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Existence and Uniqueness Results for Impulsive Hybrid Stochastic Delay Systems

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Abstract

This paper investigates the existence and uniqueness of solutions for a general class of impulsive and switched hybrid stochastic delay systems. Due to the impulse effects and time delay, solutions of an impulsive and switched hybrid stochastic delay system have to be formulated and considered to be evolving in the space of piecewise continuous functions on the delay interval. The basic theory established here can also be applied to impulsive stochastic functional differential equations as a special case.

Key words: Existence, Uniqueness, Impulsive differential equation, Stochastic differential equation, Functional differential equation, Hybrid system; Time-delay.

AMS Subject Classification: 34K50, 34K45, 93C30

1 Introduction

Impulsive differential equations or impulsive dynamical systems model real world processes that undergo abrupt changes (impulses) in the state at discrete times [9]. Particularly, impulse control and stabilization as a powerful tool to achieve stability for dynamical systems that can be highly unstable, in the absence of impulses, has gained increasing popularity and found successful applications in a wide variety of areas, such as control systems, control and synchronization of chaotic systems, complex dynamical networks, large-scale dynamical systems, secure communication, spacecraft maneuvers, population growth and biological systems, neural networks, ecosystems management, and pharmacokinetics (see [10] and references therein). Impulsive dynamical systems can be naturally viewed as a class of hybrid systems (see, e.g., [5]) that consist of three elements: a continuous differential equation, which governs the continuous evolution of the system of between impulses; a difference equation, which governs the way the system states are changed at impulse times; and an impulsive law for determining when the impulses occur.

While the area of impulsive delay differential equations is still an interesting and challenging theoretical subject yet to be fully developed [1, 9], recent interest has been shown in the stability analysis of impulsive stochastic systems with-time delay (see, e.g., [4, 14, 15, 16, 17, 18]) and impulsive stabilization of stochastic delay systems [10]. However, little attention has been paid to the existence and uniqueness of solutions for such systems.

The aim of this paper is to establish some existence and uniqueness results for a general class of systems that may exhibit all the hybrid characteristics of impulse effects, switching, stochastic dynamics, and time delays. As in the deterministic case, one of the major difficulties encountered here is caused by a fundamental difference that impulse effects bring to the studies of delay differential equation, which is that, due to the impulse effects, solutions of impulsive delay differential equations are no longer continuous functions of the time and hence they have to be considered to be evolving in the space of piecewise continuous functions. This key difference causes a myriad of problems when studying some basic properties of impulsive delay differential equations [1, 11]. These problems naturally carry to the stochastic setting and are the main issues to be addressed in this paper. The basic theory established in this paper can also be applied to impulsive stochastic delay differential equations as a special case.

2 Formulation

Let $\mathbb{Z}^+$ denote the set of nonnegative integers, $\mathbb{R}^n$ the n-dimensional real Euclidean space , and $\mathbb{R}^{n\times m}$ the space of $n\times m$ real matrices. For $x\in\mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. For $A=(a_{ij})\in\mathbb{R}^{n\times m}$, define $|A|\triangleq\sqrt{\text{trace}(A^TA)}=\sqrt{\sum_{i=1}^n\sum_{j=1}^m|a_{ij}|^2}$ and $\|A\|\triangleq\sqrt{\lambda_{\max}(A^TA)}$, i.e. $|A|$ and $\|A\|$ denote the Frobenius norm and spectral norm of the matrix A, respectively.

For $-\infty<a<b<\infty$, we say that a function from $[a,b]$ to $\mathbb{R}^n$ is *piecewise continuous*, if the function has at most a finite number of jump discontinuities on $(a,b]$ and are continuous from the right for all points in $[a,b)$. Given $r>0$, $\mathcal{PC}([-r,0];\mathbb{R}^n)$ denotes the family of piecewise continuous functions from $[-r,0]$ to $\mathbb{R}^n$. A norm on $\mathcal{PC}([-r,0];\mathbb{R}^n)$ is defined as $\|\phi\|\triangleq\sup_{-r\le s\le 0}|\phi(s)|$ for $\phi\in\mathcal{PC}([-r,0];\mathbb{R}^n)$. For simplicity, $\mathcal{PC}$ is used for $\mathcal{PC}([-r,0];\mathbb{R}^n)$ for the rest of this paper. It is easy to see that, although equipped with the norm $\|\cdot\|$, the space $\mathcal{PC}$ is not a Banach space. Let Θ be a finite subset of $(-r,0]$, and, by $\mathcal{PC}[\Theta]$, we denote a subspace of $\mathcal{PC}$, which contains functions that are continuous on $[-r,0]\backslash\Theta$ and may have jump discontinuities at the set Θ. Clearly, $\mathcal{PC}[\Theta]$ is a Banach space with respect to the norm

$\|\cdot\|$. Suppose that x is a piecewise continuous function from $\mathbb{R}$ to $\mathbb{R}^n$. For any $t \in \mathbb{R}$, let x_t be an element of $\mathcal{PC}$ defined by $x_t(\theta) := x(t+\theta)$, $-r \le \theta \le 0$.

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space with $\{\mathcal{F}_t\}_{t\ge 0}$ as a filtration satisfying the usual conditions, and $W(t)$ be an m-dimensional standard Wiener process defined on $(\Omega, \mathcal{F}, P)$ and adapted to $\{\mathcal{F}_t\}_{t\ge 0}$.

Definition 2.1. By a *$\mathcal{PC}$-valued random variable*, we mean a mapping $\xi : \Omega \to \mathcal{PC}[\Theta]$ that is $\mathcal{F}/\mathcal{B}[\Theta]$-measurable (or simply $\mathcal{F}$-measurable), where Θ is a finite subset of $(-r, 0]$ and $\mathcal{B}[\Theta]$ is the Borel σ-algebra of $\mathcal{PC}[\Theta]$. A *$\mathcal{PC}$-valued stochastic process* is a collection of $\mathcal{PC}$-valued random variables indexed by a set T, i.e. a collection $\{\phi_t\}_{t\in T}$, where each ϕ_t is a $\mathcal{PC}$-valued random variable.

For $p > 0$ and $t \ge 0$, let $\mathcal{L}^p_{\mathcal{F}_t}$ denote the family of all $\mathcal{F}_t$-measurable $\mathcal{PC}$-valued random variables ϕ such that $\mathbb{E}\left(\|\phi\|^p\right) < \infty$. Let $\mathcal{L}^{\mathrm{b}}_{\mathcal{F}_{t_0}}$ be the family of $\mathcal{PC}$-valued random variables that are bounded and $\mathcal{F}_t$-measurable. Let $\mathcal{L}^1([a,b];\mathbb{R}^n)$ denote the space of all $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b |f(t)|dt < \infty$ a.s. and $\mathcal{L}^2([t_0,T];\mathbb{R}^{n\times m})$ the space of all $\mathbb{R}^{n\times m}$-valued $\mathcal{F}_t$-adapted processes $\{g(t)\}_{a\le t\le b}$ such that $\int_a^b |g(t)|^2 dt < \infty$ a.s.

Let $\mathcal{N}_c$ and $\mathcal{N}_d$ be two arbitrary index sets. Consider the impulsive and switched stochastic delay system:

$$dx(t) = f_{i_k}(t, x_t)dt + g_{i_k}(t, x_t)dW(t), \quad t \in [t_k, t_{k+1}), \quad i_k \in \mathcal{N}_c, \quad k \in \mathbb{Z}^+, \tag{2.1a}$$
$$\Delta x(t) = I_{j_k}(t, x_{t^-}), \quad t = t_k, \quad j_k \in \mathcal{N}_d, \quad k \in \mathbb{Z}^+\backslash\{0\}, \tag{2.1b}$$
$$x_{t_0} = \xi, \tag{2.1c}$$

where ξ, the initial data, is a $\mathcal{PC}$-valued random variable; the $\mathcal{PC}$-valued stochastic process x_{t^-} is defined by $x_{t^-}(s) = x(t+s)$, for $s \in [-r, 0)$, and $x_{t^-}(0) = x(t^-)$, where $x(t^-) = \lim_{s\to t^-} x(s)$; for each $i \in \mathcal{N}_c$ and $j \in \mathcal{N}_d$,

$$f_i : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{R}^n, \quad g_i : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{R}^{n\times m}, \quad I_j : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{R}^n.$$

In (2.1), the sequence of triples $\{(t_k, i_k, j_k)\}$ together imposes the following:

(i) a sequence of indices i_k to *switch* the right-hand side of (2.1a) among the family $\{f_i : i \in \mathcal{N}_c\}$;

(ii) a sequence of indices j_k to select the *impulse* functions I_{j_k} from the family $\{I_j : j \in \mathcal{N}_d\}$ to reset the system state according to the difference equation (2.1b); and

(iii) a sequence of discrete times t_k, called the *impulse and switching times* (except the initial time t_0[1]), to determine when the switching and impulse occur.

Besides the family of functions $\{f_i : i \in \mathcal{N}_c\}$ and $\{I_i : i \in \mathcal{N}_d\}$, which govern the continuous dynamics and the discrete dynamics of system (2.1), respectively, it is expected that properties of solutions to system (2.1) (e.g. the boundedness properties to be investigated in this paper)

[1] While a switching mode is assigned by i_0 at $t = t_0$, we do not consider a solution to instantly undergo an impulse at the initial time t_0.

can also be highly affected by the sequence of triples $\{(t_k, i_k, j_k)\}$, which we call an *impulsive and switching signal*. Given an impulsive and switching signal $\{(t_k, i_k, j_k)\}$, we can define

$$\sigma(t) := i_k, \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}^+. \tag{2.2}$$

We call σ a *switching signal*, which is a function from $[t_0, \infty)$ to $\mathcal{N}_c$. While the formulation in system (2.1) emphasizes the impulsive and switching signal $\{(t_k, i_k, j_k)\}$ as a whole, defining σ by (2.2) enables us to rewrite equation (2.1a) as

$$dx(t) = f_\sigma(t, x_t)dt + g_\sigma(t, x_t)dW(t), \quad t \geq t_0, \tag{2.3}$$

and can simplify the notation in the following definition for solutions to system (2.1).

Definition 2.2. For $0 \leq t_0 < T < \infty$ and a given impulsive and switching signal $\{(t_k, i_k, j_k)\}$, an $\mathbb{R}^n$-valued process $\{x(t)\}_{t_0 - r \leq t \leq T}$ is called a *solution* to system (2.1) on $[t_0, T]$, if it satisfies the following:

(i) $x(t)$ is continuous at $t \in [t_0, T] \backslash \{t_k : k \in \mathbb{Z}^+\}$ and right-continuous at $t \in [t_0, T] \cap \{t_k : k \in \mathbb{Z}^+\}$;

(ii) $\{x_t\}_{t_0 \leq t \leq T}$ as a $\mathcal{PC}$-valued process is $\mathcal{F}_t$-adapted;

(iii) $\{f_{\sigma(t)}(t, x_t)\}_{t_0 \leq t \leq T} \in \mathcal{L}^1([t_0, T]; \mathbb{R}^n)$ and $\{g_{\sigma(t)}(t, x_t)\}_{t_0 \leq t \leq T} \in \mathcal{L}^2([t_0, T]; \mathbb{R}^{n \times m})$, where σ is defined by (2.2);

(iv) $x_{t_0} = \xi$ and, for any $t \in [t_0, T]$, it holds with probability 1 that

$$x(t) = \xi(t_0) + \int_{t_0}^{t} f_{\sigma(s)}(s, x_s)ds + \int_{t_0}^{t} g_{\sigma(s)}(s, x_s)dW(s) + \sum_{t_0 \leq t_k \leq T} I_{j_k}(t_k, x_{t_k^-}).$$

A solution $\{x(t)\}_{t_0 \leq t \leq T}$ to (2.1) on $[t_0, T]$ is said to be *unique*, if any other solution, say $\{y(t)\}_{t_0 \leq t \leq T}$, to (2.1) on $[t_0, T]$ is indistinguishable to $\{x(t)\}_{t_0 \leq t \leq T}$, i.e.

$$P\{x(t) = y(t), \ \forall t \in [t_0, T]\} = 1.$$

An $\mathbb{R}^n$-valued process $\{x(t)\}_{t_0 \leq t < \infty}$ is called a *solution* to system (2.1) on $[t_0, \infty)$, if, for any $T > 0$, the restricted process $\{x(t)\}_{t_0 \leq t \leq T}$ is a solution to system (2.1) on $[t_0, T]$.

3 Existence and Uniqueness

In this section, existence and uniqueness of solutions of system (2.1) are investigated. Before stating the main result, we introduce an assumption on the functionals f_i and g_i in system (2.1), which is a typical assumption proposed for considering existence and uniqueness of impulsive delay differential equations [1, 11].

Definition 3.1. A functional $f : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{R}^n$ (or $g : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{R}^{n \times m}$) is said to be *composite-PC* on $[t_0, T]$, where $0 \leq t_0 < T < \infty$, if, for each function x that is piecewise continuous on $[t_0 - r, T]$, the composite function $f(t, x_t)$ (or $g(t, x_t)$) is piecewise continuous on $[t_0, T]$.

Lemma 3.1. *Let $\{x(t)\}_{t_0-r\leq t\leq T}$ be a process satisfying*

(i) $x(t)$ is piecewise continuous on $[t_0,T]$ and $\{x(t)\}_{t_0\leq t\leq T}$ is $\mathcal{F}_t$-adapted; and

(ii) x_{t_0} is a $\mathcal{PC}$-valued $\mathcal{F}_{t_0}$-measurable random value.

Then $\{x_t\}_{t_0\leq t\leq T}$ as a $\mathcal{PC}$-valued process is $\mathcal{F}_t$-adapted.

Proof. Let $t\in[t_0,T]$ be arbitrarily chosen and fixed. We shall show that x_t as a $\mathcal{PC}$-valued random value is $\mathcal{F}_t$-measurable. By Definition 2.1, we have to show that there exists a finite set Θ in $(-r,0]$ such that the mapping $x_t:\Omega\to\mathcal{PC}[\Theta]$ is $\mathcal{F}_t/\mathcal{B}[\Theta]$-measurable. Since ξ is a $\mathcal{PC}$-valued $\mathcal{F}_{t_0}$-measurable random value, there exists a finite set Θ' in $(-r,0]$ such that ξ is an $\mathcal{F}_{t_0}/\mathcal{B}[\Theta']$-measurable mapping from Ω to $\mathcal{PC}[\Theta']$. Moreover, let Θ'' be the set of points of discontinuity of $x(t)$ on $[t_0,T]$. Define

$$\Theta=\{s\in[-r,0]:\, t+s\in\Theta'' \quad \text{or} \quad t+s-t_0\in\Theta'\}\,.$$

We claim that x_t defines a mapping from Ω to $\mathcal{PC}[\Theta]$ and it is $\mathcal{F}_t/\mathcal{B}[\Theta]$-measurable. The first part of the claim is obvious, since $x_t(s)$ can be discontinuous at s only if $t+s$ is a point of discontinuity of $x(t)$ or $t+s-t_0$ is a discontinuity point of ξ. To show that x_t is $\mathcal{F}_t/\mathcal{B}[\Theta]$-measurable, notice that $\mathcal{PC}[\Theta]$ is a separable Banach space. Hence any open set in $\mathcal{PC}[\Theta]$ can be written as a countable union of closed balls of the form

$$B(\phi,\varepsilon)\triangleq\{\psi\in\mathcal{PC}[\Theta]:\,\|\phi-\psi\|\leq\varepsilon\}\,,$$

where $\varepsilon>0$ and $\phi\in\mathcal{PC}[\Theta]$. It suffices to show that the inverse image of each $B(\phi,\varepsilon)$ under the mapping x_t, denoted by $x_t^{-1}(B(\phi,\varepsilon))$, is contained in $\mathcal{F}_t$. By right-continuity of the elements in $\mathcal{PC}[\Theta]$, we can rewrite $B(\phi,\varepsilon)$ as

$$B(\phi,\varepsilon)=\bigcap_{m=1}^{\infty}\{\psi\in\mathcal{PC}[\Theta]:\,|\psi(s_m)-\phi(s_m)|\leq\varepsilon\}\,,$$

where $\{s_m\}_{m=1}^{\infty}$ is a sequential representation of all the rational numbers in $[-r,0]$. Therefore,

$$x_t^{-1}(B(\phi,\varepsilon))=\bigcap_{m=1}^{\infty}\{\omega\in\Omega:\,|x(t+s_m)-\phi(s_m)|\leq\varepsilon\}\triangleq\bigcap_{m=1}^{\infty}A_m$$

Since $\{x(t)\}_{t_0\leq t\leq T}$ is $\mathcal{F}_t$-adapted, we have $A_m\in\mathcal{F}_{t+s_m}\subset\mathcal{F}_t$, if $t+s_m\geq t_0$. To show that $A_m\in\mathcal{F}_t$ still holds for $t+s_m<t_0$, define, for each $s\in[-r,0]$, a mapping $e_s:\mathcal{PC}[\Theta']\to\mathbb{R}^n$ by $e_s(\phi)=\phi(s)$, i.e. evaluating ϕ at s. It is easy to see that, for each $s\in[-r,0]$, the mapping e_s is continuous. Since $\xi(s)(\cdot)=e_s\circ\xi$, it follows that $\xi(s)(\cdot):\Omega\to\mathbb{R}^n$ is $\mathcal{F}_{t_0}$-measurable for each $s\in[-r,0]$. Hence, if $t+s_m<t_0$, we still have $A_m\in\mathcal{F}_{t_0}\subset\mathcal{F}_t$, since $x(t+s_m)=\xi(t+s_m-t_0)$ and $\xi(s)(\cdot):\Omega\to\mathbb{R}^n$ is $\mathcal{F}_{t_0}$-measurable for each $s\in[-r,0]$. The proof is complete. □

Theorem 3.1. *If*

(i) for each $i\in\mathcal{N}_c$ and $j\in\mathcal{N}_d$, $f_i(t,\phi)$, $g_i(t,\phi)$, and $I_j(t,\phi)$ are Borel measurable on $[t_0,T]\times\mathcal{PC}[\Theta]$, for each finite subset Θ of $(-r,0]$, and $f_i(t,\phi)$ and $g_i(t,\phi)$ are composite-PC on $[t_0,T]$;

(ii) there exists a positive constant K such that for $(t,\phi,\psi)\in[t_0,T]\times\mathcal{PC}\times\mathcal{PC}$ and $i\in\mathcal{N}_c$,

$$|f_i(t,\phi)-f_i(t,\psi)|+|g_i(t,\phi)-g_i(t,\psi)|\le K\|\phi-\psi\|, \tag{3.1}$$

and

$$|f_i(t,\phi)|^2+|g_i(t,\phi)|^2\le K^2(1+\|\phi\|^2); \tag{3.2}$$

(iii) ξ is a PC-valued $\mathcal{F}_{t_0}$-measurable random value,

then there exists a unique solution to system (2.1) on $[t_0,T]$.

Proof. Throughout this proof, let $\{(t_k,i_k,j_k)\}$ be a given impulsive and switching signal. The proof is completed in the following procedure **(a)-(d)**.
(a) Uniqueness. Suppose $t_0<t_1<t_2<\cdots<t_m=T$, where t_k, for $k=1,2,\cdots,m-1$, are the impulse and switching times between t_0 and T. We let $t_m=T$ to simplify the notation; if there are no impulses between t_0 and T, we can still take $t_1=T$ and the following argument holds with the same notation. Let $x(t)$ and $y(t)$ be two solutions to (2.1) on $[t_0,T]$. We first show that

$$\mathbb{E}\left(\|x_t-y_t\|^2\right)=0,\quad \forall t\in[t_0,t_1).$$

Define, for $N>0$ and $t\in[t_0,t_1)$, that

$$\chi_N(t)=\begin{cases}1, & \text{if}\quad \|x_s\|\vee\|y_s\|\le N,\quad \forall s\in[t_0,t],\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that $\chi_N(t)=\chi_N(t)\chi_N(s)$ for $s\le t$. Therefore, for $t\in[t_0,t_1)$,

$$\begin{aligned}\chi_N(t)[x(t)-y(t)]=\chi_N(t)\bigg\{&\int_{t_0}^t\chi_N(s)[f_{i_0}(s,x_s)-f_{i_0}(s,y_s)]ds\\ &+\int_{t_0}^t\chi_N(s)[g_{i_0}(s,x_s)-g_{i_0}(s,y_s)]dW(s)\bigg\},\end{aligned} \tag{3.3}$$

where $i_0=\sigma(t_0)$. By the Lipschitz condition (3.1), for $s\in[t_0,t_1)$,

$$\chi_N(s)\left(|f_{i_0}(s,x_s)-f_{i_0}(s,y_s)|+|g_{i_0}(s,x_s)-g_{i_0}(s,y_s)|\right)\le K\chi_N(s)\|x_s-y_s\|\le 2KN. \tag{3.4}$$

Using (3.4), Schwarz's inequality, and Doob's martingale inequality on (3.3), we can show that, for $t\in[t_0,t_1)$,

$$\begin{aligned}\mathbb{E}\left(\chi_N(t)\|x_t-y_t\|^2\right)&=\mathbb{E}\left(\chi_N(t)\sup_{t-r\le s\le t}|x(s)-y(s)|^2\right)\\ &\le\mathbb{E}\left(\sup_{t-r\le s\le t}|\chi_N(s)[x(s)-y(s)]|^2\right)\\ &\le 2(t_1-t_0+4)K^2\int_{t_0}^t\mathbb{E}\left(\chi_N(s)\|x_s-y_s\|^2\right)ds,\end{aligned}$$

which, by Gronwall's inequality, implies that

$$\mathbb{E}\left(\chi_N(t)\|x_t-y_t\|^2\right)=0,\quad t\in[t_0,t_1).$$

Letting $N \to \infty$, the monotone convergence theorem implies that

$$\mathbb{E}\left(\|x_t - y_t\|^2\right) = 0, \quad t \in [t_0, t_1).$$

Therefore, $x(t) = y(t)$ a.s. for $t \in [t_0, t_1)$. Since $x(t)$ and $y(t)$ both have continuous samples on $[t_0, t_1)$, we have

$$P\{x(t) = y(t),\ \forall t \in [t_0, t_1)\} = 1.$$

At $t = t_1$, we consider two cases. If t_1 is not an impulse and switching time, then $x(t_1) = y(t_1)$ a.s. by continuity of solutions at $t = t_1$. If t_1 is an impulse and switching time, then

$$x(t_1) = x(t_1^-) + I_{j_1}(t_1, x_{t_1^-}) = y(t_1^-) + I_{j_1}(t_1, y_{t_1^-}) = y(t_1), \quad \text{a.s.}$$

Therefore,

$$P\{x(t) = y(t),\ \forall t \in [t_0, t_1]\} = 1.$$

Note that now we have $x_{t_1} = y_{t_1}$ a.s. and the previous argument can be repeated on $[t_1, t_2]$ and, by induction, on $[t_k, t_{k+1})$ for $k = 1, 2, \cdots, m-1$. Hence

$$P\{x(t) = y(t),\ \forall t \in [t_0, T]\} = 1.$$

(b) Existence on $[t_0, t_1]$ **(if** $\mathbb{E}\left(\|\xi\|^2\right) < \infty$**).** We first assume that $\mathbb{E}\left(\|\xi\|^2\right) < \infty$. The general initial data will be treated in part **(c)**. Define, for $n \geq 1$ and $t \in [t_0, t_1]$,

$$x^{(n)}(t) = \xi(t_0) + \int_{t_0}^{t} f_{i_0}(s, x_s^{(n-1)})ds + \int_{t_0}^{t} g_{i_0}(s, x_s^{(n-1)})dW(s), \tag{3.5}$$

with $x^{(0)}(t) \equiv \xi(t_0)$ on $[t_0, t_1]$, and $x_{t_0}^{(n)} = \xi$. It is easy to see that the following conditions (to be referred to as conditions (A)-(C)) hold for $n = 0$:

(A) $\{x_t^{(n)}\}_{t_0 \leq t < t_1}$ as a $\mathcal{PC}$-valued process is $\mathcal{F}_t$-adapted;

(B) $\{x^{(n)}(t)\}_{t_0 - r \leq t < t_1}$ is piecewise continuous on $[t_0 - r, t_0]$ and continuous on $[t_0, t_1]$; and

(C) $\mathbb{E}\left(\int_{t_0}^{t_1} \|x_s^{(n)}\|^2 ds\right) < \infty$.

Assume, by induction, that $\{x^{(n)}(t)\}_{t_0 - r \leq t < t_1}$ satisfies conditions (A)-(C) for some $n \geq 0$. Then by the linear growth condition (3.2),

$$\mathbb{E}\left(\int_{t_0}^{t_1} |g_{i_0}(s, x_s^{(n)})|^2 ds\right) \leq K^2(t_1 - t_0) + K^2 \mathbb{E}\left(\int_{t_0}^{t_1} \|x_s^{(n)}\|^2 ds\right) < \infty,$$

$$\int_{t_0}^{t_1} |f_{i_0}(s, x_s^{(n)})| ds \leq K\sqrt{b-a}\left(\int_{t_0}^{t_1} (1 + \|x_s^{(n)}\|^2) ds\right)^{\frac{1}{2}} < \infty, \quad \text{a.s.}$$

Therefore, the integrals in (3.5) are well-defined. It follows that $\{x^{(n+1)}(t)\}_{t_0 \leq t < t_1}$ is $\mathcal{F}_t$-adapted and continuous on $[t_0, t_1]$. By Lemma 3.1, $\{x_t^{(n+1)}\}_{t_0 \leq t < t_1}$ as a $\mathcal{PC}$-valued process is

$\mathcal{F}_t$-adapted. Moveover, by virtue of the inequality $|a+b+c|^2 \le 3(a^2+b^2+c^2)$, the linear growth condition (3.2), and (3.5), we have, for $t \in [t_0, t_1]$,

$$\begin{aligned}
\mathbb{E}\left(\|x_t^{(n+1)}\|^2\right) &= \mathbb{E}\left(\sup_{t-r\le s\le t} |x^{(n+1)}(t)|^2\right) \\
&\le 3\mathbb{E}\left(\|\xi\|^2\right) + 3\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left(\int_{t_0}^{s} f_{i_0}(s, x_s^{(n)})ds\right)^2\right) \\
&\quad + 3\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left(\int_{t_0}^{s} g_{i_0}(s, x_s^{(n)})dW(s)\right)^2\right) \\
&\le 3\mathbb{E}\left(\|\xi\|^2\right) + 3(t_1-t_0)K^2\left[(t_1-t_0) + \mathbb{E}\left(\int_{t_0}^{t_1} \|x_s^{(n)}\|^2 ds\right)\right] \\
&\quad + 3\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left(\int_{t_0}^{s} g_{i_0}(s, x_s^{(n)})dW(s)\right)^2\right),
\end{aligned} \tag{3.6}$$

where, by Doob's martingale inequality and the linear growth condition (3.2),

$$\begin{aligned}
\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left(\int_{t_0}^{s} g_{i_0}(s, x_s^{(n)})dW(s)\right)^2\right) &\le 4\mathbb{E}\left(\int_{t_0}^{t_1} |g_{i_0}(s, x_s^{(n)})|^2 ds\right) \\
&\le 4K^2(t_1-t_0) + 4K^2\mathbb{E}\left(\int_{t_0}^{t_1} \|x_s^{(n)}\|^2 ds\right).
\end{aligned}$$

Hence, (3.6) shows that

$$\mathbb{E}\left(\int_{t_0}^{t_1} \|x^{(n+1)}(t)\|^2 dt\right) = \int_{t_0}^{t_1} \mathbb{E}\left(\|x^{(n+1)}(t)\|^2\right) dt < \infty.$$

Therefore, by induction, we have a sequence of processes $\{x^{(n)}(t)\}_{t_0-r\le t<t_1}$ satisfying conditions (A)-(C) for all $n \ge 0$. Next, we estimate $\mathbb{E}\left(\|x_t^{(n+1)} - x_t^{(n)}\|^2\right)$, for $n \ge 0$ and $t \in [t_0, t_1]$. For $n = 0$ and $t \in [t_0, t_1]$, by the linear growth condition (3.2) and (3.5), we have

$$\begin{aligned}
\mathbb{E}\left(\|x_t^{(1)} - x_t^{(0)}\|^2\right) &= \mathbb{E}\left(\sup_{t-r\le s\le t} |x^{(1)}(t) - x^{(0)}(t)|^2\right) \\
&\le 2\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left(\int_{t_0}^{s} f_{i_0}(s, x_s^{(0)})ds\right)^2\right) \\
&\quad + 2\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left(\int_{t_0}^{s} g_{i_0}(s, x_s^{(0)})dW(s)\right)^2\right) \\
&\le 2(t_1-t_0)K^2\left[(t_1-t_0) + \mathbb{E}\left(\int_{t_0}^{t_1} \|x_s^{(n)}\|^2 ds\right)\right] \\
&\quad + 8K^2(t_1-t_0) + 8K^2\mathbb{E}\left(\int_{t_0}^{t_1} \|x_s^{(n)}\|^2 ds\right) \\
&\le 2(t_1-t_0)K^2(t_1-t_0+4)(1+\mathbb{E}\left(\|\xi\|^2\right)) \triangleq L.
\end{aligned} \tag{3.7}$$

We claim that

$$\mathbb{E}\left(\|x_t^{(n+1)} - x_t^{(n)}\|^2\right) \leq \frac{L[C(t-t_0)]^n}{n!}, \quad n \geq 0, \quad t \in [t_0, t_1], \tag{3.8}$$

where $C = 2K^2(t_1 - t_0 + 4)$. It is shown in (3.7) that the claim holds for $n = 0$. Suppose (3.8) is true for some $n \geq 0$. Then

$$\begin{aligned}
\mathbb{E}\left(\|x_t^{(n+2)} - x_t^{(n+1)}\|^2\right) &= \mathbb{E}\left(\sup_{t-r\leq s\leq t} |x^{(1)}(t) - x^{(0)}(t)|^2\right) \\
&\leq 2\mathbb{E}\left(\sup_{t_0\vee(t-r)\leq s\leq t}\left(\int_{t_0}^{s}[f_{i_0}(s, x_s^{(n+1)}) - f_{i_0}(s, x_s^{(n)})]ds\right)^2\right) \\
&\quad + 2\mathbb{E}\left(\sup_{t_0\vee(t-r)\leq s\leq t}\left(\int_{t_0}^{s}[g_{i_0}(s, x_s^{(n+1)}) - g_{i_0}(s, x_s^{(n)})]dW(s)\right)^2\right) \\
&\leq 2K^2(t_1 - t_0 + 4)\int_{t_0}^{t}\mathbb{E}\left(\|x_s^{(n+1)} - x_s^{(n)}\|^2 ds\right) \\
&\leq C\int_{t_0}^{t}\frac{L[C(s-t_0)]^n}{n!}ds = \frac{L[C(t-t_0)]^{n+1}}{(n+1)!}.
\end{aligned} \tag{3.9}$$

Therefore, by induction, the claim is true for all $n \geq 0$. From (3.9), it follows that, for any fixed $t \in [t_0, t_1]$,

$$P\left\{\sup_{t-r\leq s\leq t} |x^{(n+1)}(s) - x^{(n)}(s)| > \frac{1}{n^2}\right\} \leq \frac{Ln^4[C(t_1-t_0)]^n}{n!}.$$

It is easy to check that $\sum_{n=0}^{\infty} \frac{Ln^4[C(t_1-t_0)]^n}{n!} < \infty$. Hence by the Borel-Cantelli lemma, for any fixed $t \in [t_0, t_1]$,

$$P\left\{\sup_{t-r\leq s\leq t} |x^{(n+1)}(s) - x^{(n)}(s)| > \frac{1}{n^2} \text{ i.o.}\right\} = 0,$$

which implies that the partial sum $\xi(t_0) + \sum_{i=0}^{n-1}[x^{(n+1)}(s) - x^{(n)}(s)] = x^{(n)}(s)$ converges uniformly on $[t-r, t]$, with probability 1, for any fixed $t \in [t_0, t_1]$. Since the interval $[t_0, t_1]$ is of finite length and $r > 0$, it actually follows that $x^{(n)}(s)$ converges uniformly on $[t_0 - r, t_1]$, with probability 1. Let the limit process be $\{x(t)\}_{t_0-r\leq t\leq t_1}$, i.e.

$$x(t) = \lim_{n\to\infty} x^{(n)}(t), \quad \text{uniformly for} \quad t \in [t_0 - r, t_1].$$

It is easy to see that the process $\{x(t)\}_{t_0-r\leq t\leq t_1}$ is piecewise continuous on $[t_0 - r, t_0]$ (with $x_{t_0} = \xi$), continuous on $[t_0, t_1]$, and adapted to $\mathcal{F}_t$ for $t \geq t_0$. It follows from Lemma 3.1 that $\{x_t\}_{t_0\leq t\leq t_1}$ as a $\mathcal{PC}$-valued process is also $\mathcal{F}_t$-adapted for $t \geq t_0$. Hence $\{x(t)\}_{t_0-r\leq t\leq t_1}$ satisfies both (i) and (ii) in Definition 2.2. Moreover, we can easily verify that

$$\mathbb{E}\left(\int_{t_0}^{t_1} |g_{i_0}(s, x_s)|^2 ds\right) \leq K^2(t_1 - t_0) + K^2\mathbb{E}\left(\int_{t_0}^{t_1} \|x_s\|^2 ds\right) < \infty,$$

$$\int_{t_0}^{t_1} |f_{i_0}(s, x_s)| ds \leq K\sqrt{b-a}\left(\int_{t_0}^{t_1}(1 + \|x_s\|^2)ds\right)^{\frac{1}{2}} < \infty, \quad \text{a.s.},$$

which implies that

$$\{f_{\sigma(t_0)}(t,x_t)\}_{t_0\le t\le t_1}\in\mathcal{L}^1([t_0,t_1];\mathbb{R}^n),\quad \{g_{\sigma(t_0)}(t,x_t)\}_{t_0\le t\le t_1}\in\mathcal{L}^2([t_0,t_1];\mathbb{R}^{n\times m}),$$

and hence (iii) in Definition 2.2 is satisfied and the right-hand side of the equation

$$x(t)=\xi(t_0)+\int_{t_0}^t f_{\sigma(s)}(s,x_s)ds+\int_{t_0}^t g_{\sigma(s)}(s,x_s)dW(s),\quad t\in[t_0,t_1], \tag{3.10}$$

is meaningful. The validity of equation (3.10) can be shown by passing the limit as $n\to\infty$ in (3.5) and the facts that

(i) $x^{(n)}(s)$ converges uniformly to $x(s)$ with probability 1 for $s\in[t_0-r,t_1]$;

(ii) $f_{\sigma(s)}(s,x_s^n)$ converges uniformly to $f_{\sigma(s)}(s,x_s)$ with probability 1 for $s\in[t_0,t_1]$; and

(iii) $g_{\sigma(s)}(s,x_s^n)$ converges uniformly to $g_{\sigma(s)}(s,x_s)$ with probability 1 for $s\in[t_0,t_1]$ and hence, as $n\to\infty$,

$$\int_{t_0}^t |g_{\sigma(s)}(s,x_s^n)-g_{\sigma(s)}(s,x_s)|^2ds\xrightarrow{P}0,\quad t\in[t_0,t_1],$$

which implies

$$\int_{t_0}^t g_{\sigma(s)}(s,x_s^n)dW(s)\xrightarrow{P}\int_{t_0}^t g_{\sigma(s)}(s,x_s)dW(s),\quad t\in[t_0,t_1],$$

as $n\to\infty$.

In summary, there exists a process $\{x(t)\}_{t_0-r\le t\le t_1}$ satisfying (i)-(iii) of Definition, $x_{t_0}=\xi$, and (3.10) on $[t_0,t_1]$.

(c) Existence on $[t_0,t_1]$ (general). For general initial data ξ that is a $\mathcal{F}_{t_0}$-measurable $\mathcal{PC}$-valued random variable, we define, for $N>0$,

$$\xi^N=\begin{cases}\xi, & \|\xi\|\le N,\\ 0, & \text{otherwise.}\end{cases}$$

According to the previous argument, there exists a process $\{x^N(t)\}_{t_0-r\le t\le t_1}$ satisfying

$$x^N(t)=\xi^N(t_0)+\int_{t_0}^t f_{\sigma(s)}(s,x_s^N)ds+\int_{t_0}^t g_{\sigma(s)}(s,x_s^N)dW(s),\quad t\in[t_0,t_1], \tag{3.11}$$

and $x_{t_0}^N=\xi^N$. Suppose that $M>N$. Define an $\mathcal{F}_{t_0}$-measurable random variable χ_N as $\chi_N=1$, for $\|\xi\|\le N$, and $\chi_N=0$, for $\|\xi\|>N$. Then $\chi_N[x_{t_0}^M-x_{t_0}^N]=0$ and we can show that, for $t\in[t_0,t_1]$,

$$\begin{aligned}\mathbb{E}\left(\chi_N\|x_t^M-x_t^N\|\right)&\le\mathbb{E}\left(\chi_N\sup_{t-r\le s\le t}|x^M(s)-x^N(s)|\right)\\&\le 2(t_1-t_0+4)K^2\int_{t_0}^t\mathbb{E}\left(\chi_N\|x_s^M-x_s^N\|^2\right)ds,\end{aligned}$$

which, by Gronwall's inequality, implies that

$$\mathbb{E}\left(\chi_N \|x_t^M - y_t^N\|^2\right) = 0, \quad t \in [t_0, t_1],$$

and hence, for $t \in [t_0, t_1]$,

$$P\left\{\sup_{M>N} \|x_t^M - x_t^N\| > 0\right\} \leq P\left\{\bigcup_{M>N} \left\{\|x_t^M - x_t^N\| > 0\right\}\right\} \leq P\left\{\|\xi\| > N\right\}.$$

Since $P\{\|\xi\| > N\} \to 0$ as $N \to \infty$, it follows from the previous inequality that, for fixed $t \in [t_0, t_1]$, the sequence $\{x_t^N\}_{N=1}^{\infty}$ is a Cauchy sequence with respect to the supremum norm $\|\cdot\|$ with probability 1, i.e. $x^N(s)$ converges uniformly on $[t-r, t]$, with probability 1, for any fixed $t \in [t_0, t_1]$. Since the interval $[t_0, t_1]$ is of finite length and $r > 0$, it actually follows that $x^N(s)$ converges uniformly on $[t_0 - r, t_1]$, with probability 1. Let the limit process be $\{x(t)\}_{t_0 - r \leq t \leq t_1}$, i.e.

$$x(t) = \lim_{N\to\infty} x^N(t), \quad \text{uniformly for} \quad t \in [t_0 - r, t_1].$$

Passing the limit as $N \to \infty$ in (3.11) (for details, see the argument in the previous case for passing the limit of $\{x^{(n)}(t)\}$ in (3.5) to obtain (3.10)) shows that $\{x(t)\}_{t_0-r\leq t\leq t_1}$ satisfies (i)-(iii) of Definition 2.2 and (3.10) with $x_{t_0} = \xi$.

(d) Existence on $[t_0, T]$. If $t_1 = T$, this part is not necessary and the proof is already complete. If $t_1 < T$, we proceed to showing that we can modify the process $\{x(t)\}_{t_0-r\leq t\leq t_1}$ at $t = t_1$ and, by induction, obtain a process $\{x(t)\}_{t_0-r\leq t\leq T}$ that is a solution of (2.1) on $[t_0, T]$, i.e. a process $\{x(t)\}_{t_0-r\leq t\leq T}$ satisfying (i)-(iv) in Definition 2.2. Redefine

$$x(t_1) = x(t_1^-) + I_{j_1}(t_1, x_{t_1^-}).$$

It follows from measurability of I that $x(t_1)$ is $\mathcal{F}_{t_1}$-measurable and hence, by Lemma 3.1, $\{x_t\}_{t_0\leq t\leq t_1}$ is $\mathcal{F}_t$-adapted. Particularly, x_{t_1} is a $\mathcal{PC}$-valued $\mathcal{F}_{t_1}$-measurable random variable. Repeat the argument for existence on $[t_0, t_1]$, we can obtain a process $\{x(t)\}_{t_1\leq t\leq t_2}$ that is continuous on $[t_1, t_2]$, $\mathcal{F}_t$-adapted, and satisfies

$$x(t) = x(t_1) + \int_{t_1}^{t} f_{\sigma(s)}(s, x_s)ds + \int_{t_1}^{t} g_{\sigma(s)}(s, x_s)dW(s), \quad t \in [t_1, t_2].$$

Redefining $x(t_2)$ to satisfy the impulse relation, repeating this procedure and by induction on the impulse times t_k, $k = 1, 2, \cdots, m$, between $[t_0, T]$, we can find processes $\{x(t)\}_{t_k\leq t\leq t_{k+1}}$, $k = 0, 1, 2, \cdots, m-1$, such that

$$x(t) = x(t_k) + \int_{t_k}^{t} f_{\sigma(s)}(s, x_s)ds + \int_{t_k}^{t} g_{\sigma(s)}(s, x_s)dW(s), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, \cdots, m-1,$$

and

$$x(t_k) = x(t_k^-) + I_{j_k}(t_k, x_{t_k^-}), \quad k = 1, 2, \cdots, m-1.$$

If $t_m = T$ is also an impulse and switching time, then $x(T)$ has to be redefined to satisfy the impulse relation as well, i.e.

$$x(T) = x(T^-) + I_{j_m}(T, x_{T^-}).$$

Piecing together these processes, we obtain $\{x(t)\}_{t_0-r\le t\le T}$, which can be verified to satisfy (i)-(iv) of Definition 2.2 on $[t_0,T]$, i.e. $\{x(t)\}_{t_0-r\le t\le T}$ is a solution to system (2.1) on $[t_0,T]$. The proof is complete. $\square$

Theorem 3.2. *Suppose that conditions (i)-(iii) in Theorem 3.1 hold. In addition, assume that* $\xi\in\mathcal{L}^p_{\mathcal{F}_{t_0}}$*, for some* $p\ge 2$*, and*

$$\mathbb{E}\left(\|\phi\|^p\right)<\infty \quad \textit{implies} \quad \mathbb{E}\left(|\phi(0)+I_j(t,\phi)|^p\right)<\infty, \tag{3.12}$$

for all $j\in\mathcal{N}_d$*,* $t\in\mathbb{R}^+$*, and* $\phi\in\mathcal{L}^p_{\mathcal{F}_t}$*. Then*

$$\mathbb{E}\left(\|x_t\|^p\right)<\infty, \quad \forall t\in[t_0,T].$$

Proof. Keep the same notation as in the proof of Theorem 3.1. We shall prove the conclusion by induction. First we show that

$$\mathbb{E}\left(\|x_t\|^p\right)<\infty, \quad t\in[t_0,t_1].$$

For each integer $n\ge 1$, define a stopping time

$$\tau_n=t_1\wedge\inf\left\{t\in[t_0,t_1]:\ \|x_t\|\ge n\right\}.$$

It is easy to see that τ_n is increasing and $\tau_n\to t_1$ a.s. as $n\to\infty$. Set $x^{(n)}(t)=x(t\wedge\tau_n)$ for $t\in[t_0-r,t_1)$. We have

$$x^{(n)}(t)=\xi(t_0)+\int_{t_0}^t f_{\sigma(s)}(s,x_s^{(n)})\chi_{[t_0,\tau_n]}(s)ds+\int_{t_0}^t g_{\sigma(s)}(s,x_s^{(n)})\chi_{[t_0,\tau_n]}(s)dW(s), \tag{3.13}$$

where $\chi_{[t_0,\tau_n]}(s)=1$, for $s\in[t_0,\tau_n]$, and $\chi_{[t_0,\tau_n]}(s)=0$ otherwise. By virtue of Hölder's inequality, the linear growth condition (3.2), and (3.13), we have, for $t\in[t_0,t_1)$,

$$\begin{aligned}
\mathbb{E}\left(\|x_t^{(n)}\|^p\right)&=\mathbb{E}\left(\sup_{t-r\le s\le t}|x^{(n)}(t)|^p\right)\\
&\le 3^{p-1}\mathbb{E}\left(\|\xi\|^p\right)+3^{p-1}\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left|\int_{t_0}^s f_{\sigma(s)}(s,x_s^{(n)})ds\right|^p\right)\\
&\quad+3^{p-1}\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left|\int_{t_0}^s g_{\sigma(s)}(s,x_s^{(n)})dW(s)\right|^p\right),
\end{aligned} \tag{3.14}$$

where, by Hölder's inequality and the linear growth condition (3.2),

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left|\int_{t_0}^{s} f_{\sigma(s)}(s,x_s^{(n)})ds\right|^p\right)\\
&\quad\le \mathbb{E}\left(\int_{t_0}^{t}|f_{\sigma(s)}(s,x_s^{(n)})|ds\right)^p\\
&\quad\le (t_1-t_0)^{\frac{p}{2}}\mathbb{E}\left(\int_{t_0}^{t}|f_{\sigma(s)}(s,x_s^{(n)})|^2ds\right)^{\frac{p}{2}}\\
&\quad\le (t_1-t_0)^{\frac{p}{2}}\mathbb{E}\left(\int_{t_0}^{t}K^2(1+\|x_s^{(n)}\|^2)ds\right)^{\frac{p}{2}}\\
&\quad\le (t_1-t_0)^{\frac{p}{2}}K^p\left[2^{\frac{p}{2}-1}(t_1-t_0)^{\frac{p}{2}}+2^{\frac{p}{2}-1}\mathbb{E}\left(\int_{t_0}^{t}\|x_s^{(n)}\|^2ds\right)^{\frac{p}{2}}\right]\\
&\quad\le (t_1-t_0)^{\frac{p}{2}}K^p\left[2^{\frac{p}{2}-1}(t_1-t_0)^{\frac{p}{2}}+2^{\frac{p}{2}-1}(t_1-t_0)^{\frac{p}{2}-1}\int_{t_0}^{t}\mathbb{E}\left(\|x_s^{(n)}\|^p\right)ds\right],
\end{aligned}$$

and, by the Burkholder-Davis-Gundy inequality and the linear growth condition (3.2),

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t_0\vee(t-r)\le s\le t}\left|\int_{t_0}^{s} g_{\sigma(s)}(s,x_s^{(n)})dW(s)\right|^p\right)\\
&\quad\le C_p\mathbb{E}\left(\int_{t_0}^{t}|g_{\sigma(s)}(s,x_s^{(n)})|^2ds\right)^{\frac{p}{2}}\\
&\quad\le C_pK^p2^{\frac{p}{2}-1}\left[(t_1-t_0)^{\frac{p}{2}}+(t_1-t_0)^{\frac{p}{2}-1}\int_{t_0}^{t}\mathbb{E}\left(\|x_s^{(n)}\|^p\right)ds\right],
\end{aligned}$$

where C_p is a constant that depends only on p. Hence, (3.14) reduces to

$$\mathbb{E}\left(\|x_t^{(n)}\|^p\right)\le\kappa_1+\kappa_2\int_{t_0}^{t}\mathbb{E}\left(\|x_s^{(n)}\|^p\right)ds,\quad t\in[t_0,t_1), \tag{3.15}$$

where

$$\kappa_1=3^{p-1}\left[\mathbb{E}\left(\|\xi\|^p\right)+K^p(t_1-t_0)^p+C_p2^{\frac{p}{2}-1}K^p(t_1-t_0)^{\frac{p}{2}}\right]<\infty,$$

and

$$\kappa_2=3^{p-1}2^{\frac{p}{2}-1}K^p\left[(t_1-t_0)^{p-1}+C_p(t_1-t_0)^{\frac{p}{2}-1}\right]<\infty.$$

By Gronwall's inequality, (3.15) implies that

$$\mathbb{E}\left(\|x_t^{(n)}\|^p\right)\le\kappa_1e^{\kappa_2(t-t_0)},\quad t\in[t_0,t_1),$$

which, by letting $n\to\infty$, yields that

$$\mathbb{E}\left(\|x_t\|^p\right)\le\kappa_1e^{\kappa_2(t-t_0)},\quad t\in[t_0,t_1).$$

Consequently, $\mathbb{E}\left(\|x_t\|^p\right)<\infty$ for all $t\in[t_0,t_1)$ and $\mathbb{E}\left(\|x_{t_1^-}\|^p\right)<\infty$. At $t=t_1$, it follows from (3.12) that

$$\begin{aligned}\mathbb{E}\left(\|x_{t_1}\|^p\right)&=\mathbb{E}\left(\sup_{t_1-r\le s\le t_1}|x^{(n)}(t)|^p\right)\\&\le\mathbb{E}\left(|x(t_1)|^p\vee\|x_{t_1^-}\|^p\right)\\&=\mathbb{E}\left(|x(t_1^-)+I_{j_1}(t_1,x_{t_1^-})|^p\vee\|x_{t_1^-}\|^p\right)\\&\le\mathbb{E}\left(|x(t_1^-)+I_{j_1}(t_1,x_{t_1^-})|^p\right)+\mathbb{E}\left(\|x_{t_1^-}\|^p\right)<\infty.\end{aligned}$$

Repeating the same argument on $[t_k,t_{k+1}]$, $k=1,2,\cdots,m-1$, we obtain the required conclusion that $\mathbb{E}\left(\|x_t\|^p\right)<\infty$ for all $t\in[t_0,T]$. The proof is complete. □

Corollary 3.1. *Suppose that conditions (i)-(iii) in Theorem 3.1 hold for each $T>0$. Then there exists a unique solution to system (2.1) on $[t_0,\infty)$. If, in addition, (3.12) is satisfied and $\xi\in\mathcal{L}^p_{\mathcal{F}_{t_0}}$, for some $p\ge 2$, then $\mathbb{E}\left(\|x_t\|^p\right)<\infty$ for all $t\ge t_0$.*

Remark 3.1. The Lipschitz condition (3.1) in Theorem 3.1 can be replaced by the following local Lipschitz condition: for each integer $n\ge 1$,

$$|f_i(t,\phi)-f_i(t,\psi)|+|g_i(t,\phi)-g_i(t,\psi)|\le K_n\|\phi-\psi\|, \tag{3.16}$$

holds for all $(t,\phi,\psi)\in[t_0,T]\times\mathcal{PC}\times\mathcal{PC}$, $i\in\mathcal{N}_c$, and $\|\phi\|\vee\|\psi\|\le n$, where K_n is a constant depending on n. The proof for uniqueness remains the same, while the existence can be proved by the standard truncation procedure (see [3] or [12]).

Remark 3.2. Following the idea of [6], it is possible to establish some existence results that guarantee the solution of system (2.1) exists under the condition that the initial date ξ satisfies $\mathbb{E}\left(\|\xi\|^4\right)<\infty$. However, this type of results are not desirable due to the impulse effects - even if the initial data ξ satisfies this moment condition, it cannot be guaranteed that the fourth moment of the solution is still finite after a certain impulse. For existence and uniqueness results on continuous stochastic functional or delay differential equations without impulses, see, e.g., [12] and [13].

Remark 3.3. The condition that f_i and g_i are composite-PC is to guarantee that the integrals in question, such as $\int_{t_0}^t f_i(s,x_s)ds$ and $\int_{t_0}^t g_i(s,x_s)ds$, are well-defined. This extra composite-PC condition is needed for impulsive delay differential equations, since x_t as a $\mathcal{PC}$-valued function of t is generally not even piecewise continuous [1], whereas, for a continuous function x, x_t as a $C([-r,0];\mathbb{R}^n)$-valued function is continuous (and hence measurable) with respect to t. In this paper, however, the composite-PC condition can be replaced by a weaker one, i.e. that the composite functions $f_i(s,x_s)$ and $g_i(s,x_s)$ are integrable with respect to t (one could call this condition a *composite-integrable* condition).

Remark 3.4. For $0<p<2$, Hölder's inequality implies that $\mathbb{E}\left(\|x_t\|^p\right)\le\left[\mathbb{E}\left(\|x_t\|^2\right)\right]^{\frac{p}{2}}$. Therefore, if (3.12) holds and $\xi\in\mathcal{L}^2_{\mathcal{F}_{t_0}}$, Theorem 3.2 also implies that $\mathbb{E}\left(\|x_t\|^p\right)<\infty$, where $0<p<2$. Particularly, if $\xi\in\mathcal{L}^{\rm b}_{\mathcal{F}_{t_0}}$, then $\mathbb{E}\left(\|x_t\|^p\right)<\infty$ for all $p>0$, which implies that the solution to system (2.1) has finite moments of any order.

Remark 3.5. As a special case of system (2.1) without switching, Theorem 3.1, Theorem 3.2 and Corollary 3.1 clearly can be applied to the following impulsive stochastic functional (delay) differential system:

$$\begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), \quad t \neq t_k, \quad t \geq t_0, \\ \Delta x(t) = I(t, x_{t^-}), \quad t = t_k, \\ \quad x_{t_0} = \xi, \end{cases} \tag{3.17}$$

where $\{t_k : k \in \mathbb{Z}^+\}$ is a strictly increasing sequence such that $t_k \to \infty$ as $k \to \infty$, called the *impulse times*.

4 Conclusions

While increasing attention has been paid to stability analysis and impulse control of stochastic delay systems, little has been paid to the existence and uniqueness of impulsive stochastic hybrid time-delay systems. It has been realized that there exist some key differences between continuous time-delay systems and impulsive time-delay systems. The same observation is true in the stochastic setting. By formulating the solutions in the space of piecewise continuous functions on the delay interval, we have established some existence and uniqueness results for general impulsive and switched hybrid stochastic time-delay systems. It should be pointed out that, in this paper, the impulsive and switching signals are considered to be purely time-dependent and deterministic. The cases of state-dependent and random impulsive and switching signals can be topics of future research.

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