha_1 \frac{k}{q_b f(q_b)\eta(q_b)\alpha_1}.$$

The right side is identical to the right side of equation (27) and is therefore positive under the same conditions. Since $J_e(y_h) > J(y_l)$, if follows that when all low productivity matches are accepted, all high productivity matches are accepted. **QED**

This concludes the proof of Proposition 2. **QED**

Proof of Proposition 3

This proof parallels that of the first part of Proposition 2. Throughout the proof we maintain the assumption that $\rho = 1 - \alpha_i$ and we drop the argument ρ from all functions. As before, we keep track of job types under the assumption that unemployed workers search for type-i jobs and employed workers search for type-j jobs, where it is understood that $i = j = 1$ throughout this proof. We first prove existence and uniqueness of the candidate equilibrium allocation. Then we show how it can be supported by a non-revealing equilibrium.

The first-order conditions for an interior solution of Problem (P1) are given in the main text. We now have that $q_e(s) = q_e(\{i, w, y_l\}) = q_e(\{i, w, y_h\})$ and it is easy to verify that $q_e(s) \in [\widehat{q}_a, \widehat{q}_b]$, where $w_e(s) \geq y_l$ if and only if $q_e(s) \geq \widehat{q}_a$ and $\widetilde{W}(q_e(s)) \leq y_l$ if and only if $q_e(s) \leq \widehat{q}_b$ and where $\widehat{q}_a$ and $\widehat{q}_b$ are given by

$$\widehat{q}_a f(\widehat{q}_a)(1-\alpha_i)\alpha_j \left(\frac{y_h - y_l}{r+\delta}\right) = k \tag{38}$$

and

$$\frac{y_h - y_l}{r+\delta} = \left(\frac{k}{\widehat{q}_b f(\widehat{q}_b)(1-\alpha_i)\alpha_j}\right)\left(1 + \left(\frac{1-\eta(\widehat{q}_b)}{\eta(\widehat{q}_b)}\right)\left(\frac{r+\delta+(1-\delta)\alpha_j f(\widehat{q}_b)}{r+\delta}\right)\right) \tag{39}$$

respectively. Clearly, $\infty > \widehat{q}_b > \widehat{q}_a > 0$.

Proceeding as before, Problem (P2) can be formulated in the present case as

$$\overline{V}(s_u) = b + V_i, \tag{$P4$}$$

where

$$V_i = \frac{\overline{V}(s_u)}{1+r} + \max_{q,q'} \left\{ f(q)\left(\frac{\widetilde{V}_0(i,q')}{1+r} - \frac{\overline{V}(s_u)}{1+r}\right)\right\}$$

subject to

$$k \leq q f(q)\left(\frac{\widetilde{M}_0(i,q')}{1+r} - \frac{\widetilde{V}_0(i,q')}{1+r} + \frac{\overline{V}(s_u)}{1+r}\right),$$

$$q' \in [q_a, q_b], \; w \leq y_h,$$

where $\widetilde{V}_0(i,q')$ and $\widetilde{M}_0(i,q')$ are defined in the main text. Let $\left\{q_u(i), q_e^l(i)\right\}$ denote a solution to Problem $(P4)$.

Noting that

$$\frac{\widetilde{V}_0(i,q')}{1+r} - \frac{\overline{V}(s_u)}{1+r} = \frac{y_h}{r+\delta} - \left(\frac{r}{r+\delta}\right)\frac{\overline{V}(s_u)}{1+r} - \frac{k}{\eta(q')q'f(q')(1-\alpha_i)\alpha_j},$$

and using (12)-(13) and the definition of $\widetilde{M}\left(\left\{i,\widetilde{W}\left(q'\right),y_l\right\}\right)$ given in the main text, one can verify that

$$\begin{aligned}
\frac{\widetilde{M}_0\left(i,q'\right)}{1+r} &= \frac{y_h}{r+\delta}-\left(\frac{r}{r+\delta}\right)\frac{\overline{V}\left(s_u\right)}{1+r} \\
&\quad -\left(1-\alpha_i\right)\left(\frac{r+\delta}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q'\right)}\right)\left(\frac{y_h-y_l}{r+\delta}\right) \\
&\quad -\left(1-\alpha_i\right)\left(1-\frac{r+\delta}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q'\right)}\right)\left(\frac{k}{q'f\left(q'\right)\left(1-\alpha_i\right)\alpha_j}\right).
\end{aligned}$$

Lemma 5 *(i) $\widetilde{W}\left(q\right)$ and $\widetilde{V}_0\left(i,q\right)$ are strictly increasing and concave functions of q on $[\widehat{q}_a,\widehat{q}_b]$. (ii) $\widetilde{M}_0\left(i,q\right)$ is a strictly concave function of q on $[\widehat{q}_a,\widehat{q}_b]\subset(0,\infty)$ and it is maximized at $q=\widehat{q}_b$; $\widetilde{M}_0\left(i,q\right)-\widetilde{V}_0\left(i,q\right)$ is a strictly decreasing and convex function of q on $[\widehat{q}_a,\widehat{q}_b]$.*

Proof: It replicates the arguments in Proposition 2 with minor changes. **QED**

The first-order conditions for an interior solution of Problem ($P4$) are given by equations (14)-(16) in the main text.

Following similar steps as in the proof of Proposition 2 one can verify that an interior solution to Problem ($P4$) satisfies

$$\frac{y_h-b}{r+\delta}-\frac{k}{\eta\left(q'\right)q'f\left(q'\right)\left(1-\alpha_i\right)\alpha_j}=\lambda_i q\left(\frac{1-\eta\left(q\right)}{\eta\left(q\right)}\right)\frac{k}{q}\left(\frac{1}{f\left(q\right)}+\frac{1}{r+\delta}\right), \tag{40}$$

where λ_i is given by (15), and

$$\begin{aligned}
\frac{k}{qf\left(q\right)} &= -\left(1-\alpha_i\right)\left(\frac{r+\delta}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q'\right)}\right)\left(\frac{y_h-y_l}{r+\delta}\right) \\
&+\left(\frac{k}{q'f\left(q'\right)\left(1-\alpha_i\right)\alpha_j}\right)\left(\frac{\left(1-\alpha_i\right)\left(r+\delta\right)}{r+\delta+\left(1-\delta\right)\alpha_j f\left(q'\right)}+\frac{1-\eta\left(q'\right)}{\eta\left(q'\right)}+\alpha_i\right).
\end{aligned} \tag{41}$$

Lemma 6 *Assume that $\left(r+\delta\right)k<\left(1-\alpha_1\right)\alpha_1\left(y_h-y_l\right)$. Equations (15), (40) and (41) have a unique solution $\left(\lambda_i,q,q'\right)$, with $q\in(0,\infty)$, $q'\in(q_c,q_d)$, and $\lambda_i q\geq 1$, where*

$$\frac{y_h-b}{r+\delta}-\frac{k}{\eta\left(q_c\right)q_c f\left(q_c\right)\left(1-\alpha_i\right)\alpha_j}=0,$$

$$\frac{\widetilde{M}_0\left(i,q_d\right)}{1+r}-\frac{\widetilde{V}_0\left(i,q_d\right)}{1+r}+\frac{\overline{V}\left(s_u\right)}{1+r}=k$$

and where $q_c<\widehat{q}_b<q_d$.

Proof: Differentiating equation (15) one can verify that the following inequality is necessary and sufficient for $\partial\lambda_i q/\partial q'<0$:

$$\frac{-\partial^2\widetilde{M}_0/\partial q'^2}{-\partial^2\widetilde{V}_0/\partial q'^2}>\frac{\partial\widetilde{M}_0/\partial q'}{\partial\widetilde{V}_0/\partial q'}.$$

The left side of the inequality is greater than one, since $\widetilde{M}_0 - \widetilde{V}_0$ *is a strictly convex function of* q'. The right side is smaller than one, since $\widetilde{M}_0 - \widetilde{V}_0$ *is a strictly decreasing function of* q'. Hence, $\partial \lambda_i q / \partial q' < 0$. Moreover, note that $\lambda_i q \geq 1$ if and only if $\partial \widetilde{M}_0 / \partial q' \geq 0$. Accordingly, (40) characterizes q as a strictly decreasing function of q', where the right side converges to 0 as q approaches ∞ and it converges to ∞ as q approaches 0. Thus, $\infty > q > 0$ if and only if $q' > q_c$. Similarly, (41) characterizes q as a strictly increasing function of q', where the left side converges to ∞ as q approaches 0 and it converges to k as q approaches ∞. Thus, $\infty > q > 0$ if and only if $q' < q_d$. Together, (40)-(41) imply that $q' \in (q_c, q_d)$ and, hence, $\infty > q > 0$.

To verify that $\widehat{q}_b < q_d$, write (39) as

$$\begin{aligned}\frac{\alpha_i k}{\eta(\widehat{q}_b)\widehat{q}_b f(\widehat{q}_b)(1-\alpha_i)\alpha_j} &= -(1-\alpha_j)\left(\frac{r+\delta}{r+\delta+(1-\delta)\alpha_j f(\widehat{q}_b)}\right)\left(\frac{y_h - y_l}{r+\delta}\right) \\ &+ \left(\frac{k}{\widehat{q}_b f(\widehat{q}_b)(1-\alpha_i)\alpha_j}\right)\left(\frac{(1-\alpha_i)(r+\delta)}{r+\delta+(1-\delta)\alpha_j f(\widehat{q}_b)} + \frac{1-\eta(\widehat{q}_b)}{\eta(\widehat{q}_b)} + \alpha_i\right).\end{aligned}$$

Comparing this with (41), it follows that $\widehat{q}_b < q_d$ if and only if

$$\frac{\alpha_i}{\eta(\widehat{q}_b)\widehat{q}_b f(\widehat{q}_b)(1-\alpha_i)\alpha_j} > 1.$$

A sufficient condition for this is $\alpha_i = \alpha_j$, which is the case here. Hence, $\widehat{q}_b < q_d$.

To verify that $\widehat{q}_b > q_c$, note that (39) implies that

$$\frac{k}{\eta(\widehat{q}_b)\widehat{q}_b f(\widehat{q}_b)(1-\alpha_i)\alpha_j} < \frac{y_h - y_l}{r+\delta},$$

which, together with the fact that $y_l > b$, implies that $\widehat{q}_b > q_c$. **QED**

If an interior non-revealing equilibrium exists, it is uniquely characterized by equations (12), (13), (15), (40) and (41). Recall that $w_e(s) \geq y_l$ if and only if $q_e(s) \geq \widehat{q}_a$ and $\widetilde{W}(q_e(s)) \leq y_l$ if and only if $q_e(s) \leq \widehat{q}_b$, but we know only that $q' \in (q_c, q_d)$. Hence, we need to verify that the candidate interior solution for q' is such that $q' \in [\widehat{q}_a, \widehat{q}_b]$.

There are only three possible solutions to problems (P1) and (P2) with $\rho = 1 - \alpha_i$. One is the interior allocation characterized above, provided that it is such that $q' \in [\widehat{q}_a, \widehat{q}_b]$. Another is the corner allocation that solves $q' \in \widehat{q}_b$, $\widetilde{W}(q') = y_l$, together with (12) and (41). In principle, a third possibility is the corner allocation such that $q' \in \widehat{q}_a$ and $w_e(s) = y_l$. However, it is easy to verify that there is a number $k_e > 0$ such that this case will never arise whenever $k \in (0, k_e)$, which is the relevant case below.

Therefore, in order to construct an equilibrium, consider the other two possible allocations and select the one that provides unemployed workers with the higher welfare. It is straightforward to characterize ψ. The unemployment rate is given by

$$\psi(s_u) = \frac{\delta}{\delta + f(q_u^*)},$$

where $q_u^* \equiv q_u(i, 1-\alpha_i)$. The wage distribution has two mass points: one wage w_u^* for workers who find jobs out of unemployment, and one wage w_e^* for workers who find jobs via on-the-job search. The mass of workers earning the wage w_u^* is

$$\psi\left(\{i, w_u^*, y_l\}\right) + \psi\left(\{i, w_u^*, y_h\}\right) = \left(\frac{f\left(q_u^*\right)}{\delta + (1-\delta)\,\alpha_j f\left(q_e^*\right)}\right)\psi\left(s_u\right),$$

where $w_u^* \equiv w_u^l(i, 1-\alpha_i) = w_u^h(i, 1-\alpha_i)$ and $q_e^* \equiv q_e\left(\{i, w_u^*, y_l\}, 1-\alpha_i\right) = q_e\left(\{i, w_u^*, y_h\}, 1-\alpha_i\right)$. The mass of workers earning the wage w_e^* is

$$\psi\left(\{i, w_e^*, y_h\}\right) + \psi\left(\{j, w_e^*, y_h\}\right) = \left(\frac{(1-\delta)\,\alpha_j f\left(q_e^*\right)}{\delta}\right)\left[\psi\left(\{i, w_u^*, y_l\}\right) + \psi\left(\{i, w_u^*, y_h\}\right)\right],$$

where $w_e^* \equiv w_e\left(\{i, w_u^*, y_h\}, 1-\alpha_i\right) = w_e\left(\{j, w_u^*, y_h\}, 1-\alpha_i\right)$.

Furthermore, one can verify that $f\left(q_u^*\right)$ is an increasing function of y_l, y_h and α_1, and $f\left(q_e^*\right)$ is an increasing function of $(y_h - y_l)$ and α_1.

It remains to prove that there are mappings Q and μ that support the candidate equilibrium allocation. The construction of Q is standard. If $s \in S^*$, beliefs must be correct. If $s \notin S^*$, beliefs are arbitrary and we may assume that employers believe that $\mu\left(s\,|x\right) = 0$ for all $s \notin S^*$. It only remains to prove that a type-i employer posting a contract offering revealing wages will not attract any unemployed workers while making non-negative profits. We show that there is a number $\widehat{k}_d > 0$ such that this is the case for all $k \in \left(0, \widehat{k}_d\right)$. To see why, it is sufficient to consider the case where potential poachers will never hire workers with $s \notin S^*$. In this case, one can verify that the value of searching for a revealing contract to an unemployed worker, denoted by V_i^d is continuous in k with

$$\lim_{k\to 0}\frac{V_i^d}{1+r} = \alpha_i\frac{y_h}{r+\delta} + (1-\alpha_i)\frac{y_l}{r+\delta} + \left(\frac{r}{r+\delta}\right)\frac{\overline{V}\left(s_u\right)}{1+r},$$

whereas the candidate equilibrium allocation has

$$\lim_{k\to 0}\frac{\widehat{V}_i}{1+r} = \left(1 - \frac{r+\delta}{r+\delta+(1-\delta)\,\alpha_j}\right)\frac{y_h}{r+\delta} + \left(\frac{r+\delta}{r+\delta+(1-\delta)\,\alpha_j}\right)\frac{y_l}{r+\delta} + \left(\frac{r}{r+\delta}\right)\frac{\overline{V}\left(s_u\right)}{1+r}.$$

Hence, $\lim\limits_{k\to 0}\widehat{V}_i > \lim\limits_{k\to 0}V_i^d$ if and only if $(1-\alpha_i)\,\alpha_j/\alpha_i > (r+\delta)\,/\,(1-\delta)$. Since $\alpha_i = \alpha_j$, all that is needed is $(1-\alpha_i)\,(1-\delta) > (r+\delta)$ as assumed in the proposition. **QED**

Proof of Proposition 4

Replicating the approach we followed in the proofs of propositions 2 and 3 for the two-job economy, one can verify that an interior equilibrium allocation where entry wages are revealing in type-2 matches, but not in type-1 matches, provided that it exists, can be constructed as follows.

Step 1. For a given value of $\widehat{c}$, find values of q_1, q_1', q_2 and q_2' such that q_2 solves

$$\begin{aligned}\frac{y_h - b}{r+\delta} - \frac{(1-\alpha_2) k}{\eta(q_b) q_b f(q_b) \alpha_1} &= \frac{F(\widehat{c})(\widehat{c} - \mathbb{E}(c | c \leq \widehat{c})) - \widehat{c}}{r+\delta} \\ &+ \frac{k}{\eta(q_2) q_2 f(q_2)} + \left(\frac{1-\eta(q_2)}{\eta(q_2)}\right) \frac{k}{(r+\delta) q_2}, \end{aligned} \tag{42}$$

with $q_2' = q_b$, and (q_1, q_1') solve

$$\begin{aligned}\frac{y_h - b}{r+\delta} - \frac{k}{\eta(q_1') q_1' f(q_1')(1-\alpha_1)\alpha_1} &= \frac{F(\widehat{c})(\widehat{c} - \mathbb{E}(c | c \leq \widehat{c}))}{r+\delta} \\ &+ \lambda_1 q_1 \left(\frac{1-\eta(q_1)}{\eta(q_1)}\right) \frac{k}{q_1} \left(\frac{1}{f(q_1)} + \frac{1}{r+\delta}\right), \end{aligned} \tag{43}$$

where λ_1 is given by the analogue of (15), for $i = 1$, and

$$\begin{aligned}\frac{k}{q_1 f(q_1)} &= -(1-\alpha_1)\left(\frac{r+\delta}{r+\delta+(1-\delta)\alpha_1 f(q_1')}\right)\left(\frac{y_h - y_l}{r+\delta}\right) \\ &+ \left(\frac{k}{q_1' f(q_1')(1-\alpha_1)\alpha_1}\right)\left(\frac{(1-\alpha_1)(r+\delta)}{r+\delta+(1-\delta)\alpha_1 f(q_1')} + \frac{1-\eta(q_1')}{\eta(q_1')} + \alpha_1\right). \end{aligned} \tag{44}$$

Step 2. Use those values of q_1, q_1', q_2 and q_2' to calculate the implied values of V_1 and V_2 as a function of $\widehat{c}$ and let

$$D(\widehat{c}) = V_2 - V_1. \tag{45}$$

Step 3. We are seeking to establish existence of a fixed point

$$D(c_0) = c_0 > 0. \tag{46}$$

Similarly, one can verify that a corner equilibrium allocation where entry wages are revealing in type-2 matches, but not in type-1 matches, with $\widetilde{W}(q_1') = y_l$, provided that it exists, can be constructed following the same steps, except that equation (43) is replaced with $q_1' = \widehat{q}_b$. As before, it is easy to verify that there is a number $k_f > 0$ such that these two cases exhaust all feasible cases whenever $k \in (0, k_f)$, which is the relevant case below.

Suppose that $\widehat{c} = 0$, in which case all arguments in propositions 2 and 3 hold. First, suppose the solution at $\widehat{c} = 0$ is interior. Clearly there is a number $\alpha_a \in (0, \alpha_1)$ such that $D(0) > 0$ for all $\alpha_2 \in (\alpha_a, \alpha_1)$. Now start increasing the value of $\widehat{c}$. Following the same arguments we used in the proof of Proposition 3, one can verify that there is a number $c_a \in (0, \infty)$ such that there is a solution to the equations above with $q' \in (\widetilde{q}_c(\widehat{c}), \widetilde{q}_d)$ where $\widetilde{q}_c(\widehat{c}) < \widehat{q}_b < \widetilde{q}_d$, for all $\widehat{c} \in [0, c_a)$. There are only two possibilities. If there exists an interior equilibrium, then there is some value $c_0 \in (0, c_a)$ such that $D(c_0) = c_0$.

Otherwise, it must be that $q_1' = \widehat{q}_b$. Replicating the arguments in the proof of Proposition 2, one can verify that, for given $\widehat{c}$, there is a number $\widehat{k}_g > 0$ such that there is a unique value of $q_2 \in (0, \infty)$ that solves equation (42), with $q_2' = q_b$, for all $k \in \left(0, \widehat{k}_g\right)$. Moreover, $q_2 > 0$, and thus, V_2 is bounded, for all $\widehat{c} \geq 0$ since $\lim_{\widehat{c} \to \infty} \left[\widehat{c} - F(\widehat{c})(\widehat{c} - \mathbb{E}(c \,|c \leq \widehat{c}))\right] = 1/\theta < \infty$. Similarly, the arguments in the proof of Proposition 3 imply that a non-revealing equilibrium allocation in the market for type-2 jobs can be supported for sufficiently small values of $k > 0$, provided that $(1 - \alpha_1)(1 - \delta) > (r + \delta)$, as assumed in the proposition. Since V_2 remains bounded as we increase $\widehat{c}$, there must exist a fixed point $D(c_0) = c_0$.

It is now straightforward to characterize ψ. The unemployment rate is given by

$$\psi(s_u) = \frac{\delta}{\delta + (1 - F(c_0)) f(q_{u1}^*) + F(c_0) f(q_{u2}^*)},$$

where $q_{ui}^* \equiv q_u(i, 1 - \alpha_1, 1)$, for $i = 1, 2$. The wage distribution across type-2 jobs consists of two mass points. The mass of workers earning the wage $w_l^* \equiv w_u^l(2, 1 - \alpha_1, 1) = y_l$ is

$$\psi(\{2, w_l^*, y_l\}) = \left(\frac{F(c_0)(1 - \alpha_2) f(q_{u2}^*)}{\delta + (1 - \delta)\alpha_1 f(q_{e2}^*)}\right) \psi(s_u),$$

where $q_{e2}^* \equiv q_e^l(2, 1 - \alpha_1, 1)$; the mass of workers earning the wage $w_h^* \equiv w_u^h(2, 1 - \alpha_1, 1)$ is

$$\psi(\{2, w_h^*, y_h\}) = \left(\frac{F(c_0)\alpha_2 f(q_{u2}^*)}{\delta}\right) \psi(s_u).$$

The wage distribution across type-1 jobs consists of three mass points. The mass of workers earning the wage $w_{e2}^* \equiv w_e(\{2, w_l^*, y_l\}, 1 - \alpha_1, 1)$ is

$$\psi(\{1, w_{e2}^*, y_h\}) = \left(\frac{(1 - \delta)\alpha_1 f(q_{e2}^*)}{\delta}\right) \psi(\{2, w_l^*, y_l\});$$

the mass of workers earning the wage $w_u^* \equiv w_u^l(1, 1 - \alpha_1, 1) = w_u^h(1, 1 - \alpha_1, 1)$ is

$$\psi(\{1, w_u^*, y_h\}) + \psi(\{1, w_u^*, y_l\}) = \left(\frac{(1 - F(c_0)) f(q_{u1}^*)}{\delta + (1 - \delta)\alpha_1 f(q_{e1}^*)}\right) \psi(s_u),$$

where $q_{e1}^* \equiv q_e^l(1, 1 - \alpha_1, 1)$ and $(1 - \alpha_1)\psi(\{1, w_u^*, y_h\}) = \alpha_1 \psi(\{1, w_u^*, y_l\})$; the mass of workers earning $w_{e1}^* \equiv w_e(\{1, w_u^*, y_l\}) = w_e(\{1, w_u^*, y_h\})$ is

$$\psi(\{1, w_{e1}^*, y_h\}) = \frac{(1 - \delta)\alpha_1 f(q_{e1}^*)}{\delta} \left(\frac{(1 - F(c_0)) f(q_{u1}^*)}{\delta + (1 - \delta)\alpha_1 f(q_{e1}^*)}\right) \psi(s_u).$$

This concludes the proof of Proposition 4. **QED**

Proof of Proposition 5

An equilibrium where both types of jobs are created must be such that $V_2 - V_1 = c_0 > 0$. Part (i) then follows from the fact that $\overline{V}(s_u, 1 - \alpha_1, 1) - b = V_1 + F(c_0)(V_2 - \mathbb{E}(c \,|c \leq c_0) - V_1)$. Part (ii) follows from the fact that an interior equilibrium where wages in type-1 jobs are non-revealing and wages in type-2 jobs are must be such that

$$V_2 - V_1 = \left(\frac{1-\eta\left(q_u\left(2,\rho_1,\rho_2\right)\right)}{\eta\left(q_u\left(2,\rho_1,\rho_2\right)\right)}\right)\frac{k}{q_u\left(2,\rho_1,\rho_2\right)} - \lambda_1 q_u\left(1,\rho_1,\rho_2\right)\left(\frac{1-\eta\left(q_u\left(1,\rho_1,\rho_2\right)\right)}{\eta\left(q_u\left(1,\rho_1,\rho_2\right)\right)}\right)\frac{k}{q_u\left(1,\rho_1,\rho_2\right)},$$

with $V_2 - V_1 > 0$ and $\lambda_1 q_u\left(1,\rho_1,\rho_2\right) \geq 1$, for $\left(\rho_1,\rho_2\right) = \left(1-\alpha_1, 1\right)$. **QED**

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