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Switching Control of Differential-Algebraic Equations with Temporal Logic Specifications

Yinan Li and Jun Liu

Abstract— **This paper studies the switching control of differential-algebraic equations (DAEs). A specific problem concerned with switched DAEs is that jumps or impulses could be induced by mode switching, which is not well understood in many applications. We aim to find the control strategies that minimize the overall magnitude of undesirable jumps or impulses while rendering the systems achieve the expected behaviors. Applying an abstraction-based hybrid controller design framework, we extend formal methods to the control synthesis for switched DAEs with the specifications expressed in linear temporal logic. Abstractions are computed utilizing incrementally globally aymptotically stable property and Lyapunov-like functions. We illustrate the control synthesis procedure using a numerical example.**

I. INTRODUCTION

Differential-algebraic equations (DAEs) are naturally used to describe the behavior of electrical circuits, where capacitors, inductances produce the differential equations and Kirchhoff's circuit laws add algebraic constraints. Power electronics-based systems rely on switching devices to perform electrical energy conversion, which motivates the study of switched DAEs. Additionally, sudden component faults that change the dynamics of the system also contribute to mode switches in DAEs. Due to its practical importance, previous work has focused on the stability of linear and nonlinear switched DAEs [1], [2], and the applications in power systems [3]. Reachability analysis for nonlinear DAEs has also been conducted [4]. Few research, however, has been done on switched DAEs in the context of switching control design.

In this paper, we are interested in designing a switching hybrid controller using abstraction-based approach for DAEs from high-level specifications. The advantages of this method is that the abstract model not only respects the low-level physical dynamics but also can be understood by the software or hardware that interacts with the physical world. In particular, control protocols that realize specifications described by high-level languages can be synthesized automatically based on the finite abstractions. Moreover, it provides a hybrid feedback control solution when implemented back to the original systems. Thus, this approach has gained increased popularity over the past decade and found applications in areas such as robot motion planning [5], automatic cruise control [6]. Switched systems have also been studied in the context of temporal logic control based on the finite abstractions. Reactive switching protocols are synthesized based on finite-state approximations using a two-player game between the system and the environment [7]. Bisimilar symbolic model is developed as the equivalence of the original switched system dynamics for hybrid controller synthesis [8] with an application in the control of DC-DC boost converters.

Symbolic models [9], [10] are computed using simulation or bisimulation relations and are one of the main types of abstractions. When equipped with certain robustness measures, such abstractions can cope with imperfections in measurements or models [11]. Another type is polytope or zonotope-based abstractions, which have been used for solving control problems of piecewise-affine systems [12], nonlinear systems [13] and switched systems [14].

Inspired by previous works, our approach relies on computing finite-state approximations of witched DAEs. Different from ordinary switched dynamical systems, the major difficulty here is the presence of switching-induced jumps. To fully describe the discontinuous property of the state transitions, the transition relation is decomposed into two steps with the first step indicating the possible jumps. We show that the proposed finite transition system is an over-approximation of the continuous switched DAE system under the assumption of incremental global aymptotic stability.

Designing a hybrid controller that minimizes switching-induced jumps, this work extends the abstraction-based control method to the switched DAEs. The cost function is chosen as the overall magnitude of non-repeated jumps, which are unesirable in practical applications. The magnitude of a single state jump is recorded in the corresponding transition of the finite-state approximation and the overall magnitude is computed during controller synthesis.

Notation: $\mathbb{R}_0^+$ and $\mathbb{R}^+$ denote the nonnegative and positive real numbers respectively. $\mathbb{N}$ denotes the nonnegative integers. We rely on the comparison functions of class $\mathcal{K}_\infty$ and class $\mathcal{KL}$ (See Definition 4.2 and 4.3 respectively in [15]). Any vector $x \in \mathbb{R}^n$ can be written as $x = [x_1, x_2, \cdots, x_n]^T$. Given a set $S \subseteq \mathbb{R}^n$ and a parameter vector $\eta \in \mathbb{R}^n$ with $\eta_i \in \mathbb{R}^+, i = 1, 2, \ldots, n$, we define $[S]_\eta = \{s \in S | \, s_i = k_i\eta_i, \, k_i \in \mathbb{Z}, \, i = 1, 2, \ldots, n\}$. $|\cdot|$ is the infinity norm, and $\|\cdot\|$ is the Euclidean norm. We denote the set of switching signals as $\Sigma = \big\{\sigma | \, \sigma(\cdot) : \mathbb{R}_0^+ \to P\big\}$.

II. DAEs AND SWITCHED DAEs

Consider switched DAEs in quasi-linear form [2]:

$$E_{\sigma(t)}(x(t))\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{1}$$

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where the switching signal $\sigma(\cdot) : \mathbb{R}_0^+ \to P$ is piecewise-constant, and $P := \{1, \ldots, m\}$, $m \in \mathbb{N}$, is a finite set of modes. Each subsystem or mode is a nonswitched DAE in the form

$$E(x)\dot{x} = f(x), \tag{2}$$

where $E : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is singular, and $f : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions. Assume there exists a unique solution $x(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}^n$ of (2) with initial state x_0, denoted as $x(t, x_0)$. Due to the singularity of E, the solution does not evolve within the whole space $\mathbb{R}^n$, but within the *consistency space* defined as $\mathfrak{C}_{E,f} = \{x_0 \in \mathbb{R}^n \,|\, \exists\, x(t, x_0)\}$.

Considering switched DAEs in form (1), if the subsystems do not have a common consistency space, discontinuity can be introduced at the time of switching, meaning that the solution under the switching signal σ contains jumps or even impulses. Switched DAEs has a unique solution for any switching signal and consistency initial values if and only if it is impulse free (see Theorem 3.3, [2]).

In this paper, we follow the impulse free assumption, and the trajectory of (1) is defined as a piecewise-smooth function $\xi(\cdot) : \mathbb{R}_0^+ \to \mathbb{R}^n$ if it satisfies $E_{\sigma(t)}(\xi(t))\dot{\xi}(t) = f_{\sigma(t)}(\xi(t))$ for all $t \in \mathbb{R}_0^+$. We denote $\xi(\tau, x_0, \sigma)$ as the state at time τ from initial state x_0 with switching signal σ. Particularly, it is written as $\xi(\tau, x_0, p)$ when $\sigma \equiv p \in P$. To characterize the jump, we use *consistency projector* Π_p, $p \in P$:

$$\Pi_p : \bigcup_{p' \in P} \mathfrak{C}_{p'} \to \mathfrak{C}_p, \quad x(t^-) \mapsto x(t^+).$$

Furthermore, for any state of the solution, there is

$$x(t^+) = \Pi_{\sigma(t)}(x(t^-)).$$

The above results also apply to linear switched DAEs

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \tag{3}$$

In this case, the exsistence of unique solutions is equivalent to the regularity of each matrix pair (E_p, A_p). The sufficient and necessary condition of being regular is the exsistence of invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SE_pT,\, SA_pT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right), \tag{4}$$

where N is nilpotent. In particular, the original DAEs can be easily decomposed into differential and algebraic part with the help of *Wong Sequences* [16]. Then the consistency projector of mode $p \in P$ is computed as

$$\Pi_p = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

The image space of Π_p constitutes the consistency space $\mathfrak{C}_p$. We can also define *flow matrix* by

$$A_p^d := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

If mode p is switched on at time $t = 0$, and lasts for a time interval τ, then the state of the linear DAE for all $t \in [0, \tau]$ is

$$\xi(t) = e^{A_p^d t}\xi(0^+) = e^{A_p^d t}\Pi_p\xi(0^-).$$

III. CONTROL PROBLEM FORMULATION

A. Formal Specification

Our objective is to design a hybrid controller from some high-level specifications for switched DAEs. Linear temporal logic (LTL) is used as the formal language to describe the specifications in our work. LTL is able to express most of the common system properties, including safety, reachability, invariance and a combination of these.

LTL is built upon atomic propositions ϕ, logical and temporal operators. Logical operators include $\neg$ (negation), $\vee$ (disjunction), and $\wedge$ (conjunction), while $\Box$ (always), $\Diamond$ (eventually), $\bigcirc$ (next) and $\mathcal{U}$ (until) are temporal operators. An LTL formula φ is formed by connecting a finite set of atomic propositions with these operators.

In this paper, we consider two LTL propositional formulas ϕ_t and ϕ_s, which represents the targeted and safe set of states respectively. The desired controlled system trajectories should enter the target set in finite time and remain within it while always satisfying the safety constraints. The corresponding LTL formula is

$$\varphi := \Box\phi_s \wedge \Diamond\Box\phi_t. \tag{5}$$

We can further define $\varphi_s := \Box\phi_s$ and $\varphi_t := \Diamond\Box\phi_t$.

B. Problem Statement

A special feature of switched DAEs is the presence of jumps at switching, which is not desirable in most of the applications, thus this paper aims to find a switching control strategy that minimizes the overall jumps.

Consider (1) with a piecewise constant switching signal σ, which is defined with respect to a switching time sequence $\{t_1, t_2, \ldots\}$, and $\sigma(t) = \sigma_k \in P$ for $\forall t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. A trajectory satisfies (5) can either asymptotically converge to a stable point or experiences periodical switching inside ϕ_t.

Let N_c be the maximum number of switches before repeated switches appear, we define the cost function of such a trajectory as

$$J(\sigma, x) := \sum_{k=1}^{N_c} \|\Pi_{\sigma_k}(x(t_k-)) - x(t_k-)\|, \tag{6}$$

which, in linear case, reduces to

$$J(\sigma, x) := \sum_{k=1}^{N_c} \|(I - \Pi_{\sigma_k})\Pi_{\sigma_{k-1}}x(t_k-)\|. \tag{7}$$

Problem 1: Given switched DAEs in the form of (1) or (3) under impulse free assumption, and an LTL specification (5), find (if there exists) an optimal switching strategy σ^* such that

$$\sigma^* = \underset{\sigma\in\Sigma_\varphi}{\operatorname{argmin}} \quad J(\sigma, x),$$

where Σ_φ is the set of switching strategies that render solutions of (1) or (3) satisfy (5), and $J(\sigma, x)$ is given in (6) or (7).

C. Hybrid Controller Design Framework

The abstraction-based control approach has been explored in various settings, e.g. switching protocols synthesis problem [7] and robot planning problem [5]. We propose a hybrid controller design framework, which is given in Fig. 1 along with a resulting online controller. First, a finite-state approximation for system (1) or (3) is constructed by partitioning each consistency space as well as building the transitions. Second, an optimal switching strategy is obtained by solving a discrete synthesis problem encoded with given specifications. Finally, the strategy is implemented as an online controller. The details of each step will be explained in the following sections.

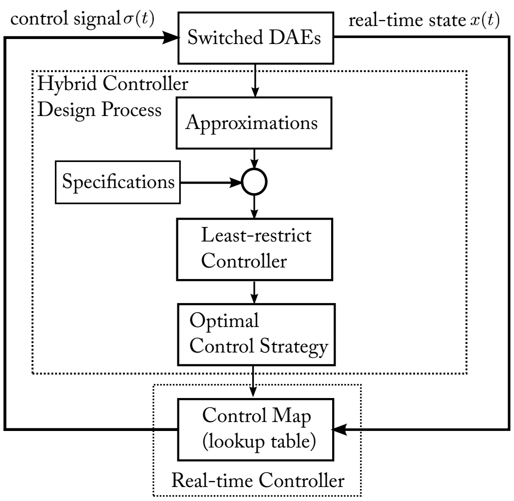

Fig. 1. Hybrid Controller design structure

IV. FINITE-STATE APPROXIMATION OF SWITCHED DAEs

A. Finite Transition System

We consider a family of *transition system* defined as a tuple $\mathcal{T} = (\mathcal{Q}, \mathcal{Q}_0, \mathcal{A}, \rightarrow_{\mathcal{T}}, \mathcal{Y}, h)$. $\mathcal{Q}$ is a set of states with $\mathcal{Q}_0 \subseteq \mathcal{Q}$ as the initial states; $\mathcal{A}$ is a set of actions; $\rightarrow_{\mathcal{T}} \subseteq \mathcal{Q} \times \mathcal{A} \times \mathcal{Q}$ is a transition relation; $\mathcal{Y}$ is a set of observations, and $h : \mathcal{Q} \to \mathcal{Y}$ is an observation map on the states. $\mathcal{T}$ is said to be finite (inifinite) if $\mathcal{Q}$, $\mathcal{A}$ and $\mathcal{Y}$ are finite (infinite); it is metric if $\mathcal{Y}$ is equipped with a metric d.

Specifically in switched DAEs, there are two types of transitions—continuous evolution inside a single mode and state jump between modes. Suppose the system is switched from mode p to p' at a switching time t_k, and then evolves in mode p' for a time duration τ, there are

$$q'' = \Pi_{p'}(q), \tag{8}$$

$$q' = \xi(\tau, q'', p'). \tag{9}$$

The resulting system state transition is from q to q' passing by an intermediate state q'', $q \in \mathfrak{C}_p$, $q', q'' \in \mathfrak{C}_{p'}$. (8) captures the state jump while (9) indicates the continuous trajectory in mode p'.

Given a sampling time $\tau_s \in \mathbb{R}^+$, and the control signal σ is restricted to be constant in this duration, which means that switching only takes place at times $t_k = k\tau_s, k \in \mathbb{N}$. The system (1) or (3) can be formulated as a transition system $\mathcal{T}_{\tau_s}(E_\sigma, f_\sigma) = (\mathcal{Q}, \mathcal{Q}_0, \mathcal{A}, \rightarrow_{\mathcal{T}}, \mathcal{Y}, h)$ with

- $\mathcal{Q} := \bigcup_{p\in P} \mathfrak{C}_p$, $\mathcal{Q}_0 \subseteq \mathcal{Q}$;
- $\mathcal{A} := P$;
- $(q, p, q') \in \rightarrow_{\mathcal{T}}$ iff $\xi(\tau_s, q, p) = q'$;
- $\mathcal{Y} := \mathcal{Q}$, $h(q) = q$, $q \in \mathcal{Q}$;

With (8) and (9), we can apply a two-step transition $q \xrightarrow{p,0} q'' \xrightarrow{p,\tau_s} q'$ instead of $(q, p, q') \in \rightarrow_{\mathcal{T}}$ as an indication that system is able to evolve from state q to q' under the control signal $\sigma(t) = p, \forall t \in [t_k, t_{k+1})$.

In order to apply the hybrid control framework, we have to construct a finite transition system for switched DAEs. It is achieved by discretizing the consistency space $\mathfrak{C}_p$ with a fixed granularity $\eta_p \in \mathbb{R}^n$, $p \in P$. Furthermore, we rely on an *incrementally jump-boundedness* property given below to define the finite transition system.

Definition 1: The system (1) or (3) is called *incrementally jump bounded* if for any modes $p, p' \in P$, there exists a function $\gamma_{p,p'}$ of $\mathcal{K}_\infty$ class such that for all $x, y \in \mathfrak{C}_p$, the following condition

$$|\Pi_{p'}(x) - \Pi_{p'}(y)| \le \gamma_{p,p'}(|x - y|) \tag{10}$$

is fulfilled.

Definition 2: Given a system (1) or (3), its finite transition system with the parameters $\tau_s \in \mathbb{R}^+$ and $\eta_p \in \mathbb{R}^n, p \in P$ is defined as $\hat{\mathcal{T}}_{\eta,\tau_s}(E_\sigma, f_\sigma) = (\hat{\mathcal{Q}}, \hat{\mathcal{Q}}_0, \hat{\mathcal{A}}, \rightarrow_{\hat{\mathcal{T}}}, \hat{\mathcal{Y}}, \hat{h})$ where

- $\hat{\mathcal{Q}} := \bigcup_{p\in P} [\mathfrak{C}_p]_{\eta_p}$, and the initial set $\hat{\mathcal{Q}}_0 \subseteq \hat{\mathcal{Q}}$;
- $\hat{\mathcal{A}} := P$;
- $\hat{q} \xrightarrow{p',0} \hat{q}'' \xrightarrow{p',\tau_s} \hat{q}'$ iff for $\forall p, p' \in \hat{\mathcal{A}}$ and $\hat{q} \in [\mathfrak{C}_p]_{\eta_p}$, $\hat{q}'', \hat{q}' \in [\mathfrak{C}_p]_{\eta_{p'}}$, there are $|\hat{q}'' - \Pi_{p'}(\hat{q})| \le \gamma_{p,p'}(\eta_p/2) + \eta_{p'}/2$, and $|\hat{q}' - \xi(\tau_s, \hat{q}'', p')| \le \eta_{p'}/2$;
- $\hat{\mathcal{Y}} := \hat{\mathcal{Q}}$, $\hat{h} = h|_{\hat{\mathcal{Q}}}$ with $d(x, y) = |x - y|$;

In some cases, there will be intersections between different consistency spaces, which leads to repeated discretizing on common part. In the definition of $\hat{\mathcal{Q}}$, we do not merge the common state space due to the importance of knowing within which mode the current state is when it falls in the intersection, and it also enable us to apply different partition parameters for different consistency spaces. Regarding the definition of $\rightarrow_{\hat{\mathcal{T}}}$, this finite transition system is nondeterministic, meaning that it is possible to have more than one successors under the same switching command from a starting state.

B. Over-approximation Relation

$\hat{\mathcal{T}}_{\eta,\tau_s}(E_\sigma, f_\sigma)$ is said to be an *ϵ-over-approximation* of switched DAEs $\mathcal{T}_{\tau_s}(E_\sigma, f_\sigma)$ if there is a transition $\hat{q} \xrightarrow{p',0} \hat{q}'' \xrightarrow{p',\tau_s} \hat{q}'$ whenever there exists a trajectory $\xi : [0, \tau_s) \to \mathbb{R}^n$ with $q = \xi(0^-), q' = \xi(\tau_s, q, p')$ such that $|\hat{q} - q| \le \epsilon$ and $|\hat{q}' - q'| \le \epsilon$.

The computation of over-approximation relation relies on *incrementally globally aymptotically stable property* (δGAS) [8], [9] of each subsystem, which requires that

$$|\xi(t,x,p)-\xi(t,y,p)| \le \beta_p(|x-y|,t), \tag{11}$$

for all x, $y \in \mathfrak{C}_p$ and all $t \ge 0$. β_p is a function of class $\mathcal{KL}$.

One way to find such function β_p is using Lyapunov-like techniques. Extending to DAE systems, we give the definition of the corresponding δGAS Lyapunov function.

Definition 3: A non-negative smooth function $V_p : \mathfrak{C}_p \times \mathfrak{C}_p \to \mathbb{R}_0^+$ is called a δGAS Lyapunov function for each DAE subsystem (E_p, f_p) if there exist functions $\underline{\alpha}_p$, $\overline{\alpha}_p$ of class $\mathcal{K}_\infty$ and $\kappa_p \in \mathbb{R}^+$, such that for all x, $y \in \mathfrak{C}_p$, it fulfills the following properties:

1) $V(x,y) = 0 \iff |x-y| = 0$, and it is bounded by $\underline{\alpha}_p(|x-y|) \le V_p(x,y) \le \overline{\alpha}_p(|x-y|)$;
2) $\frac{\partial V}{\partial x} \cdot z + \frac{\partial V}{\partial y} \cdot w \le -\kappa_p V_p(x,y), \quad \forall z \in \mathcal{T}_x(\mathfrak{C}_p) \cap E^{-1}(x)f(x), \forall w \in \mathcal{T}_y(\mathfrak{C}_p) \cap E^{-1}(y)f(y)$,

where $\mathcal{T}_x(\mathfrak{C}_p)$ denotes the tangent space of $\mathfrak{C}_p$ at $x \in \mathfrak{C}_p$, and $E^{-1}(x)$ is the pseudoinverse of $E(x)$.

When all the subsystems of the switched DAEs are regular, the function $V_p := (x-y)^T E_p^T P_p E_p (x-y)$ is a δGAS Lyapunov function if there exist symetric positive definite matrices P_p and Q_p satisfying $A_p^T P_p E_p + E_p^T P_p A_p = -Q_p$.

We additionally assume that every subsystem satisfies $V_{p'}(\Pi_{p'}(x), \Pi_{p'}(y)) \le \mu_{p,p'} V_p(x,y)$ for all $x, y \in \mathfrak{C}_p$, $p, p' \in P$, and $\mu_{p,p'} \ge 1$. In linear case (3), $\mu_{p,p'}$ can be estimated by [2]

$$\mu_{p,p'} = \frac{\lambda_{\max}(O_p^T \Pi_{p'}^T E_{p'}^T P_p E_{p'} \Pi_{p'} O_p)}{\lambda_{\min}(P_p^T E_p T P_p E_p O_p)}, \tag{12}$$

where $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimal and maximal eigenvalue of a symmetric matrix respectively, $O_p = \mathrm{im}\Pi_p = \mathfrak{C}_p$ which is the image of Π_p.

We are now ready to present how to construct a finite transition system that is an ϵ-over-approximation. Fig. 2 shows the transition relations between consistency space $\mathfrak{C}_p$ and $\mathfrak{C}_{p'}$.

Theorem 1: Consider a switched DAE system (1) or (3) with property (10) and (11). The finite transition system $\hat{\mathcal{T}}_{\eta,\tau_s}(E_\sigma, f_\sigma)$ as in Definition 2 is an ϵ-over-approximation of $\mathcal{T}_{\tau_s}(E_\sigma, f_\sigma)$ if

$$\beta_p(\frac{\eta_p}{2}, \tau_s) + \frac{\eta_p}{2} \le \epsilon \tag{13}$$

for all $p \in P$ are satisfied. Furthermore, if there exists a δGAS Lyapunov function with upper bound $\overline{\alpha}_p$ and lower bound $\underline{\alpha}_p$ for each DAE subsystem, then the parameters can be estimated by

$$\beta_p(\frac{\eta_p}{2}, \tau_s) = \underline{\alpha}_p^{-1}(e^{-\kappa_p \tau_s} \overline{\alpha}_p(\frac{\eta_p}{2})), \tag{14}$$

$$\gamma_{p,p'}(\frac{\eta_p}{2}) = \underline{\alpha}_{p'}^{-1}(\mu_{p,p'} \overline{\alpha}_p(\frac{\eta_p}{2})). \tag{15}$$

Proof: We have to show that if there is a trajectory of time τ_s in the original system, one can always find a corresponding transition in $\hat{\mathcal{T}}_{\eta,\tau_s}(E_\sigma, f_\sigma)$. It can be proved by looking at one transition from state $\hat{q}$ in mode p to $\hat{q}'$ in mode p' under the switching signal $\sigma \in \hat{\mathcal{A}}$ with $\sigma(0^-) = p$, $\sigma(t) = p'$, $\forall t \in [0, \tau_s)$.

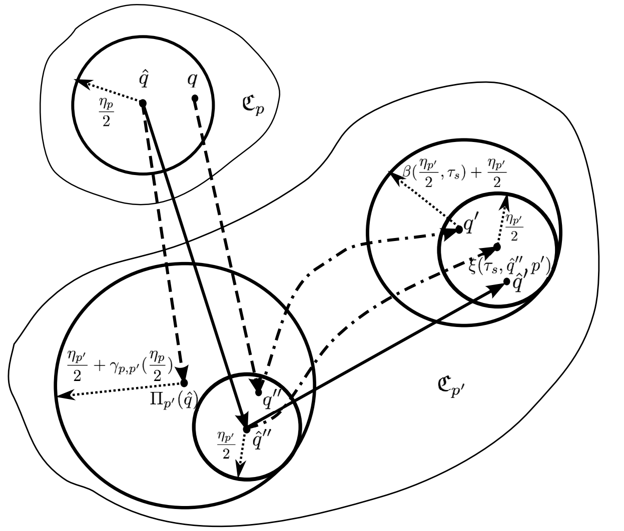

Fig. 2. Adding transition $\hat{q} \xrightarrow{p',0} \hat{q}'' \xrightarrow{p',\tau_s} \hat{q}'$ to $\to_{\hat{\mathcal{T}}}$. The first transition indicates the jump from mode p to p' while the second one represents the trajectory within mode p'.

First, we consider continuous evolution inside each mode. At mode p', by Definition 2 and property (11), given $\hat{q}'' \in [\mathfrak{C}_{p'}]_{\eta_{p'}}$, for any $q'' \in \mathfrak{C}_{p'}$ with $|q'' - \hat{q}''| \le \eta_p/2 \le \epsilon$, $q' = \xi(\tau_s, q, p')$, there is

$$\begin{aligned}|q' - \hat{q}'| &= |\xi(\tau_s, q, p') - \xi(\tau_s, \hat{q}'', p') + \xi(\tau_s, \hat{q}'', p') - \hat{q}'| \\ &\le |\xi(\tau_s, q, p') - \xi(\tau_s, \hat{q}'', p')| + |\xi(\tau_s, \hat{q}'', p') - \hat{q}'| \\ &\le \beta_p(|\hat{q}'' - q''|, \tau_s) + \frac{\eta_p}{2} \le \beta_p(\frac{\eta_p}{2}, \tau_s) + \frac{\eta_p}{2}.\end{aligned}$$

Thus, according to (13), $|q' - \hat{q}'| \le \epsilon$.

Next, we analyze the jump at mode switching from p to p'. Given $\hat{q} \in [\mathfrak{C}_p]_{\eta_p}$, for any $q \in \mathfrak{C}_p$ fulfills $|q - \hat{q}| \le \eta_p/2$, applying (10), then

$$|q'' - \Pi_{p'}(\hat{q})| \le \gamma_{p,p'}(|q - \hat{q}|) \le \gamma_{p,p'}(\frac{\eta_p}{2}).$$

In consistency space $[\mathfrak{C}_{p'}]_{\eta_{p'}}$, there must exists a $\hat{q}''$ such that $|\hat{q}'' - q''| \le \eta_{p'}/2$. Hence

$$\begin{aligned}|\hat{q}'' - \Pi_{p'}(\hat{q})| &\le |\hat{q}'' - q''| + |q'' - \Pi_{p'}(\hat{q})| \\ &\le \frac{\eta_{p'}}{2} + \gamma_{p,p'}(\frac{\eta_p}{2}).\end{aligned}$$

According to the Definition 2, $\hat{q}''$ is covered, and thus implies the existence of $\hat{q} \xrightarrow{p',0} \hat{q}''$ that connects the transition in the following space $\mathfrak{C}_{p'}$. So far, we have shown that all the trajectories in original system can be approximated in the finite transition system.

When a δGAS Lyapunov function V_p exists for subsystem p, $\forall p \in P$, Definition 3 gives

$$\begin{aligned}V_p(\hat{q}', q') &= V_p(\xi(\tau_s, \hat{q}'', p), \xi(\tau_s, q'', p)) \\ &\le e^{-\kappa_p \tau_s} V_p(\xi(0, \hat{q}'', p), \xi(0, q'', p)) \\ &\le e^{-\kappa_p \tau_s} \overline{\alpha}_p(|\hat{q}' - q'|) \le e^{-\kappa_p \tau_s} \overline{\alpha}_p(\frac{\eta_p}{2})).\end{aligned}$$

Therefore, for all $t \geq 0$

$$|\xi(\tau_s, \hat{q}'', p) - \xi(\tau_s, q'', p)| \leq \underline{\alpha}_p^{-1}(V_p(\hat{q}', q')) \leq \underline{\alpha}_p^{-1}(e^{-\kappa_p \tau_s}\overline{\alpha}_p(\frac{\eta_p}{2}))).$$

Under the assumption that for all $p, p' \in P$, there exists $\mu_{p,p'} \geq 1$, such that $\forall x, y \in \mathfrak{C}_p : V_{p'}(\Pi_{p'}(x), \Pi_{p'}(y)) \leq \mu_{p,p'} V_p(x, y)$, we get

$$V_{p'}(\hat{q}'', q'') \leq \mu_{p,p'} V_p(\hat{q}, q) \leq \overline{\alpha}_p(|\hat{q}'' - q''|) \leq \overline{\alpha}_p(\frac{\eta_p}{2}).$$

Then

$$|\hat{q}'' - q''| \leq \underline{\alpha}_{p'}^{-1}(\mu_{p,p'}\overline{\alpha}_p(\frac{\eta_p}{2})).$$

Thus, (14) and (15) hold. ■

V. CONTROLLER SYNTHESIS

The goal of this section is to design a memoryless controller that solves Problem 1. It is defined as a map

$$g : \mathcal{Q}_\varphi \to \Sigma_\varphi, \qquad (16)$$

where $\mathcal{Q}_\varphi \subseteq \cup_{p \in P} \mathfrak{C}_p$ is the largest set of initial states from which the specification φ can be satisfied and Σ_φ is the corresponding set of control signals. For sampled-data system with fixed sampling time τ_s, the control signal keeps constant in this time duration.

We define $\langle \mathcal{Q}_\varphi, \Sigma_\varphi \rangle$ as the least-restrictive controller that renders the system satisfies the control specification φ and has to be obtained at first. It gives all the possible control solutions from which we have to use optimal control strategy that minimizes the cost function (6) or (7).

To perform control synthesis in discrete level, we represent the finite-state approximation $\hat{\mathcal{T}}_{\eta,\tau_s}(E_\sigma, f_\sigma)$ as a weighted directed graph $G = (V, R)$ with the set of vertices V and edges R. Each vertex indicates a state $\hat{q} \in \hat{\mathcal{Q}}$, and the edge denotes the transition between two corresponding states. In particular, each edge is assigned a weight indicating the magnitude of the jump of this transition. By filtering out the forbidden vertices and edges of the original graph G according to the given specification, $\langle \mathcal{Q}_\varphi, \Sigma_\varphi \rangle$ can be obtained as a subgraph. Then the optimal control strategy generation is equivalent to searching for the shortest path.

The specification φ in the form of (5) can be decomposed into two fundamentals φ_s and φ_t. Invariant set $\mathcal{S}_s$ is the set of states from which the trajectories will stay within and thus satisfies φ_s. φ_t is associated with two sets $\mathcal{S}_t$, $\mathcal{S}_r$, which denotes the set of states in ϕ_t that can always stay in ϕ_t and the set of states that can reach $\mathcal{S}_t$ respectively. All these sets are computed using fixed point algorithm [17].

According to Problem 1, the optimal control strategy is the one that minimizes the overall jumps or the non-repeated jumps if the system cannot avoid periodical jumping. Synthesizing on the finite-state abstraction, we search for the shortest paths from $\mathcal{S}_r$ to $\mathcal{S}_t$ on the subgraph G_r and G_t pruned from G. Since $\mathcal{S}_t$ is finite, the accepted run must ends at a cycle path in G_t, and thus the optimal path is a straight section d_{str} connected with a cycle section d_{cyc} with the minimum total weighted distance. It is possible that not all states in $\mathcal{S}_t$ are on cycles. We separate it into two subsets $\mathcal{S}_{t1}$ (on cycles) and $\mathcal{S}_{t2}$ (not on cycles). Dijkstra algorithm is used to find the shortest paths as well as the shortest cycles. Algorithm 1 shows the overall computational procedure generating optimal controller for switched DAEs. It returns a controller $\mathcal{C} : \mathcal{Q}_\varphi \to P$. $d(s)$ represents the mode to which the system should be switched at state s in order to stay on the path d.

Algorithm 1 Optimal controller synthesis

Inputs: $\hat{T}_{\eta,\tau_s}(E_\sigma, f_\sigma)$, ϕ_s, ϕ_t

```
G = WeightedGraph(T̂_{η,τs}(E_σ, f_σ))
S_s ← InvariantSet(G, φ_s), S_t ← InvariantSet(G, φ_t)
S_r ← ReachSet(G, S_t), S_w ← S_s ∩ S_r
G_r ← SubGraph(G, S_r), G_t ← SubGraph(G, S_t)
S_t1 ← ∅, S_t2 ← ∅, C ← ∅
for each s_t ∈ S_t
    if d_cyc ← ShortestCycle(G_t, s_t) ≠ ∅
        S_t1 ← s_t
        C(s_t) ← d_cyc(s_t)
    else
        S_t2 ← s_t
    end if
end for
ShortestPath(G_t, S_t2, S_t1, C)
ShortestPath(G_r, S_w \ S_t, S_t1, C)
```

Return: $\mathcal{C}$

Algorithm 2 $ShortestPath(G, \mathcal{S}_i, \mathcal{S}_g, \mathcal{C})$

```
for each s_i ∈ S_i
    for each s_g ∈ S_g
        L_min ← ∞
        d_cyc ← ShortestCycle(G, s_g)
        if ∃ d_str ← ShortestStraight(G, s_r, s_g)
            if Length(d_str) + Length(d_cyc) < L_min
                L_min ← Length(d_str) + Length(d_cyc)
                d_tmp ← d_str + d_cyc
            end if
        end if
    end for
    C(s_i) ← d_tmp(s_i)
end for
```

VI. EXAMPLE

In this section, we demonstrate our hybrid controller design process by a numerical example from [1].

Consider a two-mode switched DAEs in the form of (3) with

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right),$$

$$(E_2, A_2) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \right).$$

The corresponding consistency projectors and flow matrices are

$$\Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_1^d = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}.$$

$$\Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_2^d = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}.$$

Each subsystem is stable and the original point $(0, 0)$ is the only stable equilibrium for both subsystems. $\mathfrak{C}_1 = \text{im}\Pi_1$, $\mathfrak{C}_2 = \text{im}\Pi_2$ are two consistency spaces.

The system is desired to reach the targeted region $\phi_t = \{x \in \bigcup_{p=1,2} \mathfrak{C}_p | 0.3 \leq x_2 \leq 0.9\}$ while the trajectory keeps inside the safe region $\phi_s = [0, 5] \times [0, 5]$. We choose $\beta_p(\Delta x, t) = e^{-t}\Delta x$, $p = \{1, 2\}$, and $\gamma_{1,2}(\Delta x) = 2\Delta x$, $\gamma_{2,1} = \Delta x$.

To construct an ϵ-over-approximation, we set the precision $\epsilon = 0.1$, sampling time $\tau_s = 0.1$. According to (14) and (15), state space granularity η_1, η_2 can be chosen within $(0, 0.11]$. In this example, we use $\eta = \eta_1 = \eta_2 = 0.05$. We run Algorithm 1 to obtain the lookup table as an embedded online controller. Fig. 3 shows the simulation result of the controled system from initial state $(0, 0.15)$. The system reaches the target region in less than 2.2s, and stays inside afterwards. As a result of minimizing jumps, the trajectory inside the invariant region is closer to the lower bound 0.3 than the upper bound 0.9.

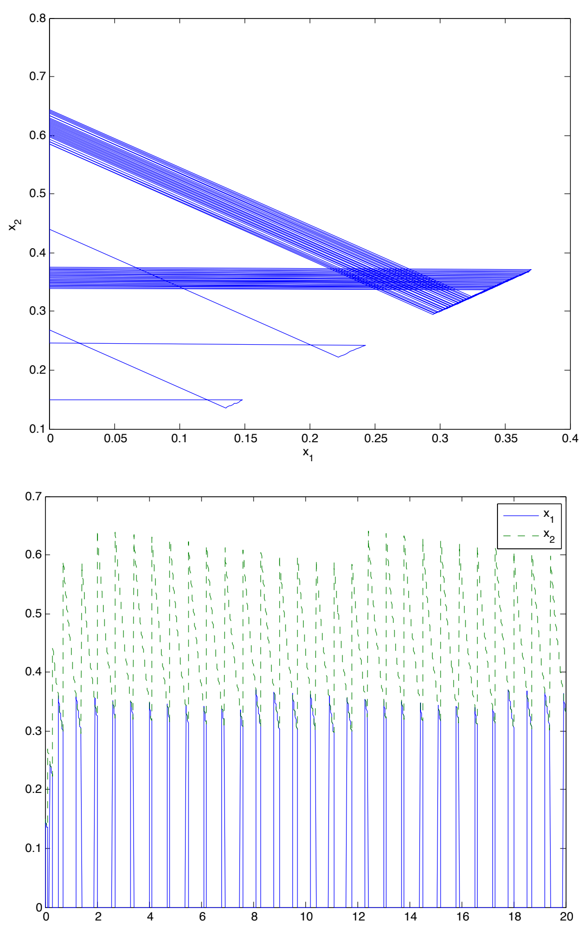

Fig. 3. Simulation result of the invented hybrid controller with initial state $(0, 0.15)$. The system state first jumps away from point $(0, 0)$ by switching, and once reach targeted region, keep staying there forever.

VII. CONCLUSIONS

In this paper, we extended the abstraction-based hybrid controller design approach to switched DAEs and dealt with its particular feature of jumps at switching. A problem of finding the control strategies that minimizes the overall magnitude of non-repeated jumps while satisfying a certain type of LTL specifications was considered. We showed that it is possible to obtain a finite-state approximation that can capture the original system behaviors by introducing a two-step transition relation. We designed a memoryless controller to solve the optimal control problem. Using the finite-state abstraction, the optimal switching strategy was equivalent to the shortest path to the goal area. The simulation result of a numerical example showed the feasibility and effectiveness of our method.

REFERENCES

[1] D. Liberzon and S. Trenn, "On stability of linear switched differential algebraic equations," in *Joint 48th IEEE Conf. Decision and Control and 28th Chinese Control Conference*, 2009, pp. 2156–2161.

[2] ——, "Switched nonlinear differential algebraic equations: solution theory, Lyapunov functions, and stability," *Automatica*, vol. 48, pp. 954–963, 2012.

[3] A. D. Domínguez-García and S. Trenn, "Detection of impulsive effects in switched daes with applications to power electronics reliability analysis," in *49th IEEE Conf. Decision and Control*, 2010, pp. 5662–5667.

[4] M. Althoff and B. H. Krogh, "Reachability analysis of nonlinear differential-algebraic systems," *IEEE trans. Automat. Control*, vol. 59, no. 2, pp. 371–383, Feb. 2014.

[5] M. Kloetzer and C. Belta, "A fully automated framework for control of linear systems from temporal logic specifications," *IEEE Trans. Automat. Control*, vol. 53, no. 1, pp. 287–297, Feb. 2008.

[6] P. Nilsson, O. Hussien, Y. Chen, and et al., "Preliminary results on correct-by-construction control software synthesis for adaptive cruise control," in *53rd IEEE Conf. Decision and Control*, 2014.

[7] J. Liu, N. Ozay, U. Topcu, and R. M. Murray, "Synthesis of reactive switching protocols from temporal logic specifications," *IEEE Trans. Automat. Control*, vol. 58, no. 7, pp. 1771–1785, 2013.

[8] A. Girard, G. Pola, and P. Tabuada, "Approximately bisimilar symbolic models for incrementally stable switched systems," *IEEE Trans. Automat. Control*, vol. 55, no. 1, pp. 116–126, 2010.

[9] G. Pola, A. Girard, and P. Tabuada, "Approximately bisimilar symbolic models for nonlinear control systems," *Automatica*, vol. 44, no. 10, pp. 2508–2516, 2008.

[10] M. Zamani, G. Pola, M. M. Jr., and P. Tabuada, "Symbolic models for nonlinear control systems without stability assumptions," *IEEE Trans. Automat. Control*, vol. 57, no. 7, pp. 1804–1809, 2012.

[11] J. Liu and N. Ozay, "Abstraction, discretization, and robustness in temporal logic control of dynamical systems," in *17th International Conference on Hybrid Systems: Computation and Control (part of CPS Week), HSCC'14, Berlin, Germany, April 15-17, 2014*, M. Fränzle and J. Lygeros, Eds. ACM, 2014, pp. 293–302.

[12] L. C. G. J. M. Habets, P. J. Collins, and J. H. van Schuppen, "Reachability and control synthesis for piecewise-affine hybrid systems on simplices." *IEEE Trans. Automat. Control*, vol. 51, no. 6, pp. 938–948, 2006.

[13] A. Girard and S. Martin, "Synthesis for constrained nonlinear systems using hybridization and robust controllers on simplices," *IEEE Trans. Automat. Control*, vol. 57, no. 4, pp. 1046–1051, 2012.

[14] N. Ozay, J. Liu, P. Prabhakar, and R. M. Murray, "Computing augmented finite transition systems to synthesize switching protocols for polynomial switched systems," in *American Control Conference (ACC), 2013*. IEEE, 2013, pp. 6237–6244.

[15] H. K. Khalil and J. Grizzle, *Nonlinear systems*. Prentice Hall, 2002.

[16] T. Berger, A. Ilchmann, and S. Trenn, "The quasi-weierstraß form for regular matrix pencils," *Linear Algebra and Its Applications*, vol. 436, no. 10, pp. 4052–4069, 2012.

[17] P. Tabuada, *Verification and Control of Hybrid Systems - A Symbolic Approach*. Springer, 2009.