Fixed Point Approaches to the Proof of the Bondareva-Shapley Theorem

Jean Guillaume Forand[*] Metin Uyanik[†]

November 29, 2017

Abstract

We provide two new proofs of the Bondareva-Shapley theorem, which states that the core of a transferable utility cooperative is nonempty if and only if the game is balanced. Both proofs exploit the fixed points of self-maps of the set of imputations, applying elementary existence arguments typically associated with noncooperative games to cooperative games.

1 Introduction

The celebrated Bondareva-Shapley Theorem (Bondareva (1962, 1963); Shapley (1967)) shows that balancedness is both necessary and sufficient for the existence of the core of a transferable utility (TU) cooperative game. In this paper, we provide two new proofs of this theorem which rely on elementary fixed point methods. Our aim is to approach the problem of core existence through analogies to common proofs of the existence of Nash equilibria in noncooperative games, where coalitional blocking of imputations in cooperative games stands for best-responses to strategy profiles in noncooperative games. While a Nash equilibrium is a strategy profile that *is* a best-response to itself, a core imputation is *not* blocked by any other imputation. Therefore, both our proofs start by assuming that some balanced TU game fails to have a core and derives a contradiction: only in this case does the blocking relation yield a well-defined "best-response" to each imputation.[1]

[*]Department of Economics, University of Waterloo, Hagey Hall of Humanities, Waterloo, Ontario, Canada N2L 3G1. Email: jgforand@uwaterloo.ca. Website: http://arts.uwaterloo.ca/∼jgforand

[†]School of Economics, University of Queensland, Colin Clark Building, Brisbane, Queensland, Australia, QLD 4072. Email: m.uyanik@uq.edu.au. Website: https://sites.google.com/site/metinuyanikjhu/home

[1]A similar strategy can establish the existence maximal elements of preference correspondences. See, e.g., Sonnenschein (1971) and Yannelis and Prabhakar (1983).

In our first proof, the analogy is to establishing the existence of Nash equilbria through fixed points of best-response correspondences (Nash (1950)). If a TU game has no core and every imputation is blocked by some coalition, then we can construct payoffs for blocking coalitions and define a (non-empty valued) correspondence from the set of imputations into itself. The existence of a fixed point of this correspondence, which corresponds to an imputation that is a convex combination of imputations that block it, is established through an elementary fixed point theorem for lower hemicontinuous correspondences due to Gale and Mas-Colell (1975). Furthermore, this fixed point identifies a balanced collection of blocking coalitions, which contradicts the balancedness of the game.

In our second proof, the analogy is to establishing the existence of Nash equilibria through an application of Brouwer's fixed point theorem to a continuous function on the set mixed strategies of the game that has tâtonnement-type properties: it tends to increase the use of pure strategies that are best-responses and decrease the use of those that are not (Nash (1951)). By adapting a construction from Zhou (1994), we provide a continuous function on the set of imputations of the game that tends to increase the payoffs of blocking coalitions and decrease the payoffs of non-blocking coalitions, and, as in our first proof, its fixed point also identifies a balanced collection of blocking coalitions. Zhou (1994) constructs this function as an intermediate step in the proof of an intersection of open covers theorem which is closely related to the K-K-M-S theorem (see Scarf (1967), Shapley (1967), Ichiishi (1981), Kannai (1992), Shapley and Vohra (1991), Krasa and Yannelis (1994), Komiya (1994) and Herings (1997)), and which he then applies to provide an alternative proof of Scarf's (1967) theorem that all balanced nontransferable utility games have nonempty cores.[2]

The original proof of the Bondareva-Shapley theorem relies on duality results from linear programming. Another proof due to Osborne and Rubinstein (1994) applies the separating hyperplanes theorem to construct a core imputation for any balanced TU game. A third proof by Aumann (1989) establishes a connection between core existence and the minimax theorem for zero-sum games: given a balanced TU game, he constructs a zero-sum game whose mixed strategy equilibrium identifies a core imputation. One distinction of our proofs is that we do not borrow results from noncooperative games to establish the Bondavera-Shapley theorem, but instead we prove the theorem through the methods used to establish the existence of equilibria in noncooperative games.

[2]Of course, the nonemptiness of the core of a balanced nontransferable utility game implies the nonemptiness of the core of a (TU) game.

2 Two Proofs of the Bondareva-Shapley Theorem

Given a set of players N, a *transferable utility (TU) game* is a function $W : \mathcal{N} \to \mathbb{R}$, where $\mathcal{N} = 2^N$. Let V be the set of imputations for this game, that is, the set of individually rational utilities attainable for the grand coalition, or

$$V = \{v \in \mathbb{R}^N : v_i \geq W(\{i\}) \text{ for all } i \in N \text{ and } \sum_{i \in N} v_i = W(N)\}.$$

Let V be endowed with the relative Euclidean topology. The *core* of the game W is defined as $\text{Core}(W) = \{v \in V | \sum_{i \in S} v_i \geq W(S) \text{ for all } S \in \mathcal{N}\}$. A collection of coalitions $\mathcal{B} \subseteq \mathcal{N}$ is *balanced* if there exists weights $\{\delta_S\}_{S \in \mathcal{B}}$ such that $\delta_S \geq 0$ for all $S \in \mathcal{B}$ and $\sum_{S \in \mathcal{B},\ i \in S} \delta_S = 1$ for all $i \in N$. The game W is *balanced* if for all balanced collections of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \delta_S W(S) \leq W(N)$.

Theorem (Bondareva-Shapley). *A TU game has a nonempty core if and only if it is balanced.*

We make a few remarks before giving our proofs of the theorem. First, given $0 < \epsilon < 1$, we normalize the game such that $W(\{i\}) = \epsilon$ for all $i \in N$. To see this, fix any game W, and consider another game $\tilde{W}$ such that $\tilde{W}(S) = W(S) + \sum_{i \in S} [\epsilon - W(\{i\})]$ for all $S \in \mathcal{N}$ along with the isomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x)_i = x_i + \epsilon - W(\{i\})$ for all $i \in N$. Clearly, a coalition $S \in \mathcal{N}$ blocks imputation $v \in V$ if and only if coalition S also blocks imputation $f(v) \in f(V)$. Second, we expand the set of imputations associated to W such that

$$\hat{V} = \{v \in \mathbb{R}^N : v_i \geq 0 \text{ for all } i \in N \text{ and } \sum_{i \in N} v_i = W(N)\},$$

and modify the definition of the core such that

$$\widehat{\text{Core}}(W) = \{v \in \hat{V} | \sum_{i \in S} v_i \geq W(S) \text{ for all } S \in \mathcal{N}\}.$$

Clearly, because for all $v \in \hat{V} \setminus V$, there exists $i \in N$ such that $v_i < W(\{i\}) = \epsilon$, we have that $\widehat{\text{Core}}(W) = \text{Core}(W)$. Third, if the game is balanced, then $W(N) > 0$ (because the collection $\{\{i\} : i \in N\}$ is balanced, with weights such that $\delta_{\{i\}} = 1$). Finally, the necessity of balancedness for a nonempty core follows from standard arguments, so that we only prove its sufficiency.

A First Proof of the Bondareva-Shapley Theorem. Towards a contradiction, assume that W

has an empty core. It follows that all $v \in \hat{V}$ are blocked by some coalition, and let $\mathcal{S}(v)$ denote the set of coalitions that block v. Define $\psi_S(v) \in \hat{V}$ such that, given any $v \in \hat{V}$ and any $S \in \mathcal{S}(v)$,

$$\psi_{S,i} = \begin{cases} \frac{W(N)\max\{v_i,\epsilon\}}{\sum_{j\in S}\max\{v_j,\epsilon\}} & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Define a correspondence $P : \hat{V} \twoheadrightarrow \hat{V}$ such that $P(v) = co\{\psi_S(v)\}_{S\in\mathcal{S}(v)}$, and note that P has nonempty and convex values.

Lemma 1. *There exists $\overline{v}$ such that $\overline{v} \in P(\overline{v})$.*

Given any imputation $v \in \hat{V}$ and coalition $S \in \mathcal{S}(v)$, the set of imputations that block v for S is open, so that P is not an upper hemicontinuous correspondence. However, P is lower hemicontinuous. Lemma 1 then follows from an application of the fixed point theorem of Gale and Mas-Colell (1975), which, as re-stated in Yannelis and Prabhakar (1983), guarantees that any non-empty and convex-valued lower semicontinuous correspondence on a nonempty, convex and compact subset of $\mathbb{R}^n$ has a fixed point. The following lemma, in obvious contradiction with Lemma 1, completes the proof of the Bondareva-Shapley theorem.

Lemma 2. *For all $v \in \hat{V}$, $v \notin P(v)$.*

Proof. Suppose, toward a contradiction, that there exists $\overline{v} \in P(\overline{v})$, and fix $i \in N$. It follows that, for all $S \in \mathcal{S}(v)$, there exist $\lambda_S \geq 0$ with $\sum_{S\in\mathcal{S}(v)} \lambda_S = 1$ such that

$$\overline{v}_i = \sum_{S\in\mathcal{S}(v)} \lambda_S \psi_{S,i}(\overline{v}).$$

First, suppose that $\overline{v}_i \geq \epsilon$, so that

$$\overline{v}_i = \sum_{S\in\mathcal{S}(v), i\in S} \lambda_S \frac{W(N)\overline{v}_i}{\sum_{j\in S}\max\{\overline{v}_j,\epsilon\}},$$

which because $\overline{v}_i > 0$ is equivalent to

$$\sum_{S\in\mathcal{S}(v), i\in S} \frac{\lambda_S W(N)}{\sum_{j\in S}\max\{\overline{v}_j,\epsilon\}} = 1.$$

Second, suppose that $\overline{v}_i < \epsilon$, so that

$$\begin{aligned}\overline{v}_i &= \sum_{S \in \mathcal{S}(v), i \in S} \lambda_S \frac{W(N)\epsilon}{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}} \\ &< \epsilon,\end{aligned}$$

which, because $\epsilon > 0$ yields that

$$\sum_{S \in \mathcal{S}(v), i \in S} \frac{\lambda_S W(N)}{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}} < 1. \tag{1}$$

We construct weights $\delta_S \geq 0$ for all $S \in \mathcal{N}$ so that the collection $\mathcal{N}$ is balanced. For all $S \in \mathcal{S}(\overline{v})$, let $\delta_S = {}^{\lambda_S}/_{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}}$. For all $i \in N$ such that $\overline{v}_i = 0$, set $\delta_{\{i\}} = 1$. For all $i \in N$ such that $0 < \overline{v}_i < \epsilon$, set

$$\delta_{\{i\}} = \lambda_{\{i\}} + \left[1 - \sum_{S \in \mathcal{S}(v), i \in S} \frac{\lambda_S W(N)}{\sum_{j \in S} \max\{\overline{v}_j, \epsilon\}}\right],$$

where $\lambda_{\{i\}}$ is well-defined because $\{i\} \in \mathcal{S}(\overline{v})$. Finally, for all other $S \in \mathcal{N}$, set $\delta_S = 0$. Note that the collection $\{\delta_S\}_{S \in \mathcal{N}}$ is well-defined because, for all $i \in N$ such that $0 < \overline{v}_i < \epsilon$, we have that

$$\begin{aligned}0 &< \delta_{\{i\}} \\ &= 1 + \lambda_{\{i\}} \left[1 - \frac{1}{\epsilon}\right] - \sum_{S \in \mathcal{S}(v) \setminus \{i\},\ i \in S} \frac{\lambda_S W(N)}{\sum_{j \in S} \max\{v_j, \epsilon\}} \\ &\leq 1,\end{aligned}$$

where the first inequality follows from (1) and the fact that $0 < \epsilon < 1$. By construction,

$\sum_{S \in \mathcal{N}, i \in S} \delta_S = 1$ for all $i \in N$, so that the collection $\mathcal{N}$ is balanced, and it follows that

$$\begin{aligned}
\sum_{S \in \mathcal{N}} \delta_S W(S) &= \sum_{S \in \mathcal{S}(\overline{v})} \delta_S W(S) + \sum_{\{i \in N : \overline{v}_i = 0\}} \delta_S W(\{i\}) \\
&> \sum_{S \in \mathcal{S}(\overline{v})} \delta_S \sum_{i \in S} \overline{v}_i + \sum_{\{i \in N : \overline{v}_i = 0\}} \delta_S \epsilon \\
&> \sum_{S \in \mathcal{N}} \delta_S \sum_{i \in S} \overline{v}_i \\
&= \sum_{i \in N} \overline{v}_i \sum_{S \in \mathcal{N}, i \in S} \delta_S \\
&= W(N),
\end{aligned}$$

contradicting the fact that the game is balanced. The first inequality follows from the facts that (i) each coalition $S \in \mathcal{S}(\overline{v})$ blocks $\overline{v}$ and $\mathcal{S}(\overline{v})$ is nonempty by assumption, and that (ii) $W(\{i\}) = \epsilon$.

□

□

A Second Proof of Bondareva-Shapley Theorem. Towards a contradiction, assume that W has an empty core. It follows that all $v \in \hat{V}$ are blocked by some coalition. For each coalition S, define a blocking benefit function $b_S : \hat{V} \to \hat{V}$ such that $b_S(v) = \max\{W(S) - \sum_{i \in S} v_i, 0\}$. Furthermore, define a modified blocking benefit function $\hat{b}_S : \hat{V} \to \hat{V}$ such that

$$\hat{b}_S(v) = \begin{cases} b_S(v) & \text{if } |S| = 1, \\ b_S(v) \max\{\min_{j \in N} v_j - \epsilon, 0\} & \text{if } |S| \geq 2, \end{cases}$$

and define a function $p : \hat{V} \to \hat{V}$ such that, for all $i \in N$

$$p(v)_i = v_i + \epsilon \left[\frac{\sum_{S \in \mathcal{S}(v), i \in S} \hat{b}_S(v) \frac{|N|}{|S|}}{\sum_{S \in \mathcal{S}(v)} \hat{b}_S(v)} - 1 \right].$$

Lemma 3 (Zhou (1994))**.** *There exists $\overline{v}$ such that $\overline{v} = p(\overline{v})$.*

The function p adapts a construction from Zhou (1994) to our simplified setting. Because p is continuous, Zhou (1994) shows that Lemma 3 follows from an application of Brouwer's fixed point theorem. The following lemma, in obvious contradiction with Lemma 3, completes the proof of the Bondareva-Shapley theorem.

Lemma 4. *For all v, $v \neq p(v)$.*

Zhou (1994)'s construction of p also ensures that the collection $\mathcal{S}(\overline{v})$ is a balanced collection, with associated weights $\delta_S = \frac{\hat{b}_S(\overline{v})\frac{|N|}{|S|}}{\sum_{S \in \mathcal{S}(\overline{v})} \hat{b}_S(\overline{v})}$ for all coalitions $S \in \mathcal{S}(\overline{v})$. It follows that

$$\begin{aligned} \sum_{S \in \mathcal{S}(\overline{v})} \delta_S W(S) &> \sum_{S \in \mathcal{S}(\overline{v})} \delta_S \sum_{i \in S} \bar{v}_i \\ &= \sum_{i \in N} \bar{v}_i \sum_{S \in \mathcal{S}(\overline{v}), i \in S} \delta_S \\ &= W(N), \end{aligned}$$

contradicting the fact that the game is balanced. The inequality follows because $\delta_S > 0$ only if $W(S) > \sum_{i \in S} \bar{v}_i$. □

References

Aumann, R. J. (1989). *Lectures on game theory*. Westview Pr.

Bondareva, O. (1962). The theory of the core in an n-person game. *Vestnik Leningrad. Univ 13*, 141–142.

Bondareva, O. N. (1963). Some applications of linear programming methods to the theory of cooperative games. *Problemy kibernetiki 10*, 119–139.

Gale, D. and A. Mas-Colell (1975). An equilibrium existence theorem for a general model without ordered preferences. *Journal of Mathematical Economics 2*(1), 9–15.

Herings, P. J.-J. (1997). An extremely simple proof of the K-K-M-S theorem. *Economic Theory 10*(2), 361–367.

Ichiishi, T. (1981). On the Knaster-Kuratowski-Mazurkiewicz-Shapley theorem. *Journal of Mathematical Analysis and Applications 81*(2), 297–299.

Kannai, Y. (1992). The core and balancedness. *Handbook of game theory with economic applications 1*, 355–395.

Komiya, H. (1994). A simple proof of K-K-M-S theorem. *Economic Theory 4*(3), 463–466.

Krasa, S. and N. C. Yannelis (1994). An elementary proof of the Knaster-Kuratowski-Mazurkiewicz-Shapley theorem. *Economic Theory 4*(3), 467–471.

Nash, J. (1951). Non-cooperative games. *Annals of mathematics*, 286–295.

Nash, J. F. (1950). Equilibrium points in n-person games. *Proceedings of the national academy of sciences 36*(1), 48–49.

Osborne, M. J. and A. Rubinstein (1994). *A course in game theory*. MIT press.

Scarf, H. E. (1967). The core of an n person game. *Econometrica*, 50–69.

Shapley, L. and R. Vohra (1991). On Kakutani's fixed point theorem, the K-K-M-S theorem and the core of a balanced game. *Economic Theory 1*(1), 108–116.

Shapley, L. S. (1967). On balanced sets and cores. *Naval Research Logistics (NRL) 14*(4), 453–460.

Sonnenschein, H. (1971). Demand theory without transitive preferences with applications to the theory of competitive equilibrium. In J. C. et al. (Ed.), *Preferences, utility and demand*. New York: Harcourt Brace Jovanovich.

Yannelis, N. C. and N. Prabhakar (1983). Existence of maximal elements and equilibria in linear topological spaces. *Journal of Mathematical Economics 12*(3), 233–245.

Zhou, L. (1994). A theorem on open coverings of a simplex and Scarf's core existence theorem through Brouwer's fixed point theorem. *Economic Theory 4*(3), 473–477.