Extending LaSalle's invariance principle to impulsive switched systems with an application to hybrid epidemic dynamics

Jun Liu[1], Xinzhi Liu[1], Wei-Chau Xie[2]
1. Department of Applied Mathematics
2. Department of Civil and Environmental Engineering
University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
E-mail: {j49liu, xzliu, xie}@uwaterloo.ca

Abstract: By introducing the notions of persistent limit set and persistent mode, we extend the classical LaSalle's invariance principle to hybrid systems exhibiting both impulses and switchings. A weak invariance principle is established for such systems, under a weak dwell-time condition on the impulsive and switching signals. This weak invariance principle is then applied to derive two asymptotic stability criteria for impulsive switched systems. As an application of the stability criteria, we investigate a switched SEIR epidemic model with pulse treatment and establish sufficient conditions for the global asymptotic stability of the disease-free solution under weak dwell-time signals.

Key Words: Switched system; Impulsive system; Hybrid system; Invariance principle; Stability; Weak dwell-time; Multiple Lyapunov functions; Hybrid SEIR model.

1 Introduction

The classical LaSalle's invariance principle [12, 13] has been extended to hybrid and switched systems by various authors (see, e.g., [1, 4, 5, 7, 8, 19, 24]). In [7], under rather general switchings (including weak dwell-time switchings), an extension of LaSalle's principle is obtained for switched linear systems. In [1], a more traditional approach is taken and the results there cover general switched nonlinear systems, while a positive dwell-time condition is assumed. In [19], the results in [1, 7] are extended and improved such that the results can deal with switched nonlinear systems with average dwell-time switching. Moreover, the weak invariance notion, which is essential to develop invariance principles for switched systems, is different from that of [1] and a more comprehensive property of the limit sets of a switched system is proved (Proposition 4.1 in [19]). The work of [24] investigates asymptotic stability of switched linear systems with dwell-time switchings using invariance-like ideas, under additional ergodicity assumption on the switching signals. The work of [5] investigates invariance principles for switched systems, following a hybrid invariance principle derived for general hybrid systems on hybrid time domains [21]. Invariance-like principles for switched systems are also obtained in [8] by exploiting the norm-observability notions, under only a weak dwell-time condition on the switching signals.

Despite the fact that there are various versions of invariance principles established in the literature for ordinary differential systems, similar invariance-like principles have not yet been well addressed for differential systems with impulse effects, which is in contrast with the fact that Lyapunov stability results on dynamical systems with impulse effects are extensively studied in the literature (e.g., [2, 10, 11, 17, 18, 20] and references therein). The only exception has been [4], in which the authors establish an invariance principle for dynamical systems with left-continuous flows, which applies to state-dependent impulsive systems as a special case. The authors of [4] point out, in the introduction, that there appear to be (at least) two difficulties to establish invariance-like principles for impulsive systems. Namely, solutions of impulsive dynamical systems are *not* continuous in time and are *not* continuous functions of the system's initial conditions, whereas these two continuity properties are essential to establish invariance principles for ordinary differential equations.

Inspired by the weak invariance principles established in [1] and [19] for switched systems using multiple Lyapunov functions, we note that, under the notion of weak invariance, those continuity properties may not be essential and, therefore, it becomes possible to establish weak invariance principles for impulsive switched systems, using a different approach from that of [4].

The main objective of this paper is to present an extension of the classical LaSalle's invariance principle to impulsive and switched hybrid systems and derive asymptotic stability criteria of impulsive switched systems as important applications of this invariance principle. It is shown that the invariance principle developed here, by using a different approach from those of [4, 5], improves various known results on switched systems by [1, 19], while assuming only rather mild restriction on the

The research for this paper was supported by the Natural Sciences and Engineering Research Council of Canada.

switching signals. The results also cover impulsive differential systems as a special case. As an interesting application of the main results, we investigate a switched SEIR model with pulse treatment and establish global asymptotic stability of the disease-free solution under weak-dwell time signals.

2 Notations and Definitions

Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space and $|x|$ the norm of a vector x in $\mathbb{R}^n$. Let $\mathscr{P}$ and $\mathscr{Q}$ be two index sets. By an *impulsive and switching signal*, we mean a sequence of triples $\{(t_k, p_k, q_k) : k = 0, 1, 2, \cdots\}$, where $0 = t_0 < t_1 < t_2 < \cdots$ and $t_k \to \infty$ as $k \to \infty$; $p_k \in \mathscr{P}$; and $q_k \in \mathscr{Q}$. The instants t_k are called the *impulse and switching times*.
Instead of the usual dwell-time conditions or average dwell-time conditions on the switching signals (see [6, 15]), only a rather mild condition, i.e., the weak dwell-time condition as in [7,8], is assumed throughout this paper and formulated as follows.

Definition 2.1. An impulsive and switching signal $\{(t_k, p_k, q_k)\}$ is said to have *weak dwell-time* $\tau > 0$ if one of the following equivalent statements holds:

(i) there exists some $p \in \mathscr{P}$ such that, for every $T \geq 0$, we can find a positive integer m such that $t_{m+1} - t_m \geq \tau$ with $t_m \geq T$ and $p_m = p$ (the pth mode is called a *persistent mode* of the signal); or

(ii) there exists some $p \in \mathscr{P}$ such that the union of all the intervals of the form $[t_m, t_{m+1})$ with length greater than τ and $p_m = p$, denoted by $\mathscr{I}_p$, has an infinite Lebesgue measure (we call $\mathscr{I}_p$ a τ-*persistent domain* for the signal).

An *impulsive switched system*, defined by a family of vector fields $\{f_p : p \in \mathscr{P}\}$, a family of impulse functions $\{I_q : q \in \mathscr{Q}\}$, both f_p and I_q are defined from $\mathbb{R}^n$ to $\mathbb{R}^n$, and an admissible set of impulsive and switching signals $\mathscr{S}$, can be written as

$$x'(t) = f_{p_k}(x(t)), \quad t \in (t_k, t_k + 1), \tag{2.1}$$

$$\Delta x(t) = I_{q_k}(x(t^-)), \quad t = t_k, \tag{2.2}$$

where $\{(t_k, p_k, q_k)\} \in \mathscr{L}$, $\Delta x(t) = x(t) - x(t^-)$, and $x(t^-)$ is the left limit of x at t. Roughly speaking, we can say that the impulsive switched system consists of the switched system (2.1) and the difference equation (2.2).
If we assume $f_p(0) = 0$ for all $p \in \mathscr{P}$ and $I_q(0) = 0$ for all $q \in \mathscr{Q}$, then (2.1) and (2.2) have a trivial solution. Let $\mathscr{S}$ be a certain set of impulsive and switching signals.

Definition 2.2. The trivial solution of (2.1) and (2.2) is said to be

($\mathbb{S}_1$) *stable* with respect to $\mathscr{S}$ if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for each solution $x(t)$ starting from x_0 with $\{(t_k, p_k, q_k)\} \in \mathscr{S}$, $|x_0| < \delta$ implies that $|x(t)| < \varepsilon$ for all $t \geq 0$;

($\mathbb{S}_2$) *asymptotically stable* with respect to $\mathscr{S}$ if ($\mathbb{S}_1$) is satisfied and there exists some $\rho > 0$ such that $|x_0| < \rho$ implies that $\lim_{t\to\infty} x(t) = 0$;

($\mathbb{S}_3$) *globally asymptotically stable* with respect to $\mathscr{S}$ if ($\mathbb{S}_2$) is satisfied with arbitrary $\rho > 0$;

($\mathbb{S}_4$) *unstable* with respect to $\mathscr{S}$ if ($\mathbb{S}_1$) fails.

Definition 2.3. A family of functions $\{V_p : p \in \mathscr{P}\}$ from $\mathbb{R}^n$ to $\mathbb{R}$ are called *multiple Lyapunov functions* for (2.1) and (2.2) on a set $G \subset \mathbb{R}^n$ if

(i) V_p is continuously differentiable at each point in G and is continuous on $\overline{G}$, the closure of G;

(ii) the derivative of each V_p along the pth mode of (2.1) satisfies
$$V_p'(x) := \nabla V_p(x) \cdot f_p(x) \leq 0,$$
for all $x \in G$, where ∇ is the gradient.

Definition 2.4. A family of functions $\{V_p : p \in \mathscr{P}\}$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called *positive definite* on $G \subset \mathbb{R}^n$ if

(i) for each $p \in \mathscr{P}$, $V_p(x) \geq 0$ for all $x \in G$;

(ii) $V_p(x) = 0$ if and only if $x = 0$.

Definition 2.5. A family of functions $\{V_p : p \in \mathscr{P}\}$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called *radially unbounded* if $V_p(x) \to \infty$ as $|x| \to \infty$.

The following assumption imposes a condition on the evolution of the functions V_p along a solution at the impulse and switching instants. This type of conditions are typically encountered in results involving multiple Lyapunov functions (see [3] and [1, 7, 8, 15, 16, 19]).

Assumption 2.1. For every pair of impulse and switching instants $t_j < t_k$ of $\{(t_k, p_k, q_k)\}$ such that $p_j = p_k = p \in \mathscr{P}$, we have
$$V_p(x(t_k)) \leq V_p(x(t_{j+1})). \tag{2.3}$$

Remark 2.1. It is worth noting that, with the presence of impulse effects, Assumption 2.1 is *not* trivially satisfied for a common Lyapunov function V. Additional conditions have to be imposed, e.g., $V(x(t_k)) \leq V(x(t_k^-))$ at all switching instants t_k as in [4]. This kind of conditions can be easily satisfied by impulse control, which, in the case of multiple Lyapunov functions, can also contribute to relax the conditions on V_p imposed by Assumption 2.1.

3 Preliminaries Results

We now formulate some preliminaries for developing invariance principles for impulsive switched systems, which is the purpose of this section.
We shall let $\mathscr{S}_{\text{weak}}(\tau)$ denote the set of impulsive and switching signals with weak dwell-time τ, for some $\tau > 0$. Unless otherwise specified, let $\{(t_k, p_k, q_k)\}$ be a fixed signal in $\mathscr{S}_{\text{weak}}(\tau)$ and x the corresponding solution to (2.1) and (2.2) with $\{(t_k, p_k, q_k)\}$. Let $p \in \mathscr{P}$ be a persistent mode of the signal. Without confusion, we may also say that p is a persistent mode of x.

Definition 3.1. Given $p \in \mathscr{P}$, a point $\eta \in \mathbb{R}^n$ is said to be a *persistent limit point* of x in the pth mode, if p is a persistent mode and there exists a sequence of $s_n \in \mathscr{I}_p$, with $s_n \to \infty$ as $n \to \infty$, such that $x(s_n) \to \eta$ as $n \to \infty$. The set of all such points is called the *persistent limit set* of x in the pth mode and is denoted by $\Omega_p(x)$.

Definition 3.2. A set $M \subset \mathbb{R}^n$ is called a *weakly invariant set* with respect to the pth mode of (2.1), if, for any $\xi \in M$, there exist a positive number r and a continuously differentiable function ϕ defined on some interval $[\alpha, \beta]$, with $\alpha \le 0 \le \beta$ and $\beta - \alpha \ge r$, such that

(i) $\phi'(t) = f_p(\phi(t))$, $\forall t \in [\alpha, \beta]$,

(ii) $\phi(0) = \xi$,

(iii) $\phi(t) \in M$, $\forall t \in [\alpha, \beta]$.

Definition 3.3. The solution x is said to *weakly approach* a set $M \subset \mathbb{R}^n$ in the pth mode as $t \to \infty$, if the pth mode persists and

$$\lim_{\substack{t\to\infty \\ t\in\mathscr{I}_p}} \operatorname{dist}(x(t), M) = 0,$$

where $\operatorname{dist}(y, M)$ for $y \in \mathbb{R}^n$ is defined by

$$\operatorname{dist}(y, M) = \inf_{z\in M} \left| y - z \right|.$$

Remark 3.1. The convergence in Definition 3.3 is called "weak approaching" because the limit is only taken for $t \in \mathscr{I}_p$, not the entire real line.

Lemma 3.1. *If x is bounded and $p \in \mathscr{P}$ denotes a persistent mode of x, then $\Omega_p(x)$ is a nonempty, compact, and weakly invariant set w.r.t. the pth mode of (2.1). Moreover, $x(t)$ weakly approaches $\Omega_p(x)$ in the pth mode as $t \to \infty$.*

Proof. Since the persistent domain $\mathscr{I}_p$ has an infinite Lebesgue measure, one can pick up a sequence $\{s_n\}$ in $\mathscr{I}_p$ such that $s_n \to \infty$ as $n \to \infty$. Since $x(t)$ is bounded on $[0, \infty)$, it follows that $\{x(s_n)\}$ is a bounded sequence and therefore has a subsequence which converges to some limit point p. By definition, p is a persistent limit point in the pth mode, which shows $\Omega_p(x)$ is nonempty.
Since $x(t)$ is bounded, it follows that $\Omega_p(x)$ is bounded. To show closedness of $\Omega_p(x)$, suppose $\xi_n \in \Omega_p(x)$ approaches ξ as $n \to \infty$. Since $\xi_n \in \Omega_p(x)$ for each n, by Definition 3.1, one can choose $s_n \in \mathscr{I}_p$, for each n, large enough such that $|x(s_n) - \xi_n| < 1/n$. Now given any $\varepsilon > 0$, choose n large enough so that $|\xi_n - \xi| < \varepsilon/2$ and $|x(s_n) - \xi_n| < \varepsilon/2$. Then $|x(s_n) - \xi| < \varepsilon$ for n large enough, which shows $\xi \in \Omega_p(x)$ and therefore $\Omega_p(x)$ is closed. It follows that $\Omega_p(x)$ is compact.
The last assertion of the lemma can be shown by contradiction. Suppose that there exists an increasing sequence of s_n in $\mathscr{I}_p$, with $s_n \to \infty$ as $n \to \infty$, and a $\delta > 0$ such that $|x(s_n) - \xi| \ge \delta$ for all $\xi \in \Omega_p(x)$. Now since $x(s_n)$ is a bounded sequence, there exists a subsequence of $x(s_n)$ which converges to some $\xi \in \Omega_p(x)$. This contradicts with the inequality above and shows that the last assertion of the lemma holds.
Finally, we show that $\Omega_p(x)$ is weakly invariant w.r.t. the pth mode of (2.1) and (2.2), i.e., for any ξ in $\Omega_p(x)$, there exist a positive number r and a continuously differentiable function ϕ defined on some interval $[\alpha, \beta]$, with $\alpha \le 0 \le \beta$ and $\beta - \alpha \ge r$, such that (i) $\phi'(t) = f_p(\phi(t))$, $\forall t \in [\alpha, \beta]$, (ii) $\phi(0) = \xi$, (iii) $\phi(t) \in \Omega_p(x)$, $\forall t \in [\alpha, \beta]$.
Since $\xi \in \Omega_p(x)$, there exists an increasing sequence of $s_n \in \mathscr{I}_p$ such that $s_n \to \infty$ and $x(s_n) \to \xi$ as $n \to \infty$. Moreover, we can pick s_n so that there exists a sequence of intervals $[\tau_{2n-1}, \tau_{2n}]$ which verifies that, for all n,

(i) $\tau_{2n} - \tau_{2n-1} \ge \tau$,

(ii) $s_n \in [\tau_{2n-1}, \tau_{2n}]$,

(iii) the pth mode is activated on $[\tau_{2n-1}, \tau_{2n}]$.

By this choice, $x(t)$ satisfies the pth subsystem on $[\tau_{2n-1}, \tau_{2n}]$ for all n, i.e., $x'(t) = f_p(x(t))$, $\forall t \in [\tau_{2n-1}, \tau_{2n}]$. Moreover, we have that $x(t)$ is continuously differentiable on (τ_{2n-1}, τ_{2n}). Putting

$$\alpha_n = \tau_{2n-1} - s_n \quad \text{and} \quad \beta_n = \tau_{2n} - s_n, \tag{3.1}$$

then $\beta_n - \alpha_n \ge \tau$ and $\alpha_n \le 0 \le \beta_n$. Define

$$\phi_n(t) := x(t + s_n), \quad t \in [\alpha_n, \beta_n]. \tag{3.2}$$

It follows that ϕ_n satisfies

$$\phi_n'(t) = f_p(\phi_n(t)), \quad t \in [\alpha_n, \beta_n], \tag{3.3}$$

$\phi_n(0) = x(s_n) \to \xi$ as $n \to \infty$, and ϕ_n is continuously differentiable on (α_n, β_n).
We claim that the sequence of intervals $[\alpha_n, \beta_n]$ has a subsequence, still designated by $[\alpha_n, \beta_n]$, which has a common subinterval $[\alpha, \beta]$, i.e., $[\alpha, \beta] \subset [\alpha_n, \beta_n]$ for all n, with $\beta - \alpha \ge \tau/2$ and $\alpha \le 0 \le \beta$. Actually, it is clear that either $\{\alpha_n\}$ has a subsequence, which we can keep the same designation, that lies in $(-\infty, -\tau/2]$ or it has a subsequence in $[-\tau/2, 0]$. In the latter case, since $\beta_n - \alpha_n \ge \tau$, one must have $\beta_n \ge \tau/2$ and therefore letting $\alpha = 0$, $\beta = \tau/2$ will give the required common subinterval; in the former case, since $\beta_n \ge 0$, letting $\alpha = -\tau/2$, $\beta = 0$ gives the required common subinterval. Now according to (3.2) and (3.3), what we have obtained is a sequence of functions ϕ_n defined on a common interval $[\alpha, \beta]$, with $\beta - \alpha \ge \tau/2$ and $\alpha \le 0 \le \beta$, such that

$$\phi_n'(t) = f_p(\phi_n(t)), \quad t \in [\alpha, \beta]. \tag{3.4}$$

We proceed to show that ϕ_n has a subsequence that uniformly converges to a function ϕ on $[\alpha, \beta]$. Since $x(t)$ is bounded, it follows that ϕ_n is uniformly bounded on $[\alpha, \beta]$. Since f_p is continuous, it follows that $\phi_n'(t) = f_p(\phi_n(t))$ is uniformly bounded on $[\alpha, \beta]$. By the mean-value theorem, this implies that the sequence ϕ_n is equicontinuous on $[\alpha, \beta]$. By the Arzela-Ascoli Theorem, there exists a subsequence of ϕ_n, still designated by ϕ_n, uniformly converges to some function ϕ on $[\alpha, \beta]$ and ϕ is continuously differentiable on (α, β). Passing the limit in (3.4) (or its equivalent integral form), one is able to see that ϕ satisfies $\phi'(t) = f_p(\phi(t))$, $t \in [\alpha, \beta]$. Moreover, $\phi(0) = \lim_{n\to\infty} \phi_n(0) = \xi$. Finally, we have $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} x(t + s_n)$ for any fixed $t \in [\alpha, \beta]$. For a fixed $t \in [\alpha, \beta]$, put $s'_n = t + s_n$. According to (3.1), $s'_n \in [\tau_{2n-1}, \tau_{2n}]$ for each n. Since the pth mode is activated on $[\tau_{2n-1}, \tau_{2n}]$ and $\tau_{2n} - \tau_{2n-1} \geq \tau$, by Definition 3.1, it follows that $\phi(t) \in \Omega_p(x)$ for all $t \in [\alpha, \beta]$ as required. Therefore $\Omega_p(x)$ is shown to be weakly invariant w.r.t. the pth mode of (2.1) and (2.2) and the proof is complete. □

4 Main Results

4.1 A weak invariance principle

Let $\{V_p : p \in \mathscr{P}\}$ be a family of multiple Lyapunov functions for (2.1) and (2.2) on $G \subset \mathbb{R}^n$ and define

$$E_p := \left\{ x \in \overline{G} : V'_p(x) = 0 \right\}.$$

Let M_p denote the largest weakly invariant set w.r.t. the pth mode of (2.1) and (2.2) in E_p.

Theorem 4.1. *Let $\{V_p : p \in \mathscr{P}\}$ be a family of multiple Lyapunov functions for (2.1) and (2.2) on G, x be a bounded solution of (2.1) and (2.2) such that $x(t)$ remains in G for $t \geq 0$, and $p \in \mathscr{P}$ be a persistent mode of x. Suppose, in addition, Assumption 2.1 is satisfied. Then x weakly approaches $M_p \cap V_p^{-1}(c)$, for some c, in the pth mode as $t \to \infty$.*

Proof. By Lemma 3.1, x has a nonempty persistent limit set in the pth mode $\Omega_p(x)$. We proceed to show that $\Omega_p(x) \subset E_p$. Let $\mathscr{I}_p$ denote the union of all the intervals of length greater than τ such that the pth mode is active. Since the pth mode persists, $\mathscr{I}_p$ must have an infinite Lebesgue measure. The conditions on V_p imply that $V_p(x(t))$ is nonincreasing on $\mathscr{I}_p$. Moreover, $V_p(x(t))$ is bounded below since $x(t)$ is bounded. Therefore, as $t \to \infty$ in $\mathscr{I}_p$, $V_p(x(t))$ yields a limit as

$$\lim_{\substack{t\to\infty \\ t\in\mathscr{I}_p}} V_p(x(t)) = c.$$

For any $\xi \in \Omega_p(x)$, there exists a sequence $s_n \in \mathscr{I}_p$ such that $s_n \to \infty$ and $x(s_n) \to \xi$ as $n \to \infty$. It follows by the continuity of V_p that

$$V_p(\xi) = \lim_{n\to\infty} V_p(x(s_n)) = \lim_{\substack{t\to\infty \\ t\in\mathscr{I}_p}} V_p(x(t)) = c.$$

Hence, $V_p(\xi) = c$ for all $\xi \in \Omega_p(x)$ and $\Omega_p(x) \subset V_p^{-1}(c)$. According to Lemma 3.1, $\Omega_p(x)$ is weakly invariant w.r.t. the pth mode, that is, for each $\xi \in \Omega_p(x)$, there exist a positive number r and a continuous differentiable function ϕ defined on some interval $[\alpha, \beta]$, with $\alpha \leq 0 \leq \beta$ and $\beta - \alpha \geq r$, such that (i) $\phi'(t) = f_p(\phi(t))$, $\forall t \in [\alpha, \beta]$, (ii) $\phi_0 = \xi$, (iii) $\phi(t) \in \Omega_p(x)$, $\forall t \in [\alpha, \beta]$. Hence, $V_p(\phi(t)) = c$ for all $t \in [\alpha, \beta]$. Differentiating $V_p(\phi(t))$ at $t = 0$ gives

$$V'_p(\xi) = \nabla V_p(\xi) \cdot f_p(\xi) = 0.$$

It follows that $\Omega_p(x) \subset E_p$. By the definition of M_p and because $\Omega_p(x)$ is weakly invariant w.r.t. the pth mode of (2.1), we have $\Omega_p(x) \subset M_p \subset E_p$. From Lemma 3.1, x weakly approaches $\Omega_p(x)$ in the pth mode and therefore it weakly approaches $M_p \cap V_p^{-1}(c)$ in the pth mode. This completes the proof. □

4.2 Stability criteria

We now apply the invariance principle established in Section 4.1 to derive some results on asymptotic stability for impulsive switched systems under weak dwell-time conditions. Let $\mathscr{S}^p_{\text{weak}}(\tau)$ denote the set of impulsive and switching signals with weak dwell-time τ and $p \in \mathscr{P}$ as a persistent mode. We use B_r to denote the ball $\left\{x \in \mathbb{R}^n : |x| \leq r\right\}$ for any $r > 0$.

Theorem 4.2. *Suppose that $\mathscr{P}$ is a finite set and there exist a family of positive definite multiple Lyapunov functions $\{V_p : p \in \mathscr{P}\}$ for (2.1) and (2.2) on B_ρ for some $\rho > 0$. Suppose, in addition, Assumption 2.1 is satisfied. Then the trivial solution of (2.1) and (2.2) is asymptotically stable w.r.t $\mathscr{S}^p_{weak}(\tau)$, provided that $M_p = \{0\}$. If $\bigcup_{p\in\mathscr{P}} M_p = \{0\}$, then the trivial solution of (2.1) and (2.2) is asymptotically stable w.r.t $\mathscr{S}_{weak}(\tau)$.*

Proof. Local stability follows from a standard argument using multiple Lyapunov functions (see, e.g., [3, 16]). Given any $\varepsilon_0 \in (0, \rho)$, let $\delta(\varepsilon_0)$ be the local stability constant such that $x_0 \in B_{\delta(\varepsilon_0)}$ implies $x(t) \in B_{\varepsilon_0} \subset B_\rho$ for all $t \geq 0$. We claim that $B_{\delta(\varepsilon_0)}$ is a domain of attraction for (2.1) and (2.2). Actually, $x(t) \in B_{\varepsilon_0}$ for all $t \geq 0$, x is clearly a bounded solution. According to Theorem 4.1, any solution staring from $B_{\delta(\varepsilon_0)}$ weakly approaches $M_p = \{0\}$ in the pth mode as $t \to \infty$. Now for any $\varepsilon \in (0, \varepsilon_0)$, let $\delta(\varepsilon) > 0$ be the local stability constant such that $x_0 \in B_{\delta(\varepsilon)}$ implies $x(t) \in B_\varepsilon$ for all $t \geq 0$. Let x be an arbitrary solution starting from $B_{\delta(\varepsilon_0)}$. Since x weakly approaches $M_p = \{0\}$ in the pth mode as $t \to \infty$, we can find T large enough in $\mathscr{I}_p$ so that $|x(T)| \leq \delta$, for any $\delta = \delta(\varepsilon) > 0$. By how we have chosen $\delta(\varepsilon)$, it follows that $|x(t)| < \varepsilon$ for all $t \geq T$. Because ε can be arbitrarily chosen, this shows $x(t) \to 0$ as $t \to \infty$ and asymptotic stability follows. □

Theorem 4.3. *Suppose that $\mathscr{P}$ is a finite set. Let $\{V_p : p \in \mathscr{P}\}$ be a family of positive definite and radially unbounded multiple Lyapunov functions for (2.1) and (2.2) on $\mathbb{R}^n$. Suppose, in addition, Assumption 2.1 is satisfied. Then the trivial solution of (2.1) and (2.2) is globally asymptotically stable w.r.t. $\mathscr{S}^p_{weak}(\tau)$, provided that $M_p = \{0\}$. If $\bigcup_{p\in\mathscr{P}} M_p = \{0\}$, then the trivial solution of (2.1) and (2.2) is globally asymptotically stable w.r.t. $\mathscr{S}_{weak}(\tau)$.*

Proof. Local stability remains the same. To show global attraction, we only have to show that the constant $\delta(\varepsilon_0)$ in the proof for Theorem 4.2 can be chosen to be arbitrarily large, provided that ε_0 is given sufficiently large. Now given any $\varepsilon_0 > 0$, we let $\delta_0 = \varepsilon_0$ and define δ_1, δ_2, $\cdots$, δ_N recursively such that $v(\delta_{j+1}) = u(\delta_j/2)$ for $j = 0, 1, 2, \cdots, N-1$, where N is the cardinality of $\mathscr{P}$. Since $u(\delta_j/2) < u(\delta_j)$, it is clear that this choice of δ_1, δ_2, $\cdots$, δ_N is in accordance with that in the proof of local stability in Theorem 4.2. Moreover, as $\varepsilon_0 \to \infty$, so is $\delta_1, \delta_2, \cdots$, and

δ_N. Therefore $\delta(\varepsilon_0) = \delta_N(\varepsilon_0)$ can be arbitrarily large if ε_0 is given sufficiently large. This completes the proof. □

Remark 4.1. Specialized to the switched system framework, two main features that make our results improve those in [1] and [19] are as follows. First, the notion of persistent limit set is introduced here, and it is therefore not required that the solution converges to its limit set (only weakly approaching is required). This weakly approaching notion together with local stability (guaranteed by the existence of multiple Lyapunov functions) actually suffices to guarantee asymptotic stability as shown in Theorem 4.2. Second, the largest invariant set M_p in each E_p is explicitly defined with respect to the pth mode, which makes the set $\bigcup_{p\in\mathscr{P}} M_p$ smaller (and in many cases remarkably smaller, as shown by several examples in Section 5) than the set M in [1], which is defined as the largest weakly invariant set (with respect to whatever mode) in the set $\bigcup_{p\in\mathscr{P}} E_p$ (according to our notation). This difference makes our results less conservative. Particularly, Theorem 4.3, specialized to switched systems without impulses, improves both Theorem 1 (in the case where a common Lyapunov function exists) and Theorem 2 of [1], the main results presented there, not only in that the dwell-time conditions are now replaced by weak dwell-time conditions, but also, more importantly, in that the set M is now remarkably smaller and, therefore, more precise convergence results can be obtained.

Remark 4.2. Even though $\bigcup_{p\in\mathscr{P}} M_p$ obtained in Theorem 4.2 and Theorem 4.3 can be remarkably smaller than the set M defined in [1], we may not yet, in some cases, be able to conclude that $\bigcup_{p\in\mathscr{P}} M_p = \{0\}$. However, if we know that the pth mode of a switching signal persists and $M_p = \{0\}$, for some $p \in \mathscr{P}$, then we are still able to show that the solution converges to zero from Theorem 4.2 and Theorem 4.3, while none of the results in [1, 7, 8, 19] can give any useful information.

5 An application to hybrid epidemic dynamics

In this section, we present an application of the main results to investigate a switched SEIR model with pulse treatment. The model is given by

$$\left.\begin{aligned} \dot S &= \mu_{p_k} - \beta_{p_k} SI - \mu_{p_k} S,\\ \dot E &= \beta_{p_k} SI - a_{p_k} E - \mu_{p_k} E,\\ \dot I &= \beta_{a_k} E - g_{p_k} I - \mu_{p_k} I,\\ \dot R &= g_{p_k} I - \mu_{p_k} R, \end{aligned}\right\} \quad t \in (t_k, t_{k+1}), \quad (5.1)$$

$$\left.\begin{aligned} S(t) &= S(t^-),\\ E(t) &= (1-q_k)E(t^-),\\ I(t) &= (1-q_k)I(t^-),\\ R(t) &= R(t^-) + q_k E(t^-) + q_k I(t^-), \end{aligned}\right\} \quad t = t_k.$$

where S represents the susceptibles, E those exposed but not yet infectious, I the infectives, and R the removed. The total population is constant and assumed to be $S + E + I + R = 1$. The variables S, E, I, and R are all positive and always take values in the meaningful domain

$$\Omega_{SEIR} := \left\{(S,E,I,R) \in \mathbb{R}_+^4 : S + E + I + R = 1\right\}.$$

Each $p_k \in \mathscr{P}$ and $\mathscr{P}$ is a finite index set. Each q_k is a real number in $[0,1]$, representing a fraction of the population which is treated and removed at time $t = t_k$. The coefficients μ_p represent the birth and death rates, β_p the contact rates, g_p the removal rates, and $1/a_p$ the latent periods; these constants are all positive and allowed to fluctuate seasonally. Choose a family of multiple Lyapunov functions $\{V_p : p \in \mathscr{P}\}$ as $V_p(E,I) = a_p E + (a_p + \mu_p) I$. Differentiating V_p along the pth mode of (5.1) gives

$$\begin{aligned} V_p'(E,I) &= a_p(\beta_p SI - a_p E - \mu_p E)\\ &\qquad + (a_p + \mu_p)(\beta_a E - g_p I - \mu_p I)\\ &= (R_p S - 1)(\mu_p + g_p)(\mu_p + a_p) I, \end{aligned} \quad (5.2)$$

where

$$R_p := \frac{\beta_p a_p}{(\mu_p + g_p)(\mu_p + a_p)}$$

is the pth reproduction number. It is easy to see that (5.1) has a disease-free equilibrium $(1,0,0,0)$. We have the following conclusion.

Theorem 5.1. *Suppose $R_p \le 1$ for all $p \in \mathscr{P}$ and Assumption 2.1 is satisfied. Then*

(i) the disease-free solution of (5.1) is globally asymptotically stable (in the meaningful domain) w.r.t. $\mathscr{S}_{weak}^{p_0}(\tau)$, if $R_{p_0} < 1$ for some $p_0 \in \mathscr{P}$;

(ii) the disease-free solution of (5.1) is globally asymptotically stable w.r.t. $\mathscr{S}_{weak}(\tau)$, if $R_p < 1$ for all $p \in \mathscr{P}$.

Proof. By (5.2), we have $E_{p_0} = \{(S,E,I,R) : I = 0\}$ in case (i). It then can be shown that $M_{p_0} = \{(1,0,0,0)\}$. The conclusions follow from Theorem 4.3. □

Remark 5.1. The main role of the pulse treatment is to control the exposed and infectious population by applying treatment at discrete times so that a non-increasing condition on this population, i.e. Assumption 2.1, is satisfied. If $a_p \equiv a$ and $\mu_p \equiv \mu$, i.e. the latent period and birth/death rate are both constant, then the multiple Lyapunov functions V_p reduce to a common (weak) Lyapunov function. Therefore, Assumption 2.1 is trivially satisfied if we do not introduce pulse treatment, and Theorem 5.1 remains true in the absence of pulse treatment, which agrees with the results reported in [23], where the author considers a single switching parameter β_p in the SEIR model without pulse treatment and the stability condition is given by $R_p < 1$ for all $p \in \mathscr{P}$.

SIMULATIONS: For simulation purpose, we choose two different modes of (5.1), motivated by different seasons, e.g. winter and non-winter seasons, and represented by two sets of parameters $\{\mu_1 = 0.2, \beta_1 = 2, g_1 = 1, a_1 = 0.3\}$ and $\{\mu_2 = 0.1, \beta_2 = 1, g_2 = 1, a_2 = 20\}$. The reproduction numbers for the two modes can be calculated as $R_1 = 1$ and $R_2 = 0.9046$. Therefore, without switching and impulses, the disease-free solution of each individual mode is globally asymptotically stable [14]. However, if we introduce a periodic switching signal, dwelling equally on the two modes, the simulation results (Figure 1) show

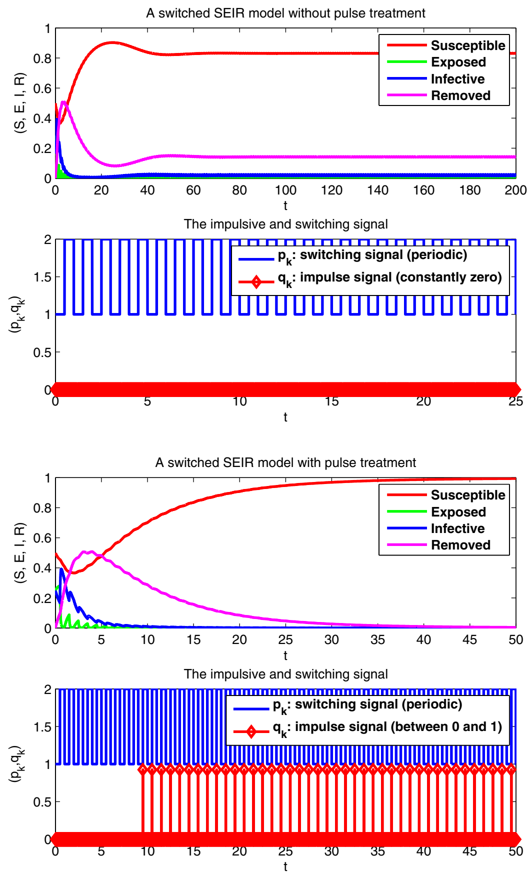

Figure 1: Numerical simulations for a switched SEIR model with and without pulse treatment.

that the disease persists, if no pulse treatment is given. This is due to a possible violation of Assumption 2.1 required by Theorem 5.1. If we introduce pulse treatment so that Assumption 2.1 is satisfied, then Theorem 5.1 can guarantee that the disease-free solution is globally asymptotically stable, i.e. the disease is eradicated by the pulse treatment, which is verified by the simulation results shown in Figure 1.

6 Conclusions

An invariance principle is established for a general class of impulsive switched systems, which generalizes the classical LaSalle's principle to the general hybrid setting of impulsive switched systems under weak dwell-time signals. Asymptotic stability results for impulsive switched systems under weak dwell-time impulsive and switching signals are derived as important applications of the invariance principle. The stability criteria are then successfully applied to study a switched SEIR model with pulse treatment.

REFERENCES

[1] A. Bacciotti and L. Mazzi, An invariance principle for nonlinear switched systems, *Systems & Control Letters*, vol. 54, pp. 1109–1119, 2005.

[2] D. Bainov and P. Simeonov, *Systems with Impulse Effects: Stability, Theory and Applications*, Ellis Horwood Limited, England, 1989.

[3] M. S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, *IEEE Trans. Automat. Conrtol*, vol. 43, pp. 475–482, 1998.

[4] V. Chellaboina, S. P. Bhat, W. M. Haddad, An invariance principle for nonlinear hybrid and impulsive dynamical systems, *Nonlinear Anal.*, vol. 53, pp. 527–550, 2003.

[5] R. Goebel, R. G. Sanfelice, and A. R. Teel, Invariance principles for switching systems via hybrid systems techniques, *Systems Control Lett.*, vol. 57, pp. 980–986, 2008.

[6] J. P. Hespanha and A. S. Morse, Stability of switched systems with average dwell time, *Proceedings of the 38th IEEE Conf. on Decision and Control*, vol. 3, pp. 2655–2660, 1999.

[7] J. P. Hespanha, Uniform stability of switched linear systems: extentions of LaSalle's invariance principle, *IEEE Trans. Automat. Control*, vol. 49, pp. 470–482, 2004.

[8] J. P. Hespanha, D. Liberzon, D. Angeli, and E. Sontag, Nonlinear observability notions and stability of switched systems, *IEEE Trans. Automat. Control*, vol. 50, pp. 154–168, 2005.

[9] H. K. Khalil, *Nonlinear Systems*, Prentice-Hall, 2002.

[10] V. Lakshmikantham, D. Bainov, and P. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[11] V. Lakshmikantham and X. Liu, On quasistability for impulsive differential systems, *Nonlinear Anal.*, vol. 13, pp. 819–828, 1989.

[12] J. P. LaSalle, The extent of asymptotic stability, *Proc. Natl. Acad. Sci. U.S.A.*, vol. 46, pp. 363–365, 1960.

[13] J. P. LaSalle, Some extensions of Liapunov's second method, *IRE Trans. Circuit Theory*, vol. 7, pp. 520–527, 1960.

[14] M. Y. Li, J. R. Graef, L. Wang, and J. Karsai, Global dynamics of a SEIR model with varying total population size, *Math. Biosci.*, vol. 160, pp. 191–213, 1999.

[15] D. Liberzon, *Switching in Systems and Control*, Boston, MA: Birkhauser, 2003.

[16] J. Liu, X. Liu, and W.-C. Xie, Uniform stability of switched nonlinear systems, *Nonlinear Anal. Hybrid Syst.*, vol. 3, pp. 441–454, 2009.

[17] X. Liu, Quasi stability via Lyapunov functions for impulsive differential systems, *Appl. Anal.*, vol. 31, pp. 201–213, 1988.

[18] X. Liu, Stability results for impulsive differential systems with applications to population growth models, *Dynam. Stability Systems*, vol. 9, pp. 163–174, 1994.

[19] J. L. Mancilla-Aguilar and R. A. Garcia, An extension of LaSalle's invariance principle for switched systems, *Systems & Control Letters*, vol. 55, pp. 376–384, 2006.

[20] A. Samoilenko and N. Perestyuk, *Impulsive Differential Equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[21] R. G. Sanfelice, R. Goebel, and A. R. Teel, Invariance principles for hybrid systems with connections to detectability and asymptotic stability, *IEEE Trans. Automat. Control*, vol. 52, pp. 2282–2297, 2007.

[22] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*, Springer-Verlag, 2000.

[23] P. Stechlinski, *A Study of Infectious Disease Models with Switching*, M. M. Thesis, University of Waterloo, 2009.

[24] J. Wang, D. Cheng, and X. Hu, An extension of LaSalle's invariance principle for a class of switched linear systems, *Systems & Control Letters*, vol. 58, pp. 754–758, 2009.