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An approach to simplifying point features on maps using the multiplicative weighted Voronoi diagram

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A number of features, such as settlements and islands, are represented using point symbols on intermediate and micro scale maps. If the maps are reduced to smaller scale, the point features need to be simplified to make the maps legible. Hence, it is necessary to develop algorithms for point feature generalisation. For the reason above, an algorithm based on the multiplicatively weighted Voronoi diagram (MWVD) is proposed in the paper. To ensure statistical, thematic, metric and topological information contained in the original point features can be transmitted correctly after simplification, the algorithm selects corresponding measures (i.e. the number of points, weight, Voronoi neighbour, Voronoi polygon and distribution range) to quantify the four types of information, and integrates the measures in the process of point feature generalisation.

First, the algorithm detects the range polygon of the given point features; second, it adds the pseudo points (i.e. the vertices of the range polygon) to the original points to form a new point set and tessellates the new point set to get the MWVD; then it computes the selection probability of each point using the area of each Voronoi polygon, and sorts all points in decreasing order by their selection probability values; after this, it marks those to-be-deleted points as 'deleted' according to their selection probability values and their Voronoi neighbouring relations, and determines if they can be physically deleted. Finally, the algorithm is ended by comparing the number of points retained on the map with that computed by the Radical Law.

The algorithm is parameter-free, automatic and easy to understand, owing to the use of the MWVD. As the experiments show, it can be used in simplification of point features arranged in clusters, such as settlements, islands and control points on topographic maps at intermediate/micro scale.

Keywords: point feature generalisation; multiplicatively weighted Voronoi diagram; range polygons; selection probability; algorithm

1. Introduction

A number of features, e.g. settlements, islands, control points and place names, are represented using point symbols on intermediate/micro scale maps. If the maps are reduced to smaller scale, conflict and congestion of map symbols become inevitable (Ruas 2001); thus, the point features need to be simplified to make the maps legible. This process is called map generalisation in the community of cartography and geographic information science. Undeniably, 'the widespread use of geographic information in computers in the context of geographic information systems has brought with it the

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Table 1. Information-based comparison of the algorithms for point feature generalisation

Algorithms	Statistical	Thematic	Metric	Topological
Settlement-spacing ratio	The number of points	Importance value	Distance, relative local density	Fixed radius neighbours
Distribution-coefficient	The number of points	Importance value	Distance	Fixed radius neighbours
Gravity-modelling	The number of points	Importance value	Distance, relative local density	
Circle-growth	The number of points	Importance value	Distance, relative local density	Fixed radius neighbours
Quadtree-based	The number of points		Distance	
Dot map simplification	The number of points		Distance	
OVD-based	The number of points	Importance value	Relative local density, distribution range	Voronoi neighbours

demand for automation of map generalization' (Jones & Ware 2005, p. 859). Hence, development of algorithms for automated point feature generalisation is of great importance (Yan & Weibel 2008).

Several algorithms for point feature generalisation have been proposed so far, five by Langran and Poicker (1986) for name selection and name placement (i.e. the set-segmentation algorithm, quadrat-reduction algorithm, settlement-spacing ratio algorithm, distribution-coefficient algorithm, and gravity-modeling algorithm), one by Van Kreveld *et al.* (1995) for settlement selection (i.e. the circle-growth algorithm), one by Burghardt *et al.* (2004) for on-the-fly generalisation of thematic point data, one by de Berg *et al.* (2004) for dot map simplification, and one by Yan and Weibel (2008) for point cluster generalisation. The set-segmentation algorithm and the settlement-spacing ratio algorithm are not fully automatic; therefore they will not be further discussed in this paper.

It is commonly accepted by cartographers that one of the most important functions of maps is for transmitting information (Sukhov 1967, 1970; Neumann 1994; Bjørke 1996; Li & Huang 2002), and readers generally capture four types of information from maps, i.e. statistical, thematic, metric and topological information (Li & Huang 2002). Hence, whether the four types of information can be transmitted correctly should be the criterion in evaluating algorithms for point feature generalisation.

Table 1 summarises the parameters used in each algorithm for denoting corresponding types of information contained in point features. If no parameter is used to denote a type of information in an algorithm, the corresponding grid in Table 1 is left blank. It is clear that (1) not all of the four types of information are taken into account in the gravity-modeling algorithm, the quadtree-based algorithm and the dot map simplification algorithm, and (2) the settlement-spacing ratio algorithm, the distribution-coefficient algorithm and the circle-growth algorithm denote topological information using 'fixed radius neighbours' which has been proved inferior to 'Voronoi neighbours', (Ahuja & Tuceryan 1989), and (3) only the algorithm based on OVD (ordinary Voronoi Diagram) takes into account all of the four types of information. Nevertheless, OVD cannot correctly describe the influenced areas of weighted points compared with MWVD (multiplicatively weighted Voronoi diagram), so the OVD-based algorithm uses some strategies to compensate for its disadvantage (Yan & Weibel 2008), which makes the point generalisation procedures complicated.

Table 2. Measures for types of information in the new algorithm

Types of information	Selected measures
Statistical	The number of points
Thematic	Weight
Topological	Voronoi neighbor
Metric	Voronoi polygon, range polygon

After the introduction, OVD and MWVD are discussed and the strategies used in the new algorithm are presented (section 2). Then an MWVD-based algorithm is proposed (section 3), and some experiments are shown to illustrate the validity of the algorithm (section 4). This paper ends with conclusions and an outlook on our further research (section 5).

2. Improvements in the new algorithm

It has been mentioned in the previous section that existing algorithms for point feature generalisation either use an inferior tool (i.e. OVD) to denote influenced areas of points, or do not take into account all of the four types of information contained in point clusters, or use inappropriate parameters (i.e. fixed radius neighbours) to describe topological information. Thus, it is reasonable to make improvements in the following aspects: (1) a better tool should be employed to denote influenced areas of point features so that the point feature generalisation process can be simpler to implement and easier to understand; (2) measures should be selected to quantify the four types of information so that each type of information is controllable in map generalisation; and (3) appropriate strategies should be utilised to ensure each type of information can be transmitted correctly.

MWVD used in lieu of OVD

An OVD is a kind of decomposition of a metric space determined by distances to a specified discrete set of objects in the space. In the simplest case, a set of points S in the plane are given. Each point s is called a Voronoi site or a Voronoi generator. Each Voronoi diagram $V(s)$ consists of all points closer to s than to any other site (Green & Sibson 1978).

An MWVD in n-dimensional space is a Voronoi diagram for which the Voronoi cells are defined in terms of a distance defined by some common metrics modified by positive weights assigned to generator points. The edges of the MWVD are circular arcs and straight line segments (Galvão *et al.* 2006). A polygon of an MWVD may be non-convex, disconnected and have holes (Aurenhammer & Edelsbrunner 1984).

All generators in OVD are identical and each generator has the same weight. However, this assumption may not be true in practical applications. In many cases, it is better to assume that generators have different weights reflecting their variable properties (e.g. population in a settlement, the number of customers of a shopping centre, the amount of emissions from a polluter, or the size of an atom in a crystal structure, etc.). Therefore, it is more reasonable in theory and reliable in practice to use the MWVD in lieu of the OVD for denoting the influenced areas of point features.

Measures selected for each type of information

The measures for quantifying each type of information in the new algorithms are listed in Table 2. Detailed explanations of the reasons are presented as follows.

- 'The number of points' is the only measure for describing statistical information.
- For thematic information, existing approaches utilise 'importance value' or 'significance value' (van Kreveld *et al.* 1995; Yan & Weibel 2008). In essence, both of them are the weights of points; thus, 'weight' is employed in this paper.

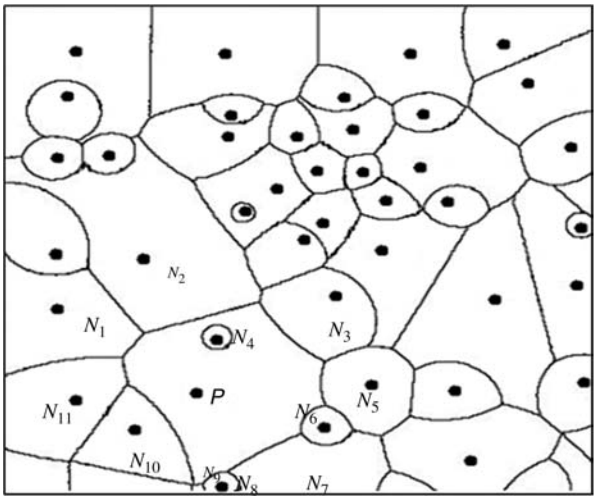

Figure 1. The Voronoi neighbours of point P

- For topological information, 'Voronoi neighbour' is used. Its priority among various measures, including 'fixed radius neighbour', 'k-nearest neighbour' and 'nearest neighbour', has been thoroughly analysed (Ahuja 1982; 1989). The Voronoi neighbours of point *P* are defined as the points whose Voronoi polygons are adjacent to or contained by that of *P*.

  For example, in Figure 1, point *P* has 11 Voronoi neighbours. One neighbour's (i.e. N_4) Voronoi polygon is contained by the Voronoi polygon of *P* and the other 10 neighbours' polygons are adjacent to the Voronoi polygon of *P*.
- 'Distribution range' and 'relative local density' are traditionally used to describe metric information contained in point clusters (Sadahiro 1997).

'Range polygon', but not 'border polygon' (Figure 2), is selected to represent the distribution range of a point set here, because the scope enclosed by the range polygon is more reasonable than that enclosed by the border polygon for denoting the influenced area of the point cluster in 2-dimensional space. For example, the border polygon of the control points in a specific area is obviously too small to cover the area controlled by the control points in surveying; whereas the range polygon which covers a larger area is more appropriate for representing the surveyed scope. On the other hand, the use of 'range polygon' can make each divergent Voronoi polygon of the point at the outlier layer of the original point set (Green & Sibson 1978; Boots & South 1997) convergent, which facilitates the calculation of the local relative density of the points. This will be seen in Section 4.

The relative local density of the i^{th} point is defined as (Sadahiro 1997; Yan & Weibel 2008):

$$r_i = \frac{R_i}{\sum_{k=1}^{n} R_k} \tag{1}$$

Where r_i is the relative local density of the i^{th} point, n is the number of the points, R_i is the absolute local density of the i^{th} point is and defined as:

$$R_i = \frac{1}{A_i} \tag{2}$$

Where A_i is the area of the Voronoi polygon containing the i^{th} point.

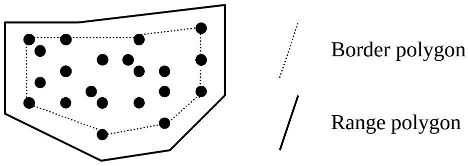

Figure 2. The range polygon and border polygon of a point set

Strategies employed in point cluster simplification

To integrate the selected measures into the new algorithm for transmitting the four types of information, the following strategies are employed.

- Statistical information: The Radical Law (Topfer & Pillewizer 1966) is the most commonly accepted method for calculating the number of objects on a target scale map from that of the original scale map. Thus, it is employed to determine the number of points that should be retained on target maps (Formula (3)).

$$N_f = N_o \cdot \sqrt{\frac{s_o}{s_f}} \tag{3}$$

where N_f is the number of points on the final map, N_o is the number of points on the original map, s_o is the denominator of the original map scale, and s_f is the denominator of the final map scale.

- Thematic information: Weights of the points are integrated in constructing the MWVD of the point features and the rule 'the larger a multiplicatively weighted Voronoi polygon is, the more probable the original point in the polygon can be retained' is abided by so that the points with greater importance values have a greater probability of appearing on the final map.

  Weight values of the points are obtained from their thematic attributes stored in databases in the form of ordinal (e.g. rating the service of staff: excellent, good and poor), interval (e.g. Fahrenheit temperature) and/or ratio (e.g. population of a city) data. Calculation of weight values may be a complicated combination of ordinal, interval and/or ratio for constructing empirical formulas (it is difficult to address this issue in detail here) or simply a use of a single ordinal/interval/ratio data.
- Topological information: The rule 'do not delete any two Voronoi neighbours' is obeyed in point feature generalisation so that topological information (i.e. the adjacency relationship of points) can be transmitted well. This is based on the following two reasons: (1) simultaneous deletion of adjacent points is generally unacceptable by cartographers in practice if the scale reduction is rather small (e.g. from 1:10K to 1:25K); (2) this operation, in theory, obviously acts contrary to the First Law of Geography: 'everything is related to everything else, but near things are more related than distant things' (Tobler 1970, p. 234).
- Metric information: 'Selection probability' calculated by 'local relative density' and 'range polygon' will be employed in the new algorithm. In the process of point feature generalisation, the greater the selection probability of a point, the more probable it is that the point can be retained on the final map. This ensures that the metric information contained in point features can be transmitted correctly.

3. An MWVD-based algorithm

Previous sections presented the disadvantages of existing algorithms and the strategies that will be used in the new algorithm. Thus, it should be pertinent to introduce the new algorithm in this section.

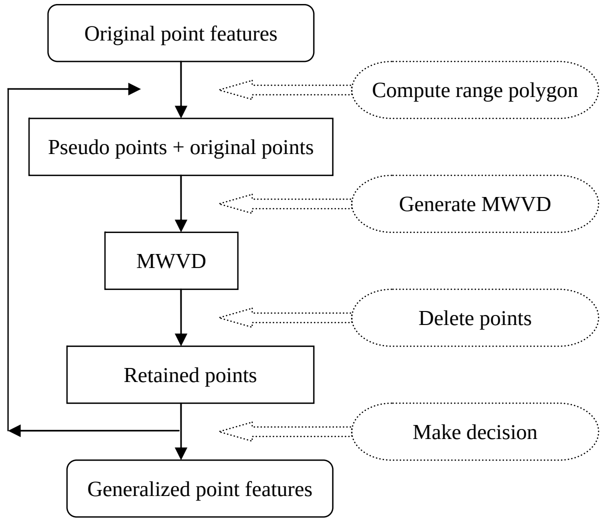

Figure 3. Flowchart of the MWVD-based algorithm

Framework of the MWVD-based algorithm

The flowchart of the new algorithm is shown in Figure 3, including the following four procedures:

- computation of range polygon;
- MWVD generation;
- point deletion; and
- decision making.

To facilitate presenting the new algorithm clearly, the generalisation procedures are demonstrated using an example. The original map is at scale 1:10K, with 18 points whose weight is 1 and 19 points whose weight is 2 (Figure 4(a)). The objective of the new algorithm is to get a map at scale 1:20K.

Computation of range polygon

This procedure includes four steps.

(1) The points are triangulated (Figure 4 (b)) using a Delaunay triangulation method (Guibas *et al.* 1992).
(2) After triangulation, the border polygon can be obtained by sequentially connecting the hull points of the outlier triangles.
(3) The range polygon for describing the distribution range of the point features is computed (Figure 4(c)). Each vertex of the range polygon must be on the extension line of the line segment connecting the centroid of the border polygon and the corresponding vertex of the border polygon. The length of the extended line segment is the mathematically weighted mean value of the lengths of the triangle edges connected to the corresponding vertex. The vertices of the range polygon are named ‘pseudo points’. The weight of each pseudo point is equal to that of the corresponding vertex of the border polygon.
(4) Add all vertices of the range polygon to the original point set to form a new point set (Figure 4 (d)).

MWVD generation

Several algorithms have been proposed for generating an MWVD of point sets by far, either vector-based or raster-based (Galvão *et al.* 2006). Here, the one proposed by Aurenhammer and Edelsbrunner (1984) is

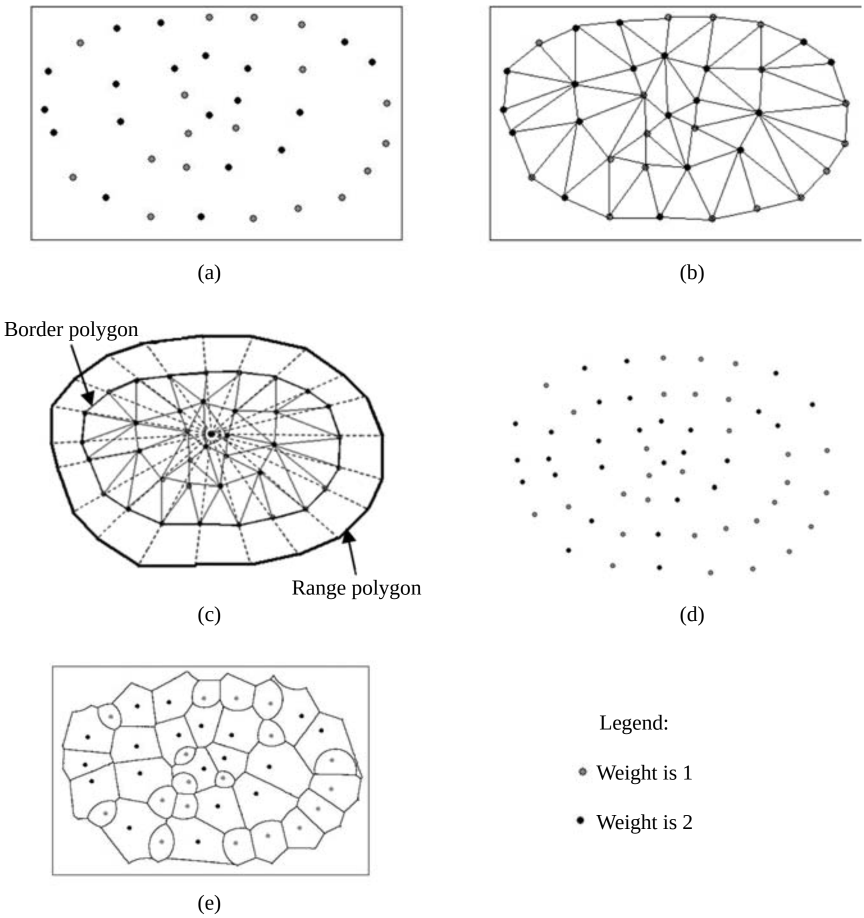

Figure 4. Generation of MWVD. (a) original point set; (b) triangulation; (c) border polygon and range polygon; (d) new point set with pseudo points; and (e) multiply weighted Voronoi diagrams of the points

employed. It should be noticed that all Voronoi polygons of the original points are convergent (Figure 4(e)), owing to the addition of the pseudo points.

Point deletion

(1) Calculate and sort the selection probability values.

The more important a point feature is, the more probable the point should be retained on the resulting map. Thus, it is crucial to determine the selection probability of each point. On the other hand, the bigger a Voronoi polygon and the greater the weight of the point, the more important the point is. In other words, the area of the Voronoi polygon is proportional to the selection probability of the point. So the selection probability value of each point can be calculated by:

$$P_i = \frac{A_i}{A_{\max}} \tag{4}$$

Where P_i is the selection probability of the i^{th} point feature,A_i is the area of the

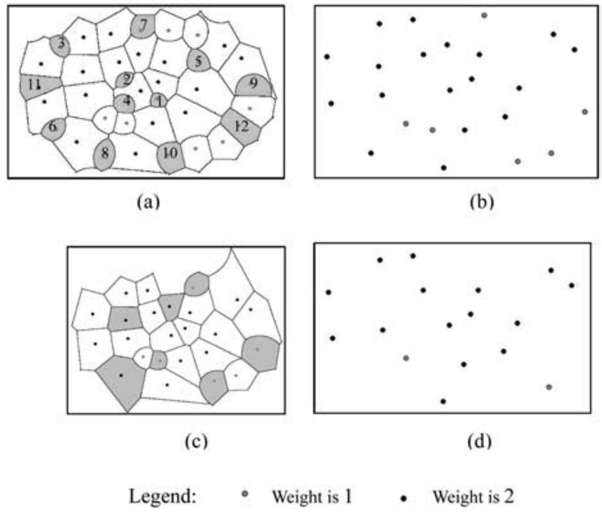

Figure 5. Point feature generalisation. (a) the first round of point deletion. Deletion sequence of the points is labeled; (b) retained points after the first round of deletion; (c) the second round of point deletion; and (d) retained points after the second round of deletion

Voronoi polygon owned by the i^{th} point feature, and $A_{\max}$ is the maximum area value in all of the Voronoi polygons.

The selection probability values are sorted in increasing order and saved in a one-dimensional array, say*P*.

(2) Mark the to-be-deleted points.

Each of the points may be in one of the three statuses, 'free', 'fixed', or 'deleted', in this step. At the beginning, all points are marked as 'free'. Then examine each point in *P* starting from the one with the least selection probability value and change the status of the points. If the following three requirements are all satisfied a point may be marked as 'deleted':

(a) it is marked as 'free';
(b) its selection probability value is the least in the 'free' points; and
(c) none of its Voronoi neighbours are marked as 'deleted'.

If a point is marked as 'deleted', it means this point is a candidate that will be deleted but not that it is deleted from the point set at once.

If a point has been marked as 'deleted', its Voronoi neighbours should be marked as 'fixed' (Figure 5(a)). 'Fixed' points cannot be marked as 'deleted' in the same round of point deletion, which ensures that no adjacent points can be deleted simultaneously. This step is repeated until no points can be marked as 'deleted'.

It should be noticed that the rule 'do not delete any two Voronoi neighbours' has been obeyed in this procedure.

Decision making

After a round of deletion, mark all points as 'free' except the ones that are marked as 'deleted'. Suppose that after this round of deletion the number of 'free' point is N_c and after the last round of deletion the number of 'free' points is N_l; the theoretical number of points on the resulting map is N_f (calculated by formula (3)). Then whether the algorithm may be ended or not can be determined.

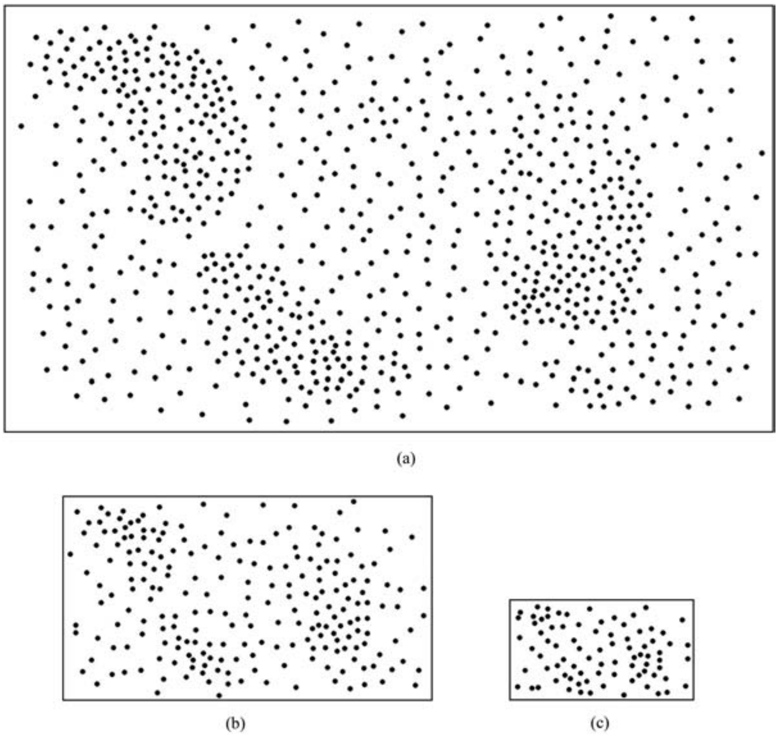

Figure 6. Experiment 1. (a) source data at scale 1:10K with 500 points; (b) generalised point data at scale 1:50K, with 221 points retained; and (c) generalised point data at scale 1:500K with 83 points retained. The weight values of all points are 1. Maps are not shown exactly to scale

- If $N_c > N_f$, go to the second procedure (i.e. MWVD generation); else,
- if $|N_f - N_c| > |N_f - N_l|$, physically delete those points that are marked as 'deleted' after the last round of deletion, and take the remaining N_l 'free' points as the ones on the resulting map and end the procedure; else,
- physically delete those points that are marked as 'deleted' after this round of deletion, and take the remaining N_c 'free' points as the ones on the resulting map and end the procedure.

In the example (Figure 4), $N_f \approx 23$. After the first round of deletion, $N_c = 24$ (Figure 5 (b)). Therefore, the deletion procedure will be repeated. After the second round of deletion (Figure 5(c) and Figure 5(d)), $N_c = 17$ and $N_l = 24$. Obviously, $|N_f - N_c| > |N_f - N_l|$. So the algorithm may be ended and the map that contains the 24 points retained after the first round of deletion is the resulting map (Figure 5(b)).

Experimental studies and discussions

The MWVD-based approach has been implemented by the authors in Visual C^{++} (Version 6.0), and various point data have been utilised to test its correctness. Here, three experiments with different characteristics are selected to illustrate the algorithm (Figure 6, Figure 7 and Figure 8). The data used in experiment 1 are simulated, and those used in the other two experiments are field surveying data provided by the Chinese Academy of Surveying and Mapping, Beijing, China.

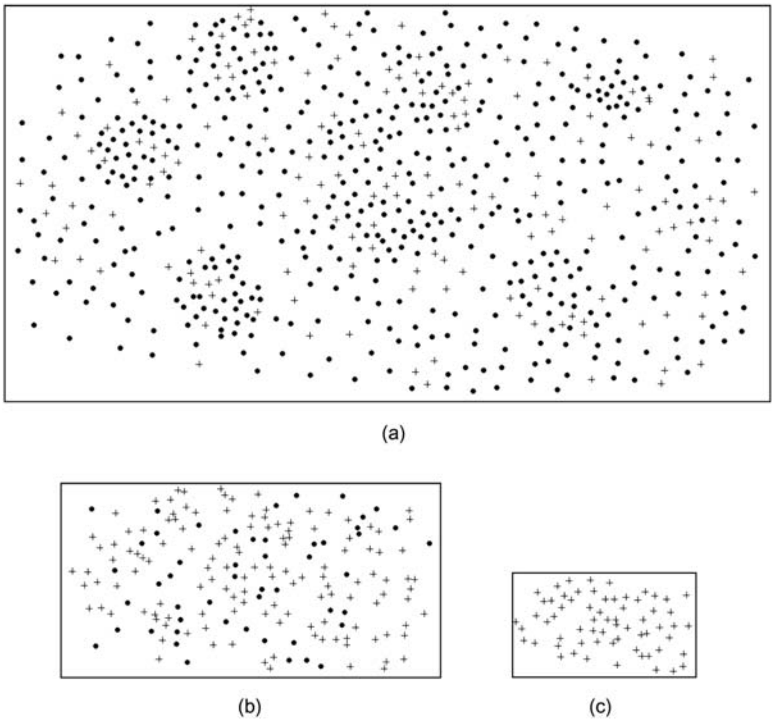

Figure 7. Experiment 2. (a) Source data at scale 1:10K with 650 points, of which 496 are weighted 1 and 154 are weighted 2; (b) generalised point data at scale 1:100K, with 185 points retained, of which 51 are weighted 1 and 134 are weighted 2; and (c) generalised point data at scale 1:1M with 65 points retained, the weight value of each point being 2. The maps are not shown exactly to scale

As is mentioned in the previous sections, the purpose of maps is for transmitting information to readers; hence, whether the statistical, thematic, topological and metric information contained in the maps can be transmitted well should be the criterion for evaluating the generalisation results. To be exact, the following indices are employed to evaluate the algorithm.

- For statistical information, the deviation percentage (D_p) between the theoretical number (N_t) and the resulting number (N_r) of the points on the target map can be used. It is defined as:

$$D_p = \frac{|N_r - N_t|}{N_t} \qquad (5)$$

The less D_p is, the better the statistical information has been transmitted.

- For thematic information, the deviation of the mean weight value (D_v) between the mean weight value of the original points ($\overline{W_o}$) and that of the resulting points ($\overline{W_r}$) is used. It is defined as:

$$D_v = \overline{W_r} - \overline{W_o} \qquad (6)$$

A greater D_v means that more points with greater weights (in other words, they are more important) have been retained on the resulting map. Hence, the greater D_v is, the better the thematic information has been transmitted.

- For topological information, the mean change of topological neighbours (D_t) for the points on the resulting map is

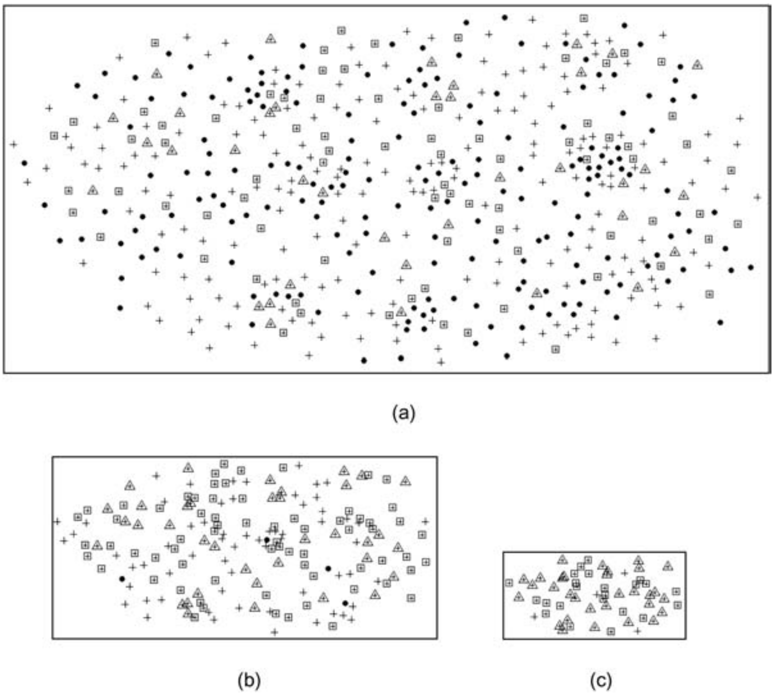

Figure 8. Experiment 3. (a) Source data at scale 1:10K with 500 points, of which 189 are weighted 1, 194 are weighted 2, 80 are weighted 4, and 37 are weighted 8; (b) generalised point data at scale 1:100K, with 173 points retained, of which 4 are weighted 1, 63 are weighted 2, 69 are weighted 4, and 37 are weighted 8; and (c) generalised point data at scale 1:1M with 58 points retained, of which 4 are weighted 2, 23 are weighted 4, and 31 are weighted 8. The maps are not shown exactly to scale

employed. It can be calculated by:

$$D_t = \overline{V_o} - \overline{V_r} \quad (7)$$

Where $\overline{V_r}$ is the mean number of original Voronoi neighbours of the points on the resulting map and $\overline{V_o}$ is the mean number of Voronoi neighbours of the points on the original map.

The lower $D_t$is, the better the topological information has been transmitted.

- For metric information, the change of range polygon (C_p) and the change of local relative density (C_l) should be evaluated. They are defined as follows.

$$C_p = \left| \frac{P_o - P_r}{P_o} \right| \quad (8)$$

Where P_o is the area of the original range polygon; and P_r is the area of the range polygon on the resulting map.

The lower C_p is, the better the topological information has been transmitted.

The following method can be used to calculate C_l. Firstly the points on the resulting map are sorted in increasing order according to their local relative density and saved in array A_r; secondly the corresponding points on the original map are also sorted in the increasing order according to their local relative density and saved in array A_o; then the points in A_r and A_o are compared one by one. If point p is at the i^{th} position in A_r

Table 3. Indices used in the experiments for evaluating the new algorithm

Experiments	Resulting Map scales	D_p	D_v	D_t	C_p	C_l
Experiment 1	1:50K	0.9%	0	2.44	3.89%	2.3%
	1:500K	16.9%	0	1.58	9.60%	4.8%
Experiment 2	1:100K	10.8%	0.49	3.77	6.79%	4.3%
	1:1M	0%	0.76	0.84	12.41%	7.7%
Experiment3	1:100K	9.5%	1.67	3.28	7.55%	2.9%
	1:1M	16.0%	3.614	1.03	13.06%	3.4%

and at the j^{th} position in A_o ($i \neq j$), it is named a disagreement of the two arrays. Suppose that the total number of disagreements is N_{dis}, we have:

$$C_l = \frac{N_{dis}}{N_r} \tag{9}$$

It is obvious that the lower C_l and C_p are, the better the topological information has been transmitted.

The results of the indices calculated using the three experiments are listed in Table 3. A number of insights can be gained from the indices and the experiments.

- First of all, it can be seen from D_p in Table 3 that the number of points retained on the resulting map in the experiments is not equal to but approximate to that calculated by the Radical Law. The new approach can also be slightly modified to obtain the specified number of points on the target map if needed (this can be achieved by stopping the algorithm in the point iterative deletion process when the marking of a to-be-deleted point results in the remaining number of the points equalling N_t). However, this is generally unnecessary, because N_t is a statistic-based value (Topfer & Pillewizer 1966), and it needs to be modified by adding a coefficient in some practical applications (Wang 2006). In this sense, the transmission of statistical information in the experiments should be acceptable.
- Second, the deviation of the mean weight value (D_v) increases with the reduction of map scale (except in experiment 1). This demonstrates that the thematic information has been transmitted well.
- Third, the mean change of Voronoi neighbours (D_t) increases with the reduction of map scale, which means the more the point features are generalised, the more the topological relations among the features are damaged.
- Fourth, both the change of range polygon (C_p) and the change of local relative density (C_l) are small. According to our survey of 12 experienced cartographers in the Geomatics Centre of Lanzhou, China, these changes are acceptable.
- Finally, the weight value in the new algorithm is directly given, because it can easily be calculated from the attributes of the map features. For example, the weight of each control point in surveying may be determined by the class of the point (Figure 7 and Figure 8), and the weight of each settlement can be calculated from the ratio of the number of people living in it and the total number of people living in the neighbourhood.

In sum, this algorithm can transmit statistical, thematic, topological and metric information well. If compared with existing algorithms, it has the following three advantages: (1) it can be used to simplify both weighted and non-weighted point features; (2) the scale span (ratio between the original scale and the final scale) can reach 100 (e.g. Figure 8); and (3) it is parameter-free in programming.

5. Conclusions

This paper proposed an MWVD-based algorithm for simplifying point features on maps automatically. Three improvements have been made in this algorithm compared with existing ones. First, appropriate measures for quantitatively expressing statistical, thematic, topological and metric information are selected. Second, a couple of strategies are employed to integrate the measures into the algorithm to control the transmission of the four types of information in the process of point feature generalisation. Third, MWVD is used in lieu of OVD, which makes the algorithm simpler and easier to understand. Correctness of the algorithm has been demonstrated using a number of experiments.

It is our future work to implement the other existing algorithms and use the same point set to test them along with this new algorithm, because it is of great interest not only to cartographers but also to some researchers in the field of computational geometry. A detailed comparison and evaluation of the existing algorithms may make it clear for users to select the most appropriate algorithm for their special-purpose map generalisation. In addition, it is also a potential development to extend the use of this new algorithm from personal computers to mobile devices coded with IOS or Android as the OS.

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