Computing maximal invariant sets for switched nonlinear systems

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Abstract— **Invariant sets play a fundamental role in the analysis and control design of dynamical systems. In this paper, we consider computation of maximal invariant sets for discrete-time switched nonlinear systems. Switched nonlinear systems are described as a family of nonlinear systems with a discrete variable that decides which mode will be active at a specific time. We consider two scenarios: one leads to the computation of switching controlled invariant sets and the other leads to computation of invariant sets subject to arbitrary switching. The main result of the paper establishes that the outer approximations computed using interval analysis indeed converge to the maximal invariant sets for general switched nonlinear systems without any stability assumptions. The proposed method is illustrated with nonlinear systems with polynomial dynamics.**

I. INTRODUCTION

For a dynamical system, a subset of the state space is called (positively) invariant if the state evolution of the system starting from an initial state within this set remains in this set for all (future) time. Invariant sets play an important role in dynamical systems analysis and control. For this reason, they have been investigated extensively within the systems and control community (see, e.g., [1]–[6] and, in particular, the excellent survey paper [7] and references therein).

Traditionally, invariant sets are closely related to stability analysis of dynamical systems by Lyapunov functions. In particular, the sublevel sets of Lyapunov functions naturally define positively invariant sets and are often used for estimating the domain of attraction for dynamical systems. As is well known, the construction of Lyapunov functions for nonlinear systems is a very difficult problem in general [8].

In practice, computational methods are often sought to approximate invariant sets of interest. Ellipsoidal and polyhedral sets [7] received special attention in the literature, namely . Ellipsoids are popular because of their close relationship with linear problems and quadratic forms. Polyhedral sets are routinely used in control because many physical constraints can be naturally described by polyhedral sets. Moreover, both ellipsoidal and polyhedral sets are convex and hence technically more appealing. There is, however, one major drawback in computing invariant sets based on ellipsoidal or polyhedral sets—the shape of such sets has to be preserved during the set computation. While this is possible for linear systems [5], it is general not the case for nonlinear systems. Computing invariant sets for nonlinear systems remains a challenging problem because of strong mathematical difficulties. Recent research on symbolic approach, which is based on bisimilar discrete abstractions of nonlinear dynamics, provides an algorithmic solution to this question for the systems that are incrementally stable [9], [10]. Abstractions without this stability assumption [11] have no guarantee on finding the maximal controlled invariant set even if it exists for the original system.

In this paper, we focus on switched systems with nonlinear dynamics. Switched systems are an important class of hybrid dynamical systems that have found numerous applications in many fields (see [12]). We propose to compute invariant sets for switched nonlinear systems by interval methods. Interval analysis, or interval computation, refers to computational methods that use interval arithmetic with the aim to yield reliable results. Such methods have been developed since the 1960s [13] and successfully applied in different fields [14].

The main contribution of this paper is to provide an interval-based computational method that can arbitrarily approximate maximal controlled invariant sets within a given compact set of states for switched nonlinear systems. It is given in Algorithm 1. No further assumptions on the shape of sets need to be made, since any compact sets can be arbitrarily approximated by finite unions of intervals (because of the finite covering theorem). This enables us to prove some general convergence results without any stability assumptions on the nonlinear dynamics, in contrast with most results that require asymptotic stability [3], [5], [6], [9], [10] or controllability assumptions [15], even for linear systems. This conclusion is presented in Proposition 7.

The rest of the paper is organized as follows. In Section II, we introduce switched systems and define invariant sets. In Section III, we prove that the maximal invariant sets can be characterized as the set limits of sequences. In Section IV, we present an interval method for computing outer approximations of maximal invariant sets and prove its convergence. In Section V, we illustrate the results using polynomial dynamical systems.

II. PRELIMINARIES

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $|\cdot|$ denotes a specific norm in $\mathbb{R}^n$; given $\varepsilon \geq 0$ and $x \in \mathbb{R}^n$, define $\mathcal{B}_\varepsilon(x) := \{y \in \mathbb{R}^n : |y - x| \leq \varepsilon\}$;

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given $y \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, the distance from y to A is defined by $d(y, A) := \inf_{x\in A} |y - x|$; given two sets $A, B \subset \mathbb{R}^n$, $B \setminus A := \{x \in B : x \notin A\}$; an interval vector (or simply an interval) in $\mathbb{R}^n$ is denoted by $[x]$, where $[x] := [x_1] \times \cdots \times [x_n] \subset \mathbb{R}^n$ and $[x_i] = [\underline{x}_i, \overline{x}_i] \subset \mathbb{R}$ for $i = 1, \cdots, n$; the width of the interval $[x]$ is defined as $w([x]) := \max_{1\le i\le n}\{\overline{x_i} - \underline{x_i}\}$; the set of all interval vectors in $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$.

A. Discrete-time switched systems

We consider discrete-time switched systems of the form:

$$x_{k+1} = f_{p_k}(x_k), \quad k \in \mathbb{Z}_{\ge 0}, \tag{1}$$

where x_k, $x_{k+1} \in \mathbb{R}^n$ denote the system states at time k and $k+1$, respectively, and $p_k \in \mathcal{P}$ is the mode of the system at time k. It is assumed that the set of modes $\mathcal{P}$ is finite. The family of functions $\{f_p\}_{p\in\mathcal{M}} : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be continuous, and they determine the nonlinear dynamics for all subsystems.

Any infinite sequence in $\mathcal{P}$ defines a *switching signal* for system (1). We often denote a particular switching signal by $\sigma := \{p_k\}_{k=0}^{\infty}$, where $p_k \in \mathcal{P}$ for all $k \ge 0$.

Given a switching signal $\sigma := \{p_k\}_{k=0}^{\infty}$ and an initial state $x_0 \in \mathbb{R}^n$, the solution of system (1) is the unique sequence $\{x_k\}_{k=0}^{\infty}$ in $\mathbb{R}^n$ such that (1) is satisfied.

B. Invariant sets

To define invariant sets for (1), we need to specify whether the switching mode is treated as a control variable or an uncontrolled disturbance. To spell out this difference, let us consider the following two definitions.

Definition 1: A set $\Omega \subset \mathbb{R}^n$ is said to be a *controlled invariant set* for system (1) if, for any initial state $x_0 \in \Omega$, there exists a switching signal σ such that the resulting solution $\{x_k\}_{k=0}^{\infty}$ of (1) satisfies $x_k \in \Omega$ for all $k \ge 0$.

Definition 2: A set $\Omega \subset \mathbb{R}^n$ is said to be an *arbitrarily invariant set* for system (1) if, for any initial state $x_0 \in \Omega$ and any switching signal σ, the resulting solution $\{x_k\}_{k=0}^{\infty}$ of (1) satisfies $x_k \in \Omega$ for all $k \ge 0$.

Clearly, when the set of modes $\mathcal{P}$ is a singleton, the above definitions reduce to the same one; that is, invariant sets for a nonlinear system $x_{k+1} = f(x_k)$. Note that we only consider what are known as *positively invariant sets*, i.e. sets that are invariant with respect to positive time. For brevity, they are referred to simply as invariant sets in this paper.

C. Maximal invariant sets

Given a set $\Omega \subset \mathbb{R}^n$, it is often of interest to characterize the subset within Ω from which the solutions of (1) can remain in Ω. This is know as the maximal invariant set in Ω. Let us consider the following definitions.

Definition 3: Given a set $\Omega \subset \mathbb{R}^n$, define the set $\mathcal{R}^\infty(\Omega) \subset \Omega$ as follows: $x_0 \in \mathcal{R}^\infty(\Omega)$ if and only if there exists a switching signal σ such that the resulting solution $\{x_k\}_{k=0}^{\infty}$ of (1) satisfies $x_k \in \Omega$ for all $k \ge 0$.

Definition 4: Given a set $\Omega \subset \mathbb{R}^n$, define the set $\mathcal{W}^\infty(\Omega) \subset \Omega$ as follows: $x_0 \in \mathcal{W}^\infty(\Omega)$ if and only if, for any switching signal σ, the resulting solution $\{x_k\}_{k=0}^{\infty}$ of (1) satisfies $x_k \in \Omega$ for all $k \ge 0$.

It will be shown in Section III that the sets $\mathcal{R}^\infty(\Omega)$ and $\mathcal{W}^\infty(\Omega)$ are indeed invariant, and also maximal in the sense that they contain any other controlled and arbitrarily invariant sets in Ω, respectively.

D. Set convergence

We introduce set convergence in the Painlevé-Kuratowski sense [16, Chapter 4].

Definition 5: For a sequence $\{X_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}^n$. The *outer limit* of $\{X_n\}_{n=1}^{\infty}$ is defined by

$$\limsup_{n\to\infty} X_n = \left\{ x \in \mathbb{R}^n : \liminf_{n\to\infty} d(x, X_n) = 0 \right\}.$$

The *inner limit* of $\{X_n\}_{n=1}^{\infty}$ is defined by

$$\liminf_{n\to\infty} X_n = \left\{ x \in \mathbb{R}^n : \limsup_{n\to\infty} d(x, X_n) = 0 \right\}.$$

The *(set) limit* of $\{X_n\}_{n=1}^{\infty}$ exists if and only if the outer and inner limit sets are equal:

$$\lim_{n\to\infty} X_n = \limsup_{n\to\infty} X_n = \liminf_{n\to\infty} X_n.$$

III. Characterization of maximal invariant sets

In this section, we present some basic results that characterize maximal invariant sets of (1) in terms of the set limits of sequences of sets.

A. Invariance and maximality of $\mathcal{R}^\infty(\Omega)$ and $\mathcal{W}^\infty(\Omega)$

We first prove that the sets defined in Definitions 3 and 4 are indeed invariant and maximal.

Proposition 1: Given a set $\Omega \subset \mathbb{R}^n$, if the set $\mathcal{R}^\infty(\Omega)$ is nonempty, then it is the maximal controlled invariant set inside Ω in the sense that

(i) $\mathcal{R}^\infty(\Omega)$ is controlled invariant; and

(ii) if $\Omega' \subset \Omega$ is another controlled invariant set, then $\Omega' \subset \mathcal{R}^\infty(\Omega)$.

Proof: We first prove that $\mathcal{R}^\infty(\Omega)$ is controlled invariant. This follows directly from the definition. Given any $x_0 \in \mathcal{R}^\infty(\Omega)$, there exists a switching signal $\sigma = \{p_k\}_{k=0}^{\infty}$ such that the resulting solution $\{x_k\}_{k=0}^{\infty}$ of (1) satisfies $x_k \in \Omega$ for all $k \ge 0$. It follows that $x_k \in \mathcal{R}^\infty(\Omega)$ for all $k \ge 0$, because with switching signal $\{p_n\}_{n=k}^{\infty}$, the resulting solution $\{x_n\}_{n=k}^{\infty}$ satisfies $x_n \in \Omega$ for all $n \ge k$.

If $\Omega' \subset \Omega$ is another controlled invariant set, then for all $x_0 \in \Omega'$, there exists a switching signal such that the resulting solution $\{x_k\}_{k=0}^{\infty}$ of (1) satisfies $x_k \in \Omega' \subset \Omega$ for all $k \ge 0$. By the definition $\mathcal{R}^\infty(\Omega)$, this shows $x_0 \in \mathcal{R}^\infty(\Omega)$. Thus $\Omega' \subset \mathcal{R}^\infty(\Omega)$. ■

Similarly, we can prove the following result.

Proposition 2: Given a set $\Omega \subset \mathbb{R}^n$, if the set $\mathcal{W}^\infty(\Omega)$ is nonempty, then it is the maximal arbitrarily invariant set inside Ω in the sense that

(i) $\mathcal{W}^\infty(\Omega)$ is controlled invariant; and
(ii) if $\Omega' \subset \Omega$ is another arbitrarily invariant set, then $\Omega' \subset \mathcal{W}^\infty(\Omega)$.

B. Controlled and uncontrolled predecessors

To characterize invariant sets, we define two set mappings between subsets of $\mathbb{R}^n$.

Definition 6: Given a set $\Omega \subset \mathbb{R}^n$, the *controlled predecessor* of Ω with respect to system (1) is defined by

$$\mathrm{Pre}_\exists(\Omega) := \left\{x \in \mathbb{R}^n : \exists p \in \mathcal{P} \text{ such that } f_p(x) \in \Omega\right\}.$$

Definition 7: Given a set $\Omega \subset \mathbb{R}^n$, the *uncontrolled predecessor* of Ω with respect to system (1) is defined by

$$\mathrm{Pre}_\forall(\Omega) := \left\{x \in \mathbb{R}^n : f_p(x) \in \Omega, \forall p \in \mathcal{P}\right\}.$$

In other words, we have

$$\mathrm{Pre}_\exists(\Omega) = \bigcup_{p\in\mathcal{P}} f_p^{-1}(\Omega)$$

and

$$\mathrm{Pre}_\forall(\Omega) = \bigcap_{p\in\mathcal{P}} f_p^{-1}(\Omega),$$

where $f_p^{-1}(\Omega)$ is the pre-image of Ω by f_p, defined by $f_p^{-1}(\Omega) := \left\{x \in \mathbb{R}^n : f_p(x) \in \Omega\right\}$.

We have the following proposition. The proof is straightforward and hence omitted.

Proposition 3: Let $\Omega \subset \mathbb{R}^n$ and $A \subset B \subset \mathbb{R}^n$. Then
(i) $\mathrm{Pre}_\forall(\Omega) \subset \mathrm{Pre}_\exists(\Omega)$;
(ii) $\mathrm{Pre}_\exists(A) \subset \mathrm{Pre}_\exists(B)$ and $\mathrm{Pre}_\forall(A) \subset \mathrm{Pre}_\forall(B)$;
(iii) if Ω is closed, so are $\mathrm{Pre}_\exists(\Omega)$ and $\mathrm{Pre}_\forall(\Omega)$.

C. Characterization of invariance

Proposition 4: A set $\Omega \subset \mathbb{R}^n$ is controlled invariant for system (1) if and only if $\Omega \subset \mathrm{Pre}_\exists(\Omega)$.

Proof: If $\Omega \subset \mathrm{Pre}_\exists(\Omega)$, then, for any $x_0 \in \Omega \subset \mathrm{Pre}_\exists(\Omega)$, there exists $p_0 \in \mathcal{P}$ such that $f_p(x_0) \in \Omega$. Continuing this indefinitely will give a switching signal $\{p_k\}_{k=0}^\infty$ such that the resulting solution $\{x_n\}_{n=k}^\infty$ of (1) satisfies $x_n \in \Omega$ for all $n \geq k$. Hence Ω is controlled invariant.

If Ω is controlled invariant, by definition, for any $x_0 \in \Omega$, there exists a switching signal σ such that the resulting solution $\{x_k\}_{k=0}^\infty$ of (1) satisfies $x_k \in \Omega$ for all $k \geq 0$. In particular, $x_1 = f_{p_0}(x_0) \in \Omega$. This shows $x_0 \in \mathrm{Pre}_\exists(\Omega)$. Hence $\Omega \subset \mathrm{Pre}_\exists(\Omega)$. ∎

Similarly, we can prove the following result.

Proposition 5: A set $\Omega \subset \mathbb{R}^n$ is arbitrarily invariant for system (1) if and only if $\Omega \subset \mathrm{Pre}_\forall(\Omega)$.

D. Characterization of maximal invariance

We define two new mappings between subsets of $\mathbb{R}^n$ as follows:

$$R(X) = \mathrm{Pre}_\exists(X) \cap X, \quad W(X) = \mathrm{Pre}_\forall(X) \cap X \qquad (2)$$

where $X \subset \mathbb{R}^n$. Let R^n and W^n ($n \geq 1$) denote the n-times repeated compositions of the mappings R and W, respectively. By convention, $R^0(X) = W^0(X) = X$.

The following result characterizes the maximal controlled and arbitrarily invariant sets for system (1).

Proposition 6: Suppose Ω is closed. Then
(i) $\mathcal{R}^\infty(\Omega) = \lim_{n\to\infty} R^n(\Omega)$;
(ii) $\mathcal{W}^\infty(\Omega) = \lim_{n\to\infty} W^n(\Omega)$.

Proof: (i) First, it is straightforward to check that $R^n(\Omega)$ is a monotonically decreasing sequence of sets in the sense that $R^{n+1}(\Omega) \subset R^n(\Omega)$ for all $n \geq 1$. Moreover, $R^n(\Omega)$ is a closed set for all $n \geq 1$. Thus the set limit of $R^n(\Omega)$ exists and is given by $\bigcap_{n=1}^\infty R^n(\Omega)$ (see [16, p. 111]).

Second, it is easy to check that $\mathcal{R}^\infty(\Omega) \subset R^n(\Omega)$ for all $n \geq 1$ by induction, using the fact that $\mathcal{R}^\infty(\Omega)$, if nonempty, is a controlled invariant subset of Ω and Proposition 4. Hence, $\mathcal{R}^\infty(\Omega) \subset \bigcap_{n=1}^\infty R^n(\Omega) = \lim_{n\to\infty} R^n(\Omega)$.

Third, we claim that $\lim_{n\to\infty} R^n(\Omega) \subset \mathcal{R}^\infty(\Omega)$. If $\lim_{n\to\infty} R^n(\Omega)$ is empty, this trivially holds. If not, pick any $x_0 \in \lim_{n\to\infty} R^n(\Omega)$. Then $x_0 \subset R^n(\Omega)$ for all $n \geq 1$. It follows that there exists $p_n \in \mathcal{P}$ such that $f_{p_n}(x_0) \in R^{n-1}(\Omega)$ for all $n \geq 1$. Since $\mathcal{P}$ is finite, the sequence $\{p_n\}_{n=1}^\infty \subset \mathcal{P}$ must admit a constant subsequence. In other words, there exists $p \in \mathcal{P}$ such that $f_p(x_0) \subset R^{n-1}(\Omega)$ for infinitely many $n \geq 1$. By monotonicity of the sequence $R^n(\Omega)$, this implies $f_p(x_0) \in \bigcap_{n=1}^\infty R^n(\Omega) = \lim_{n\to\infty} R^n(\Omega)$. Hence $\lim_{n\to\infty} R^n(\Omega)$ is a controlled invariant subset of Ω. By Proposition 1, $\lim_{n\to\infty} R^n(\Omega) \subset \mathcal{R}^\infty(\Omega)$. Hence, $\mathcal{R}^\infty(\Omega) = \lim_{n\to\infty} R^n(\Omega)$ and the proof is complete.

(ii) The proof is similar to that for part (i) and hence omitted. ∎

Remark 1: The main result of this section, Proposition 6, appeared in the literature in various forms (see [1] for similar results proved in a more general setting and remarks after [15, Theorem 2] for a more detailed discussion). Here, by focusing on system (1), our aim is to present the section in a more concise and self-contained manner. To point out a slight difference, in Proposition 6, we only assume that Ω is closed, whereas in [1, Proposition 4] and remarks therein, it is assumed that Ω is compact. The difference lies in that we use set convergence in the Painlevé-Kuratowski sense [16, Chapter 4], compared with the usual Hausdorff metric used in [1].

IV. Interval approximations of maximal invariant sets

In this section, we present an interval method for computing outer approximations of the maximal invariant sets for system (1). More specifically, we use unions of intervals to contain and approximate the maximal invariant sets $\mathcal{R}^\infty(\Omega)$ and $\mathcal{W}^\infty(\Omega)$, for a given compact set $\Omega \subset \mathbb{R}^n$.

Let us first introduce a very useful notion routinely used in interval analysis.

Definition 8: Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$ and an interval function $[f] : \mathbb{IR}^n \to \mathbb{IR}^m$. The function $[f]$ is called a *convergent inclusion function* of f if the following two conditions hold:

- $f([x]) \subset [f]([x])$ for all $[x] \in \mathbb{IR}^n$;
- $\lim_{w([x]) \to 0} w([f]([x])) = 0$.

There are various methods to obtain such inclusion functions. The natural inclusion function, which is obtained by replacing variables and operations in the expression for f by their interval counterparts, is perhaps the simplest choice. Other types of inclusion functions include centered, mixed centered, and Taylor inclusion functions, with different convergence rates [14]. All the inclusion functions above are convergent under mild assumptions [17].

The following algorithm is inspired by the set inversion algorithm presented in [18]. Set inversion is a procedure to approximate the pre-image X of a set Y under a function f, i.e., to approximate $X = f^{-1}(Y)$. We modify the algorithm to compute the controlled predecessor $\mathrm{Pre}_{\exists}(\Omega)$ and use the iteration defined in (2) to obtain an arbitrarily precise outer approximation of $\mathcal{R}^{\infty}(\Omega)$.

Algorithm 1 Approximation of $\mathcal{R}^{\infty}(\Omega)$

Require: $\{[f_p]\}_{p \in \mathcal{P}}, \Omega, \varepsilon$
Ensure: $\mathcal{R}^{\infty}(\Omega) \subset Y$ and $Y \to \mathcal{R}^{\infty}(\Omega)$ as $\varepsilon \to 0$

$Stack \leftarrow \{[\Omega]\}, Y \leftarrow \Omega$
while $X_c \neq \varnothing$ **do**
  $\underline{X} \leftarrow \varnothing, \Delta X \leftarrow \varnothing, X_c \leftarrow \varnothing$
  while $Stack \neq \varnothing$ **do**
    $[x] \leftarrow Pop(Stack)$
    if $[f_p]([x]) \subset Y$ for some $p \in \mathcal{M}$ **then**
      $\underline{X} \leftarrow \underline{X} \cup [x]$;
    else if $[f_p]([x]) \cap Y = \varnothing$ for all $p \in \mathcal{M}$ **then**
      $X_c \leftarrow X_c \cup [x]$
    else
      if $w([x]) < \varepsilon$ **then**
        $\Delta X \leftarrow \Delta X \cup [x]$;
      else
        $\{L[x], R[x]\} = Bisect([x])$
        $Push(Stack, \{L[x], R[x]\}$
      end if
    end if
  end while
  $\overline{X} \leftarrow \underline{X} \cup \Delta X$
  $Stack \leftarrow \overline{X}$
  $Y \leftarrow \cup_{[x] \in \overline{X}} [x]$
end while
return Y

The algorithm takes as input a compact set Ω, which is assumed to be a union of a finite number of intervals, denoted by $[\Omega]$. This is without loss of generality, because any compact set can be arbitrarily approximated by a union of intervals (because of the finite covering theorem). At each iteration of the inner while loop, Algorithm 1 checks if the image $[f_p]([x])$ of a particular box $[x]$ is contained in Y, i.e., the outer approximation obtained in the previous iteration, for some $p \in \mathcal{P}$, or completely outside of Y for any $p \in \mathcal{P}$. If neither and the box size is greater than ε, then $[x]$ is deemed to be undetermined and divided into two subintervals $L[x]$ and $R[x]$ by bisection, which are given by

$$L[x] = [\underline{x}_1, \overline{x}_1] \times \cdots \times [\underline{x}_j, (\underline{x}_j + \overline{x}_j)/2] \times \cdots \times [\underline{x}_n, \overline{x}_n],$$
$$R[x] = [\underline{x}_1, \overline{x}_1] \times \cdots \times [(\underline{x}_j + \overline{x}_j)/2, \overline{x}_j] \times \cdots \times [\underline{x}_n, \overline{x}_n],$$

where j is the dimension in which the box x attains its width. A box will not go through subdivision once its size is less than ε, which is used to control the smallest size of the intervals, and thus control the precision of the set $\mathrm{Pre}_{\exists}(Y)$ at each iteration. In each iteration, $\underline{X}$ denotes the list of intervals that entirely belongs to $\mathrm{Pre}_{\exists}(Y)$, and X_c contains the intervals that will be mapped outside of Y by $[f_p]$ for any $p \in \mathcal{P}$ while those that are partly inside $\mathrm{Pre}_{\exists}(Y)$, i.e., undetermined intervals, are collected in ΔX.

We prove that the output of Algorithm 1 indeed approximates $\mathcal{R}^{\infty}(\Omega)$ in the following sense.

Proposition 7: Suppose that Ω is a compact set and $[f_p]$ is a convergent inclusion function for f_p for each $p \in \mathcal{P}$. For a given $\varepsilon > 0$, Algorithm 1 terminates in finite steps. Let Y^{ε} denote the output of Algorithm 1. Then

(i) if $0 < \varepsilon_1 < \varepsilon_2$, $\mathcal{R}^{\infty}(\Omega) \subset \overline{Y}^{\varepsilon_1} \subset \overline{Y}^{\varepsilon_2}$;
(ii) $\lim_{\varepsilon \to 0} Y^{\varepsilon} = \mathcal{R}^{\infty}(\Omega)$.

Proof: We use subscript j to denote the corresponding sets in jth iteration ($j \in \mathbb{Z}_{\geq 0}$), i.e., $X_{c,j}$, ΔX_j, and Y_j represent X_c, ΔX and Y in the jth iteration, respectively.

We first prove the finite termination. In each iteration, $X_{c,j}$ is removed from Y_j, yielding $Y_j \subset Y_{j-1}$. Hence, $\{Y_j\}$ is an non-increasing sequence of sets with $Y_0 = \Omega$. Under a given precision $\varepsilon > 0$, Y_j is represented by a union of intervals with minimum width ε. Suppose for all $j \in \mathbb{Z}_{\geq 0}$, $X_{c,j} \neq \varnothing$. Hence, there are always some intervals being removed from Y_{j-1}, and $\{Y_j\}$ is strictly decreasing. Then there must exist a positive integer N such that $Y_N = \varnothing$, which means $X_c^N = \varnothing$ and the algorithm stops at step N, since Y_j is comprised of a finite number of intervals (Ω is compact).

For (i), we first consider $\mathcal{R}^{\infty}(\Omega) \subset Y^{\varepsilon}$. For the sake of contradiction, let $y \in \mathcal{R}^{\infty}(\Omega)$ but $y \notin Y_N$, for a positive integer N. Then $y \in \Omega \setminus Y_N$. According to the algorithm, $\forall z \in \Omega \setminus Y_N$, there must be a step $0 < j \leq N$ such that $[f_p](z) \cap \Omega = \varnothing$ for all $p \in \mathcal{P}$. This indicates that $z \notin \mathcal{R}^{\infty}(\Omega)$, which is a contradiction. Thus $\mathcal{R}^{\infty}(\Omega) \subset Y^{\varepsilon}$. Next we prove $Y^{\varepsilon_1} \subset Y^{\varepsilon_2}$ by induction. Consider the first two steps: $Y_0^{\varepsilon_1} = Y_0^{\varepsilon_2} = \Omega$. Since $0 < \varepsilon_1 < \varepsilon_2$, some intervals in $\Delta X_1^{\varepsilon_2}$ will be divided into finer boxes and are possible to be included in $X_{c,1}^{\varepsilon_1}$, and thus $X_{c,1}^{\varepsilon_2} \subset X_{c,1}^{\varepsilon_1}$. Together with $Y_1^{\varepsilon_1} = Y_0^{\varepsilon_1} \setminus X_{c,1}^{\varepsilon_1}$, and $Y_1^{\varepsilon_2} = Y_0^{\varepsilon_2} \setminus X_{c,1}^{\varepsilon_2}$, we have $Y_1^{\varepsilon_1} \subset Y_1^{\varepsilon_2}$. Assume $Y_j^{\varepsilon_1} \subset Y_j^{\varepsilon_2}$ for any step $1 \le j < N$. Similarly, we have $X_{c,j}^{\varepsilon_2} \subset X_{c,j}^{\varepsilon_1}$. That gives $Y_{j+1}^{\varepsilon_1} \subset Y_{j+1}^{\varepsilon_2}$.

For (ii), consider a decreasing sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ with $\varepsilon_j > 0$ and $\lim_{j\to\infty} \varepsilon_j = 0$. Since Y^{ε_j} is compact, $\lim_{\varepsilon_j \to 0} Y^{\varepsilon_j}$ exists and is given by the compact set $\bigcap_{j=1}^{\infty} Y^{\varepsilon_j}$. Let $Y = \bigcap_{j=1}^{\infty} Y^{\varepsilon_j}$. If every Y^{ε_j} is nonempty, then Y is nonempty. By (ii), $\mathcal{R}^{\infty}(\Omega) \subset Y^{\varepsilon_j}$ for all $j \ge 1$. Then it is clear that $\mathcal{R}^{\infty}(\Omega) \subset Y$. Next, we claim that $Y \subset \mathcal{R}^{\infty}(\Omega)$. If this is not true, then there exists $y \in Y$ such that $f_p(y) \notin Y$ for all $p \in \mathcal{P}$, i.e., $f_p(X) \in Y_c$, which is the complement of Y_c and is open. Then it follows that there exists $\delta > 0$ such that $\mathcal{B}_{\delta}(f_p(y)) \subset Y_c$ for all $p \in \mathcal{P}$. Furthermore, by the definition of set limit, there exists a J_1 sufficiently large such that $B_{\delta}(f_p(y)) \cap Y^{\varepsilon_j} = \varnothing$ for all $p \in \mathcal{P}$ and $j \ge J_1$. Then it is only possible that $y \in [x] \in \Delta X_j, j \ge J_1$. Since f_p is a continuous inclusion function, there exists a J_2 such that $[f_p]([x]) \subset \mathcal{B}_{\delta}(f_p(y))$ for all $p \in \mathcal{P}$ and $[x] \in \Delta X_j, j \ge J_2$. Then for all $j \ge \max\{J_1, J_2\}$, we have $[f_p]([x]) \cap Y_N = \varnothing$, which is a contradiction. ■

Remark 2: The only parameter we need to configure is the precision requirement ε. The smaller ε is, the closer the outer approximation to the real maximal controlled invariant set, but more iterations are required.

Given any system (1) with dynamics $[f_p]_{p\in\mathcal{P}}$ or any compact set Ω, Algorithm 1 does not always return an nonempty set. If the returned $Y^{\varepsilon} = \varnothing$, by Proposition 7 (i), we can conclude that there is no controlled invariant set within Ω. On the other side, if there exists a controlled invariant set contained in Ω for a system (1), then the maximal controlled invariant set $\mathcal{R}^{\infty}(\Omega)$ can be approached in arbitrary precision by using a sufficiently small ε.

By a slight modification, i.e., require "for some $p \in \mathcal{P}$" in line 6 and "for all $p \in \mathcal{P}$" in line 8, Algorithm 1 can be reused to compute outer approximations for the maximal arbitrarily invariant set $\mathcal{W}^{\infty}(\Omega)$. We have the following result on the output of Algorithm 1. The proof is similar to that for Proposition 7 and hence omitted.

Proposition 8: Suppose that Ω is a compact set and $[f_p]$ is a convergent inclusion function for f_p for each $p \in \mathcal{P}$. Let Y^{ε} denote the output of Algorithm 1. Then

(i) For a given $\varepsilon > 0$, Algorithm 1 terminates in finite steps;
(ii) if $0 < \varepsilon_1 < \varepsilon_2$, $\mathcal{R}^{\infty}(\Omega) \subset Y^{\varepsilon_1} \subset Y^{\varepsilon_2}$;
(iii) $\lim_{\varepsilon\to 0} Y^{\varepsilon} = \mathcal{W}^{\infty}(\Omega)$.

V. EXAMPLES—POLYNOMIAL DYNAMICAL SYSTEMS

In the following two examples in this section, natural inclusion functions are used to compute $[f_p]([x])$.

A. Maximal controlled invariant set

Consider a discrete-time, two-dimensional switched system with four modes of polynomial dynamics given by:

$$f_1(x) = \begin{bmatrix} 0.85x_1 - 0.1x_2 - 0.05x_1^3 \\ x_2 - 0.1x_2^2 + 0.1x_1 + 0.2 \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} 0.85x_1 - 0.1x_2 - 0.05x_1^3 \\ 0.9x_2 + 0.1x_1 \end{bmatrix},$$

$$f_3(x) = \begin{bmatrix} 0.99x_1 - 0.02x_2 - 0.01x_1^3 + 0.04 \\ x_2 + 0.02x_1 + 0.2 \end{bmatrix},$$

$$f_4(x) = \begin{bmatrix} 0.99x_1 - 0.02x_2 - 0.01x_1^3 - 0.03 \\ x_2 + 0.02x_1 - 0.2 \end{bmatrix}.$$

We would like to check and approximate the maximal controlled invariant set within $\Omega = [0.2, 3] \times [-2, -0.5]$, which only contains an unstable fixed point $([0.8952, -1.7015])$ of mode 1. By Proposition 4 and 5, Ω is neither a controlled invariant set nor an arbitrarily invariant set. Moreover, there is no arbitrarily invariant set contained in Ω.

Using a precision requirement $\varepsilon = 0.001$, the result returned by Algorithm 1 is shown in Figure 1. The algorithm is implemented using Matlab and run on a 2.4 GHz Intel Core i5 processor. As observed in Figure 1, the outer approximation is refined at the boundary in order to approximate the maximal controlled invariant set $\mathcal{R}^{\infty}(\Omega)$ with higher precision.

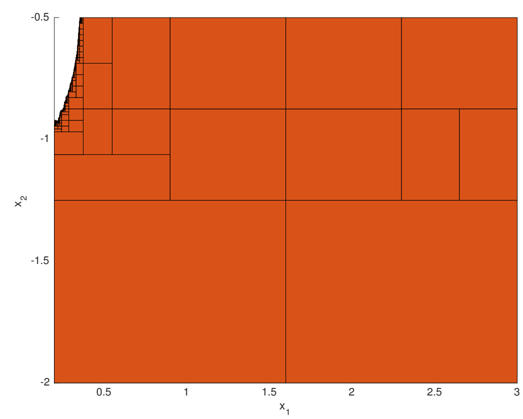

Fig. 1. An outer approximation of $\mathcal{R}^{\infty}(\Omega)$ (the red area).

B. Maximal arbitrarily invariant set

The second example is a two-dimensional and two modes switched polynomial system:

$$f_1(x) = \begin{bmatrix} 0.95x_1 - 0.1x_1x_2^2 \\ 0.9x_2 - 0.05x_1^2x_2 \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} 0.9x_1 + 0.05x_1x_2^2 \\ 0.9x_2 - 0.05x_1^2x_2 \end{bmatrix}.$$

The origin is a stable fixed point for both modes.

We use modified Algorithm 1 to compute the arbitrarily invariant set $\mathcal{W}^{\infty}(\Omega)$ for $\Omega = [-2, 2] \times [-2, 2]$.

The precision parameter is set to be 5% of the width of Ω. Within 4 iterations, we are able to obtain a reasonable shape of $\mathcal{W}^{\infty}(\Omega)$. It is observed that the outer approximation, denoted by Y_4, is symmetric with respect to the origin, which is consistent with the dynamics.

To check the closeness of this approximation, we simulate the trajectories under random switchings from four different initial conditions: $[-1.87, 1.75]$, $[1.96, 1.5]$, $[-1, -1.7]$, and $[1.9, -1.9]$. First three are deliberately chosen to be close enough to the boundary of Y_4, and the last one is not inside. The simulated trajectories are shown in Figure 2. All four trajectories converge to the origin as expected. In particular, the trajectory starting from $[1.9, -1.9]$ (which is not in Y_4 but close to its boundary) evolves out of Ω for some time under arbitrary switching. This indicates that our outer approximation is tight.

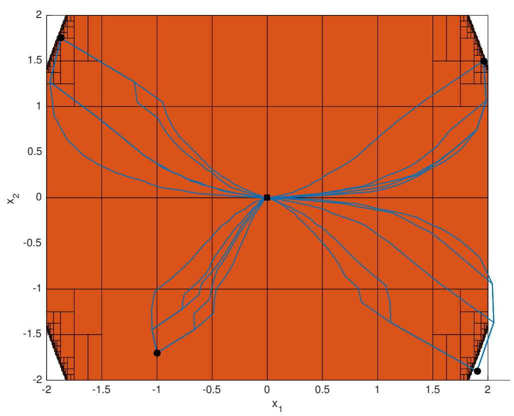

Fig. 2. An outer approximation of $\mathcal{W}^{\infty}(\Omega)$ (the red area).

VI. Conclusions

We have presented in this paper an interval method for computing maximal invariant sets of switched nonlinear systems. Both switching controlled invariant sets and invariant sets under arbitrary switching are considered. The computation relies on an iterative procedure that provides outer approximations of the maximal invariant sets with increasing precision. It has been shown that the outputs of the iterative procedure indeed converge to the maximal invariant sets in set convergence sense, as the number of iterations tend to infinity. This general convergence result does not require the switched nonlinear systems to be asymptotically stable, in contrast with most results on this topic that require such stability conditions even for linear systems.

We would like to point out that despite the general convergence guarantee, the convergence rates for such methods can be highly influenced by the stability properties of underlying dynamical systems. For future work, we will analyze the convergence rates under stronger assumptions on the system dynamics. Robustness issues will be discussed as well as the relationship between invariant set computation for discrete-time and continuous-time dynamical systems (see [4], [6], [15], [19]). The ideas will be explored further for control synthesis for both safety and more general specifications (e.g., switching control for temporal logic specifications [20]).

VII. Acknowledgments

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