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On the (h_0, h)-stabilization of switched nonlinear systems via state-dependent switching rule [☆]

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ABSTRACT

This paper considers switching stabilization of some general nonlinear systems. Assuming certain properties of a convex linear combination of the nonlinear vector fields, two ways of generating stabilizing switching signals are proposed, i.e., the minimal rule and the generalized rule, both based on a partition of the time-state space. The main theorems show that the resulting switched system is globally uniformly asymptotically stable and globally uniformly exponentially stable, respectively. It is shown that the stabilizing switching signals do not exhibit chattering, i.e., two consecutive switching times are separated by a positive amount of time. In addition, under the generalized rule, the switching signal does not exhibit Zeno behavior (accumulation of switching times in a finite time). Stability analysis is performed in terms of two measures so that the results can unify many different stability criteria, such as Lyapunov stability, partial stability, orbital stability, and stability of an invariant set. Applications of the main results are shown by several examples, and numerical simulations are performed to both illustrate and verify the stability analysis.

1. Introduction

Switched systems are appropriate models for many systems encountered in applied fields that exhibit switching between several subsystems depending on various environmental factors. A switched system usually consists of a family of continuous subsystems and a logic-based rule that determines the activation of one of the subsystems at certain time-interval. Switched systems form a special class of hybrid systems and have numerous applications in control of mechanical systems, automotive industry, aircraft and air traffic control, switching power converters, and many other fields (see [8,9,11,12] and references therein). Therefore, switched systems and, more generally, hybrid systems are currently becoming a large and growing interdisciplinary area of research. While major advances in this topic have been made by applied mathematicians, control engineers, and, more recently, computer scientists, many important questions related to the stability analysis of such systems still remain unanswered, even for linear systems [13].

In [9,13], some basic problems related to the stability issues of switched systems are surveyed, among which we note, in particular, the problem of constructing stabilizing switching rules for a family of unstable systems. In [15] (see also [17]), the following problem is addressed:

Given two linear system $x' = A_1x$ and $x' = A_2x$, where A_1 and A_2 are not Hurwitz in that they both have some eigenvalues in the right half plane, determine if there exists a switching rule such that the resulting switched system is stable.

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It is established in [15] that, if there exists a Hurwitz convex linear combination of A_1 and A_2, i.e., there exists some $\alpha \in [0,1)$ such that $\alpha A_1 + (1-\alpha)A_2$ is Hurwitz, then a stabilizing switching rule does exist. This result can be easily generalized to the case of a switched system with finitely many subsystems. Namely, consider a family of linear systems with coefficient matrices $A_1, A_2, \ldots, A_N$ and assume there exists a Hurwitz convex linear combination of these matrices, i.e., there exist real numbers $\alpha_i \in (0,1)$ with $\sum_{i=1}^{N}\alpha_i = 1$ such that $\sum_{i=1}^{N}\alpha_i A_i$ is Hurwitz. Then a stabilizing switching rule can also be constructed. The idea of proof involves constructing a common quadratic Lyapunov function and the stability obtained is actually quadratic stability. In [14], time-dependent fast periodic switching rules are constructed for families of linear systems, based on a more direct approach analyzing the state transition matrix.

Later, the work of [15] is extended in [16], by using piecewise quadratic Lyapunov functions as opposed to quadratic Lyapunov functions. It is also extended in [1] to the cases when there exists a convex combination of A_1 and A_2 with the property that all of its eigenvalues have nonpositive real parts and any eigenvalues on the imaginary axis are simple. Moreover, it is shown in [5,10] that the result of [15] is actually robust with respect to small time-delay, uncertainties, and stochastic perturbation.

All the results above, however, rely on the existence of a quadratic Lyapunov function and only apply to switched linear systems. It is natural to ask if similar results can be obtained for switched nonlinear systems. The current paper aims to address this problem. The main results will deal with general switched nonlinear systems. Under a suitably modified assumption on the property of a convex linear combination of the nonlinear vector fields, it is shown that a state-dependent switching rule can be constructed so that the resulting switched system is asymptotically stable. It is formally proved that the stabilizing switching signal, which is generated to stabilize the family of individually unstable subsystems, does not exhibit chattering. Moreover, we can further prove that Zeno behavior can be prevented by introducing a generalized rule to generate stabilizing switching signals. It should be mentioned that, in the recent paper [2], the authors also investigate the stabilization of nonlinear systems using discontinuous feedback, corresponding to switching stabilization considered in the current paper. Their results can be applied to general families of nonlinear systems. A key difference between the results in [2] and the results in the current paper is that their results deal with solutions in the Krasowski sense and a switching signal is only required to be measurable, whereas we focus on classical solutions and require that the stabilizing switching signals are piecewise constant and well-defined in the classical sense.

Since the concept of stability in terms of two measures provides a unified notion for Lyapunov stability, partial stability, orbital stability, and stability of an invariant set of nonlinear systems [6,7], it would be desirable that we can formulate the stability results of the current paper in terms of two measures. It is worth noting that this notion has been adopted in the framework of switched systems by Chatterjee and Liberzon [4].

The rest of this paper is organized as follows. In Section 2, necessary notations and definitions are given, including the (h_0,h)-stability notion for switched systems. The main results are presented in Section 3. Both the minimal rule and generalized rule are introduced, and two types of stabilizing switchings are proposed, both based on a partition of the time-state space. A sequence of propositions are presented to establish some important properties of both the minimal rule and the generalized rule. Particularly, it is shown that the switching signals constructed do not exhibit chattering, and, under the general rule, Zeno behavior is also excluded. The main theorems then show that the resulting switched system is indeed (h_0,h)-globally uniformly asymptotically stable and (h_0,h)-globally uniformly exponentially stable, respectively. Applications of the main theorems are shown by several examples in Section 4, where the simulation results are also presented to both illustrate and verify the stability analysis. Particularly, following Example 4.1, an in-depth discussion on the comparison between a minimal switching solution and a generalized rule switching is presented. Finally, the paper is concluded by Section 5.

2. Notation and definitions

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $|x|$ denote the norm of a vector x in $\mathbb{R}^n$. Let $C[M,N]$ denote the set of all continuous functions from $M \subset \mathbb{R}^m$ to $N \subset \mathbb{R}^n$. The following special function classes are introduced for later use

$$\begin{aligned}
\mathcal{K} &:= \{\alpha \in C[\mathbb{R}^+,\mathbb{R}^+] : \alpha \text{ is strictly increasing and } \alpha(0)=0\},\\
\mathcal{K}_\infty &:= \left\{\beta \in \mathcal{K} : \lim_{s\to\infty}\beta(s)=\infty\right\},\\
\Gamma &:= \left\{h \in C[\mathbb{R}^+\times\mathbb{R}^n,\mathbb{R}^+] : \inf_{(t,x)} h(t,x)=0\right\}.
\end{aligned}$$

Consider a family of nonlinear nonautonomous systems

$$x' = f_i(t,x), \quad i \in I, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $I := \{1,2,\ldots,N\}$ is a finite index set, and $f_i \in C[\mathbb{R}^+\times\mathbb{R}^n,\mathbb{R}^n]$. A *switching signal* is a piecewise constant and right-continuous function from $\mathbb{R}^+$ to I, and it is said to be *well-defined* if it has only finitely many discontinuities on every bounded subinterval of $\mathbb{R}^+$. A *switched system* generated by the family (2.1) and a switching signal σ can be written as

$$x' = f_\sigma(t,x), \quad x(t_0)=x_0, \qquad t \geqslant t_0, \tag{2.2}$$

where $t_0 \geqslant 0$ is the initial time.

Let $h_0, h \in \Gamma$ and $x(t) := x(t; t_0, x_0)$ denote a solution of system (2.2). Then we define uniform stability, global uniform asymptotic stability and global exponential stability in terms of two measures for the switched system (2.2). Other stability notions in terms of two measures can be defined similarly based on those given in [6].

Definition 2.1. The switched system (2.2) is said to be

($\mathbf{S}_1$) (h_0, h)-*uniformly stable* if, for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies that $h(t, x(t)) < \varepsilon$ for all $t \geqslant t_0$;

($\mathbf{S}_2$) (h_0, h)-*globally uniformly asymptotically stable* if (S_1) is satisfied and, for each $\rho > 0$ and each $\varepsilon > 0$, there exists $T = T(\rho, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \rho$ implies that $h(t, x(t)) < \varepsilon$ for all $t \geqslant t_0 + T$;

($\mathbf{S}_3$) (h_0, h)-*globally uniformly exponentially stable* if there exists positive constants K and λ, independent of t_0 and x_0, such that

$$h(t, x(t)) \leqslant K h_0(t_0, x_0) e^{-\lambda(t - t_0)},$$

for all $t \geqslant t_0$.

For a continuously differentiable function $V \in C^1[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+]$, the derivative of V along the ith subsystem in the family (2.1) is given by

$$\left.\frac{dV}{dt}\right|_i (t, x) := \frac{\partial V}{\partial t}(t, x) + \nabla V(t, x) \cdot f_i(t, x),$$

where $\nabla V(t, x)$ is the gradient of V with respect to x.

3. Main results

3.1. (h_0, h)-Global uniform asymptotic stabilization

The following assumption plays an important role in the construction of a stabilizing switching rule.

Assumption 3.1. There exists a convex linear combination of $\{f_i : i \in I\}$, i.e.:

$$f_\alpha := \sum_{i=1}^{N} \alpha_i f_i,$$

with $\alpha_i \in [0, 1)$ and $\sum_{i=1}^{N} \alpha_i = 1$, a function $V \in C^1[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+]$, and functions $h_0,\ h \in \Gamma,\ a,\ b \in \mathcal{K}_\infty,\ c \in \mathcal{K}$ such that

$$a(h(t, x)) \leqslant V(t, x) \leqslant b(h_0(t, x)), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \tag{3.1a}$$

$$\frac{\partial V}{\partial t}(t, x) + \nabla V(t, x) \cdot f_\alpha(t, x) \leqslant -c(h_0(t, x)), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{3.1b}$$

In addition, V satisfies the following:

(i) a solution that satisfies any of the subsystems in the family (2.1) with $(t_0, x(t_0)) \in \Gamma_0 := \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : V(t, x) = 0\}$ will stay in Γ_0 for all $t \geqslant t_0$; and

(ii) the set

$$U(\varepsilon, \rho, [t_0, T]) := \{(t, x) \in [t_0, T] \times \mathbb{R}^n : \varepsilon \leqslant V(t, x) \leqslant \rho\},$$

where $0 < \varepsilon \leqslant \rho < \infty$ and $0 \leqslant t_0 < T < \infty$, is bounded.

Remark 3.1. Inequality (3.1a) states that the function V is h-positive definite, h-radially unbounded, and h_0-decrescent. Inequality (3.1b), inspired by the well-known results in [15–17] for linear systems, essentially proposes an assumption on the nonlinear vector fields $\{f_i : i \in I\}$ in terms of a smooth Lyapunov function for a linear convex combination of the nonlinear vector fields. Part (i) of the additional conditions says that the set Γ_0 is positively invariant with respect to each of the subsystems. In particular, if $h_0 = h = |x|$, this corresponds to that the trivial solution $x = 0$ is the unique solution for each of the subsystems in (2.1) on $[t_0, \infty)$ with $x(t_0) = 0$. In part (ii), boundedness of the set $U(\varepsilon, \rho, [t_0, T])$ is very easy to check and usually follows from radially unboundedness of the function V. Moreover, continuity of V and boundedness of the set $U(\varepsilon, \rho, [t_0, T])$ implies that it is compact, a key property to be used later.

Based on this assumption, we proceed to construct a switching signal. Define

$$\Omega_i := \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : \left.\frac{dV}{dt}\right|_i (t, x) \leqslant -c(h_0(t, x)) \right\}.$$

A switching signal will be constructed based on this partition of the time-state space. The idea of hysteresis-based switching [9] is important here to prevent chattering and maintain the property that the switching signal function $\sigma(t)$ has only a finite number of discontinuities on every bounded time-interval. To define the hysteresis switching, the regions Ω_i are enlarged so that they can have some overlaps near the boundaries. Define Ω_i' as

$$\Omega_i' := \left\{ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{dV}{dt}\bigg|_i (t,x) \leqslant -\frac{1}{\zeta} c(h_0(t,x)) \right\},$$

where $\zeta > 1$ can be arbitrarily chosen. A switching signal σ can be constructed by the following *minimal rule*.

($\mathbf{R}_1$) Starting from some $t = t_0$, if $V(t_0, x_0) = 0$, let $\sigma(t) \equiv i_0$ for all $t \geqslant t_0$, where $i_0 \in I$ can be arbitrarily chosen. If $V(t_0, x_0) \neq 0$, let

$$\sigma(t_0) = \arg\min_i \frac{dV}{dt}\bigg|_i (t_0, x(t_0)),$$

where arg denotes the value of the argument i such that the minimal is attained;

($\mathbf{R}_2$) Maintain $\sigma = i$ as long as $(t, x(t)) \in \Omega_i'$ and $\sigma(t^-) = i$;

($\mathbf{R}_3$) Once $(t_1, x(t_1))$ hits the boundary $\partial\Omega_i'$ of Ω_i' for some t_1, let $t_0 = t_1$ and start over according to ($\mathbf{R}_1$).

To guarantee that the switching is well-defined, we first show that $\{\Omega_i\}$ forms a covering of the time-state space.

Proposition 3.1. *If Assumption 3.1 holds, then* $\cup_{i=1}^N \Omega_i = \mathbb{R}^+ \times \mathbb{R}^n$.

Proof. Suppose $\cup_{i=1}^N \Omega_i \neq \mathbb{R}^+ \times \mathbb{R}^n$, i.e., there exists a pair $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$ such that $(t,x) \notin \cup_{i=1}^N \Omega_i$. By the definition of Ω_i, this implies

$$\frac{dV}{dt}\bigg|_i (t,x) = \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_i(t,x) > -c(h_0(t,x)).$$

It follows that

$$\begin{aligned}\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_\alpha(t,x) &= \frac{\partial V}{\partial t}(t,x)\sum_{i=1}^N \alpha_i + \nabla V(t,x) \cdot \sum_{i=1}^N \alpha_i f_i(t,x) = \sum_{i=1}^N \alpha_i \left[\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_i(t,x)\right] \\ &> \sum_{i=1}^N \alpha_i[-c(h_0(t,x))] = -c(h_0(t,x)),\end{aligned} \tag{3.2}$$

where the coefficients α_i are from Assumption 3.1. Since (3.2) contradicts inequality (3.1b) in Assumption 3.1, one must have $\cup_{i=1}^N \Omega_i = \mathbb{R}^+ \times \mathbb{R}^n$ if Assumption 3.1 holds. This completes the proof. □

The following proposition, as an immediate consequence of Assumption 3.1, plays an important role in the sequel. Denote

$$\Gamma_i := \left\{ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \arg\min_i \frac{dV}{dt}\bigg|_i (t,x) = i \right\}, \quad i \in I.$$

Clearly, each Γ_i is a closed set and $\cup_{i \in I} \Gamma_i = \mathbb{R}^+ \times \mathbb{R}^n$.

Proposition 3.2. *Under Assumptions 3.1, the set* $\partial\Omega_i' \cap \Gamma_j \cap \partial\Omega_j' \cap U(\varepsilon, \rho, [t_0, T])$ *is empty for all* $i,j \in I$, $0 < \varepsilon \leqslant \rho < \infty$, *and* $0 \leqslant t_0 < T < \infty$.

Proof. It suffices to show that $\partial\Omega_i' \cap U(\varepsilon, \rho, [t_0, T])$ and $\Gamma_i \cap U(\varepsilon, \rho, [t_0, T])$ are disjoint for all $i \in I$. Suppose this is not true and let (t^*, x^*) be an element in both $\partial\Omega_i' \cap U(\varepsilon, \rho, [t_0, T])$ and $\Gamma_i \cap U(\varepsilon, \rho, [t_0, T])$. It is clear that $V(t^*, x^*) > 0$ and hence $c(h_0(t^*, x^*)) > 0$. Note that $(t^*, x^*) \in \partial\Omega_i'$ implies

$$\frac{dV}{dt}\bigg|_i (t^*, x^*) = -\frac{1}{\xi} c(h_0(t^*, x^*)),$$

and $(t^*, x^*) \in \Gamma_i$ implies

$$\frac{dV}{dt}\bigg|_j (t^*, x^*) \geqslant \frac{dV}{dt}\bigg|_i (t^*, x^*), \quad \forall j \in I.$$

It follows that

$$\frac{dV}{dt}\bigg|_j (t^*, x^*) \geqslant -\frac{1}{\xi} c(h_0(t^*, x^*)), \quad \forall j \in I,$$

which in turn implies that

$$\frac{\partial V}{\partial t}(t^*,x^*) + \nabla V(t^*,x^*)\cdot f_\alpha(t^*,x^*) \geqslant -\frac{1}{\xi}c(h_0(t^*,x^*)) > -c(h_0(t^*,x^*)),$$

where $c(h_0(t^*,x^*)) > 0$, which contradicts Assumption 3.1. □

The following proposition formally confirms that no chattering occurs in a switching signal σ constructed according to the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$.

Proposition 3.3. *Under Assumption 3.1, a switching signal σ constructed according to the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ obeys the following:*

(i) Let t^* be a switching time of σ. There exists a positive constant ε, which may depend on $(t^*,x(t^*))$, such that no switching occurs within $[t^*,t^*+\varepsilon)$;
(ii) The switching signal σ has a positive dwell-time as long as

$$(t,x(t)) \in U(\varepsilon,\rho,[t_0,T]),$$

where $U(\varepsilon,\rho,[t_0,T])$ is from Assumption 3.1, i.e., there exists a constant $\delta > 0$, which may depend on the set $U(\varepsilon,\rho,[t_0,T])$, such that any two switching times of σ on $[t_0,T]$ is separated by at least δ unit of time;
(iii) Given $t' \geqslant t \geqslant t_0$, we have $V(t',x(t')) \leqslant V(t,x(t)) \leqslant V(t_0,x(t_0))$.

Proof

(i) Note that t^* being a switching time of σ implies $(t^*,x(t^*)) \in \partial\Omega'_i$ for some $i \in I$. If $V(t^*,x(t^*)) = 0$, the conclusion is true, since it is defined by the minimal rule that $\sigma(t) \equiv i_0$ for some fixed $i_0 \in I$ and all $t \geqslant t^*$. Now suppose $V(t^*,x(t^*)) \neq 0$ and $(t^*,x(t^*)) \in \partial\Omega'_i$. It is clear that we can choose a set $U(\varepsilon,\rho,[t_0,T])$ such that $(t^*,x(t^*)) \in U(\varepsilon,\rho,[t_0,T])$. Moreover, the minimal rule chooses the next mode to be j by letting $\sigma(t^*) = j$, where

$$j = \arg\min_i \left.\frac{dV}{dt}\right|_i (t^*,x(t^*)),$$

i.e., $(t^*,x(t^*)) \in \Gamma_j$. Proposition 3.2 says that $(t^*,x(t^*)) \notin \partial\Omega'_j$. Therefore, it takes a positive amount of time for $(t,x(t))$ to reach Ω'_j and for the next switching to take place.

(ii) Let t^* be a switching time of σ, which implies that $(t^*,x(t^*)) \in \partial\Omega'_i$ for some $i \in I$. As shown in part (i), the next mode $j, j \neq i$, is determined by the minimal rule such that $(t^*,x(t^*)) \in \Gamma_j$, and the next switching time is when $(t,x(t)) \in \partial\Omega'_j$ occurs. According to Proposition 3.2, $\partial\Omega'_i \cap \Gamma_j$ and $\partial\Omega'_j$ are disjoint within $U(\varepsilon,\rho,[t_0,T])$, and $U(\varepsilon,\rho,[t_0,T])$ is compact by Assumption 3.1. Therefore, as long as $(t,x(t))$ remains in $U(\varepsilon,\rho,[t_0,T])$, it takes a minimal time for $(t,x(t))$ to travel from $\partial\Omega'_i \cap \Gamma_j$ to $\partial\Omega'_j$, considering that $\partial\Omega'_i \cap \Gamma_j$ and $\partial\Omega'_j$ are both closed sets and disjoint in $U(\varepsilon,\rho,[t_0,T])$, and that each element of the vector fields $\{f_i : i \in I\}$ is continuous and hence bounded on the compact set $U(\varepsilon,\rho,[t_0,T])$. Note that each $\partial\Omega'_i$ can be finitely partitioned as $\partial\Omega'_i = \cup_{i\in I}(\partial\Omega'_i \cap \Gamma_j)$ and the index set I is finite. It follows that a positive dwell-time exists if $(t,x(t))$ remains in $U(\varepsilon,\rho,[t_0,T])$.

(iii) Differentiating $V(t,x(t))$ with respect to t gives

$$\frac{dV}{dt}(t,x(t)) = \frac{\partial V}{\partial t}(t,x(t)) + \nabla V(t,x(t))\cdot f_{\sigma(t)}(t,x(t)).$$

According to the construction of σ, one has

$$\frac{dV}{dt}(t,x(t)) \leqslant -\frac{1}{\zeta}c(h_0(t,x(t))) \leqslant 0, \tag{3.3}$$

which implies $V(t,x(t))$ is nonincreasing for $t \geqslant t_0$. The proof is complete. □

Although Proposition 3.3 shows that a switching signal generated by the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ has no chattering, the minimal rule alone cannot exclude the possibility of *Zeno behavior*, i.e., the switched system undergoes an infinite number of switchings within a finite interval of time. To guarantee that no Zeno behavior occurs, based on the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$, we further propose a *generalized rule*. Define

$$\mathcal{D}_k = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : 2^k < V(t,x) \leqslant 2^{k+1}\right\}, \quad k \in \mathbb{Z},$$

where V is from Assumption 3.1 and $\mathbb{Z}$ denotes the set of all integers. A switching signal can be constructed by the following generalized rule. Let $\tau > 0$ be an arbitrary positive real number.

($\mathbf{GR}_1$) Starting from some $t = t_0$, if $V(t_0, x_0) = 0$, let $\sigma(t) \equiv i_0$ for all $t \geqslant t_0$, where $i_0 \in I$ can be arbitrarily chosen. If $V(t_0, x_0) \neq 0$ and $(t_0, x_0) \in \mathcal{D}_{k_0}$ for some $k_0 \in \mathbb{Z}$, maintain the minimal rule ($\mathbf{R}_1$)–($\mathbf{R}_3$) until $(t_1, x(t_1)) \in \overline{\mathcal{D}_{k_0-2}}$ for some t_1 and proceed to ($\mathbf{GR}_2$);

($\mathbf{GR}_2$) If $t_1 - t_0 \geqslant \tau$, let $t_1 = t_0$ and start over according to ($\mathbf{GR}_1$). If $t_1 - t_0 < \tau$, maintain σ unchanged until $t_2 - t_0 \geqslant \tau$ or $(t_2, x(t_2)) \in \overline{\mathcal{D}_{k_0}}$ for some t_2, whichever occurs first. If $t_2 - t_0 \geqslant \tau$ occurs first, let $t_0 = t_2$ and start over according to ($\mathbf{GR}_1$). If $(t_2, x(t_2)) \in \overline{\mathcal{D}_{k_0}}$ occurs first, proceed to ($\mathbf{GR}_3$);

($\mathbf{GR}_3$) Maintain the minimal rule ($\mathbf{R}_1$)–($\mathbf{R}_3$) until $(t_3, x(t_3)) \in \overline{\mathcal{D}_{k_0-2}}$ for some t_3. Let $t_1 = t_3$ and proceed according to ($\mathbf{GR}_2$).

We call it a *cycle* of the generalized rule ($\mathbf{GR}_1$)–($\mathbf{GR}_3$) and, unambiguously, a cycle of a switching signal generated by ($\mathbf{GR}_1$)–($\mathbf{GR}_3$), once we start over according to ($\mathbf{GR}_1$). Each cycle consists of two portions of time: a portion spent while the minimal rule ($\mathbf{R}_1$)–($\mathbf{R}_3$) is engaged, which we call the *minimal rule time*, and the other spent without the minimal rule engaged, which we call the *wandering time*. The positive constant τ is called the *(minimal) cycle time*. We observe the following crucial properties of the generalized rule.

Proposition 3.4. *Under Assumption 3.1, a switching signal constructed according to the generalized rule* ($\mathbf{GR}_1$)–($\mathbf{GR}_3$) *obeys the following:*

(i) The switching signal is well-defined;
(ii) Let t_0 be the starting time of any cycle. We have $V(t, x(t)) \leqslant V(t_0, x(t_0))$ for all $t \geqslant t_0$;
(iii) Let $t_0 < t_0' < t_0''$ be the starting times of any three consecutive cycles. We have

$$V(t_0'', x(t_0'')) \leqslant \frac{1}{2} V(t_0, x(t_0));$$

(iv) Each cycle lasts for at least τ unit of time. If a cycle contains an interval of the form $[t' - \tau, t' + T]$ with some $T > 0$ (i.e., the duration of this cycle is strictly greater than τ), then the interval $[t', t' + T]$ is spent continuously as minimal rule time. Moreover, picking any $t^* \in [t', t' + T]$, we have $V(t, x(t)) \leqslant V(t^*, x(t^*))$ for all $t \geqslant t^*$.

Proof

(i) Note that each cycle lasts for at least τ unit of time. Fix any finite interval of the form $[T_1, T_2] \subset \mathbb{R}^+$. We have to show that at most finitely many switchings can occur during this interval. Since there are only a finite number of cycles contained in $[T_1, T_2]$, it suffices to show that each cycle in $[T_1, T_2]$ contains at most finitely many switching times. Let $[\tau_1, \tau_2]$ denote such a cycle, where τ_2 is either the end of this cycle or T_2 (the end of the finite interval considered). Without loss of generality, suppose $V(\tau_1, x(\tau_1)) \neq 0$ and $(\tau_1, x(\tau_1)) \in \mathcal{D}_{k_0}$ for some $k_0 \in \mathbb{Z}$ (the trivial case $V(\tau_1, x(\tau_1)) = 0$ would imply that $[\tau_1, \tau_2]$ is continuously spent as minimal rule time and there is no switching). Note that, during $[\tau_1, \tau_2]$, the minimal rule is engaged only when $(t, x(t)) \in \overline{\mathcal{D}_{k_0-1} \cup \mathcal{D}_{k_0}}$, where $k_0 \in \mathbb{Z}$ is fixed for each cycle, which implies that the minimal rule time of this cycle is spent (not necessarily continuously) in $U\left(2^{k_0-1}, 2^{k_0+1}, [\tau_1, \tau_2]\right)$. According to Proposition 3.3 (ii), only a finite number of switchings can occur during the minimal rule time in $[\tau_1, \tau_2]$. In addition, note that there is no switching during wandering time. The only possibility to have infinitely many switchings during $[\tau_1, \tau_2]$ is switching back and forth between the wandering time and minimal rule time infinitely many times. However, note that the wandering time starts from $(t, x(t)) \in \overline{\mathcal{D}_{k_0-2}}$ and ends when either the minimal cycle time τ is achieved or $(t, x(t)) \in \overline{\mathcal{D}_{k_0}}$. In the former case, the cycle also ends. In the latter case, since k_0 is fixed, it can be shown that it takes a minimal amount of time for $(t, x(t))$ to travel from $\overline{\mathcal{D}_{k_0-2}}$ to $\overline{\mathcal{D}_{k_0}}$. Therefore, the end point τ_2 is reached through a finite number of switching out from the wandering time. It has been shown that it is impossible to have infinitely many switchings during $[\tau_1, \tau_2]$, and so is it during $[T_1, T_2]$, an arbitrarily fixed finite subinterval of $\mathbb{R}^+$. The proof of part (i) is complete.

(ii)–(iv) The proof follows from the construction of a switching signal according to the generalized rule ($\mathbf{GR}_1$)–($\mathbf{GR}_3$) and the monotonicity property of the minimal rule shown in Proposition 3.3 (iii). The proof is complete. □

The above proposition first confirms that a switching signal generated by the generalized rule ($\mathbf{GR}_1$)–($\mathbf{GR}_3$) is actually well-defined, and then establishes some important monotonicity properties of generalized rule for later use.

Remark 3.2. The intuitive idea behind the generalized rule ($\mathbf{GR}_1$)–($\mathbf{GR}_3$) is to prevent $V(t, x(t))$ from converging to 0 through an infinite number of switching within a finite interval of time. The minimal rule is maintained if $V(t, x(t))$ is converging to 0 "not too fast", which in this case means not faster than a certain exponential decay. If $V(t, x(t))$ is decreasing too fast, the minimal rule time is succeeded by a portion of "wandering time" on which the switching signal is maintained constant without engaging the minimal rule. During the wandering time, it is expected that $V(t,x(t))$ would rise, since each of the subsystem is assumed to be unstable. Therefore, the simple monotonicity property of the minimal rule, as shown in Proposition 3.3, can be violated. Nevertheless, we can establish certain monotonicity properties of the generalized rule, as shown in the above proposition.

Remark 3.3. The problem whether a state-dependent switching rule based on a partition of the state space is actually well-defined and can be converted to a piecewise time-dependent signal is difficult. Very few results are available on this topic. In [3], the author investigates systems with finite valued discontinuous feedbacks and gives some necessary conditions for the cause of Zeno phenomenon in the finite valued feedback laws. Our generalized rule ($\mathbf{GR}_1$)–($\mathbf{GR}_3$) not only considers the partition of the time-state space, but also takes advantage of the properties of the Lyapunov function V. As shown in Propositions 3.3 and 3.4, both chattering and Zeno phenomenon are prevented under the generalized rule. Therefore, the switching signals constructed and the resulting switched systems are well-defined in the classical sense. It should also be pointed out that the generalized rule ($\mathbf{GR}_1$)–($\mathbf{GR}_3$) introduces a dependence on the time (due to the minimal cycle time), while the minimal rule ($\mathbf{R}_1$)–($\mathbf{R}_3$) is purely state-dependent (although with memory due to the hysteresis switching).

Before stating the main theorem, we make the following remark regarding the existence of solution by the boundedness of $h(t,x(t))$ in future time.

Remark 3.4. In the classical Lyapunov stability theorems, the existence of every solution in the future time is guaranteed by the fact that the theorem conditions imply solutions are bounded in the future and hence exist globally. Similar thing can be said for (h_0,h)-stability analysis, i.e., the boundedness of $h(t,x(t))$ implies future existence of the solution $x(t)$. For example, $h(t,x)=|\dot q|$, where $x^T=(q,\dot q)$; $h(t,x)=d(x(t),A)$, where A is a compact invariant set and $d(\cdot)$ is the distance function. Thus we assume in this paper that, if $h(t,x(t))$ is bounded for all $t\geqslant t_0$, then the solution $x(t)$ exists for all $t\geqslant t_0$.

Theorem 3.1. *If Assumption 3.1 holds, then the switched system (2.2) is (h_0,h)-globally uniformly asymptotically stable*

(i) under the switching signal σ constructed according to ($\mathbf{R}_1$)–($\mathbf{R}_3$), with the possibility that σ has a finite accumulation point t^* for its switching times, and $x(t)=0$ for all $t\geqslant t^*$;
(ii) under the switching signal σ constructed according to ($\mathbf{GR}_1$)–($\mathbf{GR}_3$), and σ is well-defined.

Proof

(i) Since V in Assumption 3.1 is h-positive definite, h-radially unbounded, and h_0-decrescent, there exist functions $u,v\in\mathcal{K}_\infty$ such that

$$u(h(t,x))\leqslant V(t,x)\leqslant v(h_0(t,x)),\quad \forall (t,x)\in\mathbb{R}^+\times\mathbb{R}^n. \tag{3.4}$$

Given $\varepsilon>0$ and $t_0\geqslant 0$, choose $\delta=\delta(\varepsilon)$ independent of t_0 and sufficiently small such that $v(\delta)<u(\varepsilon)$. We proceed to show that every solution $x(t;t_0,x_0)$ of (2.2) with $h_0(t_0,x_0)<\delta$ verifies

$$h(t,x(t))<\varepsilon,\quad t\geqslant t_0. \tag{3.5}$$

According to Proposition 3.3 (iii) and (3.4), one has

$$u(h(t,x(t)))\leqslant V(t,x(t))\leqslant V(t_0,x_0)\leqslant v(h_0(t_0,x_0))<v(\delta)<u(\varepsilon) \tag{3.6}$$

for all $t\geqslant t_0$. Since $u\in\mathcal{K}_\infty$, (3.5) is verified and (h_0,h)-uniform stability follows. We proceed to show (h_0,h)-global attraction. Still let $\varepsilon>0$ and t_0 be arbitrarily fixed. Choose $\delta=\delta(\varepsilon)$ independent of $t_0\geqslant 0$ and sufficiently small such that $v(\delta)<u(\varepsilon)$. For an arbitrarily fixed $\rho>0$, let $T(\rho,\varepsilon)=\zeta v(\rho)/c(\delta)+1$ and $x(t;t_0,x_0)$ be an arbitrary solution with $h_0(t_0,x_0)<\rho$. Since $u\in\mathcal{K}_\infty$, one can choose η large such that $u(\eta)>v(\rho)$. Then, by (3.4), one has

$$u(h(t,x(t)))\leqslant V(t,x(t))\leqslant V(t_0,x_0)\leqslant v(h_0(t_0,x_0))<v(\rho)<u(\eta)$$

for all $t\geqslant t_0$, which implies $h(t,x(t))$ is bounded in the future and hence, by the assumption in Remark 3.4, $x(t)$ exists for all $t\geqslant t_0$.
Claim: There exists $t^*\in[t_0,t_0+T]$ such that $h_0(t^*,x(t^*))<\delta$.
Suppose the claim is not true. Then $h_0(t,x(t))\geqslant\delta$ for all $t\in[t_0,t_0+T]$. It follows, from (3.3) and (3.4), that

$$0\leqslant V(T,x(T))\leqslant V(t_0,x_0)-\frac{1}{\zeta}\int_{t_0}^{t_0+T}c(h_0(s,x(s)))ds\leqslant v(\rho)-\frac{c(\delta)T}{\zeta}<0,$$

which is a contradiction and the claim is proved.
Since $\delta>0$ is the same (h_0,h)-stability constant as above, it follows from (3.6) that $h(t,x(t))<\varepsilon$ for all $t\geqslant t^*$ and hence for $t\geqslant t_0+T$. This completes the proof for part (i).

(ii) The proof of part (ii) is based on that of part (i). The proof for (h_0, h)-uniform stability remains the same, since the estimate (3.6) remains valid in view of Proposition 3.4 (iii). To show (h_0, h)-global attraction, let $\varepsilon > 0,\ t_0,\ \delta,\ \rho,\ T$, and $x(t; t_0, x_0)$ be the same as in the proof of part (i).
We aim to show that there exists some $T' > 0$, independent of t_0, such that $h(t, x(t)) < \varepsilon$ for all $t \geqslant t_0 + T'$. It is clear that there exists a positive integer k_0 such that $2^{-k_0} v(\rho) < u(\varepsilon)$. We choose $T' = (2k_0 + 1)(T + \tau)$ and show that $h(t, x(t)) < \varepsilon$ for all $t \geqslant t_0 + T'$. Suppose there are more than $2k_0 + 1$ cycles of the generalized rule between t_0 and $t_0 + T'$. Then (iii) of Proposition 3.4 implies that

$$V(t, x(t)) \leqslant V(t^*, x(t^*)) \leqslant 2^{-k_0} V(t_0, x_0) \leqslant 2^{-k_0} v(\rho) < u(\varepsilon)$$

for all $t \geqslant t_0 + T' \geqslant t^*$, where t^* is the starting time of the $(2k_0 + 1)$-th cycle, which in turn implies that $h(t, x(t)) < \varepsilon$ for all $t \geqslant t_0 + T'$. Suppose there are at most $2k_0$ cycles of the generalized rule between t_0 and $t_0 + T'$. Then there must exist at least one cycle lasting at least time $T + \tau$. By Proposition 3.4 (ii), during this cycle, at least time T is spent continuously on the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$, say on some interval $[t', t' + T] \subset [t_0, t_0 + T']$. Following how we proved the claim in part (i), we can show that there exists $t^* \in [t', t' + T]$ such that $h_0(t^*, x(t^*)) < \delta$. It follows from Proposition 3.4 (ii) that

$$V(t, x(t)) \leqslant V(t^*, x(t^*)) \leqslant v(h_0(t^*, x(t^*))) \leqslant v(\delta) < u(\varepsilon)$$

for all $t \geqslant t_0 + T' \geqslant t^*$, which implies that $h(t, x(t)) < \varepsilon$ for all $t \geqslant t_0 + T'$. The proof is complete. $\square$

3.2. (h_0, h)-Global exponential stabilization

In this section, we propose a stronger version of Assumption 3.1, and show that under this stronger assumption, we can construct an exponentially stabilizing switching signal for system (2.2), based on the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and generalized rule $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$.

Assumption 3.2. Assumption 3.1 is satisfied with $\mathcal{K}_\infty$-functions $a(s) := c_1 s^p,\ b(s) := c_2 s^p$, and $c(s) := c_3 s^p$, where $c_1,\ c_2,\ c_3$, and p are positive constants.

The corresponding partitions of the time-state space are replaced by

$$\Delta_i := \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{dV}{dt}\bigg|_i (t, x) \leqslant -c_3 h_0^p(t, x) \right\}$$

and

$$\Delta_i' := \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{dV}{dt}\bigg|_i (t, x) \leqslant -\frac{c_3}{\zeta} h_0^p(t, x) \right\},$$

where $\zeta > 1$ can be arbitrarily chosen. The minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and the generalized rule $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$ can now be employed to construct a switching signal.

The following theorem is the similar to Theorem 3.1, except that now we have (h_0, h)-globally uniformly exponentially stability, which is a more desirable stability property in many applications.

Theorem 3.2. *If Assumption 3.2 holds, the switched system (2.2) is (h_0, h)-globally uniformly exponentially stable*

(i) under the switching signal σ constructed according to $(\mathbf{R}_1)$–$(\mathbf{R}_3)$, with the possibility that σ has a finite accumulation point t^* for its switching times, and $x(t) = 0$ for all $t \geqslant t^*$;
(ii) under the switching signal σ constructed according to $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$, and σ is well-defined.

Proof

(i) Assumption 3.2 says that

$$c_1 h^p(t, x) \leqslant V(t, x) \leqslant c_2 h_0^p(t, x), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \tag{3.7}$$

and

$$\frac{dV}{dt}(t, x) = \frac{\partial V}{\partial t}(t, x) + \nabla V(t, x) \cdot f_\alpha(t, x), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

which, according to the construction of σ, implies that

$$\frac{dV}{dt}(t, x(t)) = \frac{\partial V}{\partial t}(t, x(t)) + \nabla V(t, x(t)) \cdot f_{\sigma(t)}(t, x(t)) \leqslant -\frac{c_3}{\zeta} h_0^p(t, x) \leqslant -\frac{c_3}{\zeta c_2} V(t, x(t)), \quad \forall t \geqslant t_0.$$

Therefore,

$$V(t,x(t)) \leqslant V(t_0,x_0)e^{-\frac{c_3}{\xi c_2}(t-t_0)}, \quad \forall t \geqslant t_0 \tag{3.8}$$

and

$$h(t,x(t)) \leqslant \left(\frac{c_2}{c_1}\right)^{1/p} h_0(t_0,x_0)e^{-\frac{c_3}{\xi c_2 p}(t-t_0)}, \quad \forall t \geqslant t_0,$$

i.e., system (2.2) is (h_0,h)-globally uniformly exponentially stable.

(ii) Note that (3.8) gives an estimate of $V(t,x(t))$ in terms of an exponential decay. We show that under the assumption of the theorem, each nontrivial cycle (by which we mean a cycle starting with $V(t_0,x(t_0)) \neq 0$) of the generalized rule has an upper bound. Suppose there exists a nontrivial cycle with its duration strictly greater than τ and this cycle contains an interval of the form $[t'-\tau,t+T]$, where $T>0$. According to Proposition 3.4 (iv), on $[t',t'+T]$, the minimal rule is engaged. Similar to the argument in part (i), we can show

$$V(t'+T,x(t'+T)) \leqslant V(t',x(t'))e^{-\frac{c_3}{\xi c_2}T}. \tag{3.9}$$

Moreover, during the minimal rule time, we have $(t,x(t)) \in \overline{\mathcal{D}_{k_0-1} \cup \mathcal{D}_{k_0}}$, and the cycle would end once $(t,x(t))$ hits the boundary of $\mathcal{D}_{k_0-2}$ (since the minimal cycle time is already achieved). Therefore, we must have in the same time

$$V(t',x(t')) \leqslant 2^{k_0+1}, \quad V(t'+T,x(t'+T)) \geqslant 2^{k_0-1},$$

which, according to (3.9), implies that

$$T \leqslant \frac{\xi c_2 \ln 2}{c_3},$$

which further implies that a cycle lasts for at most $\hat{\tau} := \tau + \frac{\xi c_2 \ln 2}{c_3}$ unit of time. Recall that Proposition 3.4 (iii) shows the value of $V(t,x(t))$ decreases at least by half every two cycles. We claim that

$$V(t,x(t)) \leqslant MV(t_0,x_0)e^{-\mu(t-t_0)}, \quad \forall t \geqslant t_0,$$

where $\mu := \frac{\ln 2}{2\hat{\tau}}$ and $M := e^{2\mu\hat{\tau}}$ are positive constants independent of t_0 and x_0. We denote the consecutive nontrivial cycles by $I_1, I_2, \ldots$, and $\tau_1 = t_0, \tau_2, \ldots$ are the corresponding starting times. On I_1 and I_2, we clearly have, from Proposition 3.4 (ii)

$$V(t,x(t)) \leqslant V(t_0,x_0) \leqslant MV(t_0,x_0)e^{-\mu(t-t_0)}.$$

It follows from Proposition 3.4 (ii) and (iii) that, on I_3 and I_4, we have

$$V(t,x(t)) \leqslant V(\tau_3,x(\tau_3)) \leqslant \frac{1}{2}V(t_0,x_0) \leqslant \frac{1}{2}MV(t_0,x_0)e^{-\mu(\tau_3-t_0)} \leqslant \frac{1}{2}e^{2\mu\hat{\tau}}MV(t_0,x_0)e^{-\mu(t-t_0)} = MV(t_0,x_0)e^{-\mu(t-t_0)}.$$

This procedure can be carried on for all subsequent nontrivial cycles. The proof can be completed by induction. Note that, if there exists a subsequent trivial cycle, i.e., $V(t,x(t))$ becomes 0 in a finite time, it follows from the minimal rule and Assumption 3.2 that $V(t,x(t))$ stays 0 for all future time, and therefore the claim still holds. □

4. Examples

We shall discuss a few examples in this section to illustrate our results obtained in Section 3.

Example 4.1. (*Lyapunov Stability*[1]). Consider two subsystems given by

$$x' = f_1(t,x) = \begin{pmatrix} -2x_1^{\frac{1}{3}} \\ x_2^{\frac{1}{3}} \end{pmatrix} \tag{4.1}$$

and

$$x' = f_2(t,x) = \begin{pmatrix} x_1^{\frac{1}{3}} \\ -2x_2^{\frac{1}{3}} \end{pmatrix}. \tag{4.2}$$

Both subsystems are decoupled and admit explicit solutions. The solution for subsystem (4.1) with initial condition (t_0,x_0), where $x_0 = [x_{10}\ x_{20}]$, is given by

[1] This example is suggested by the anonymous reviewer.

$$x_1^{\frac{2}{3}} = \begin{cases} 0, & t \geqslant t_0 + \frac{3}{4}x_{10}^{\frac{2}{3}}, \\ x_{10}^{\frac{2}{3}} - \frac{4}{3}(t - t_0), & t_0 \leqslant t < t_0 + \frac{3}{4}x_{10}^{\frac{2}{3}}, \end{cases}$$

$$x_2^{\frac{2}{3}} = \begin{cases} 0, & t \geqslant t_0,\ x_{20} = 0, \\ x_{20}^{\frac{2}{3}} + \frac{2}{3}(t - t_0), & t \geqslant t_0,\ x_{20} \neq 0 \end{cases}$$

and the solution for subsystem (4.2) with the same initial condition is given by

$$x_1^{\frac{2}{3}} = \begin{cases} 0, & t \geqslant t_0,\ x_{10} = 0, \\ x_{10}^{\frac{2}{3}} + \frac{2}{3}(t - t_0), & t \geqslant t_0,\ x_{10} \neq 0, \end{cases}$$

$$x_2^{\frac{2}{3}} = \begin{cases} 0, & t \geqslant t_0 + \frac{3}{4}x_{20}^{\frac{2}{3}}, \\ x_{20}^{\frac{2}{3}} - \frac{4}{3}(t - t_0), & t_0 \leqslant t < t_0 + \frac{3}{4}x_{20}^{\frac{2}{3}}. \end{cases}$$

It is clear that, for both subsystems, one of the components of a nontrivial solution goes to infinity and the other goes to 0 in finite time. Therefore, both subsystems are unstable. To apply the switching stabilization method of this paper, one can let

$$f(t,x) = \frac{1}{2}f_1(t,x) + \frac{1}{2}f_2(t,x) = -\frac{1}{2}\begin{pmatrix} x_1^{\frac{1}{3}} \\ x_2^{\frac{1}{3}} \end{pmatrix}$$

and $V(t,x) = x_1^2 + x_2^2$. Then

$$\nabla V(t,x) \cdot f(t,x) = -x_1^{\frac{4}{3}} - x_2^{\frac{4}{3}} \leqslant -V^{\frac{2}{3}}(t,x)$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, and the set Γ_0 in Assumption 3.1 is given by $\Gamma_0 = \mathbb{R}^+ \times \{0\}$. A solution for either (4.1) or (4.2), starting from $(t_0, 0)$, certainly stays 0 for all $t \geqslant t_0$. It is easy to see that Assumption 3.1 is satisfied with $h_0 = h = |x|,\ a(s) = b(s) = s^2$, and $c(s) = s^{\frac{4}{3}}$, where $s > 0$. By choosing $\zeta = 2$, one can define

$$\begin{aligned} \Omega_1' &= \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \nabla V(t,x) \cdot f_1(t,x) \leqslant -\frac{1}{2}|x|^{\frac{4}{3}}\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : -8x_1^{\frac{4}{3}} + 4x_2^{\frac{4}{3}} + |x|^{\frac{4}{3}} \leqslant 0\right\} \\ &= \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : |x_2| \leqslant c|x_1|\} \end{aligned}$$

and

$$\begin{aligned} \Omega_2' &= \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \nabla V(t,x) \cdot f_2(t,x) \leqslant -\frac{1}{2}|x|^{\frac{4}{3}}\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : 4x_1^{\frac{4}{3}} - 8x_2^{\frac{4}{3}} + |x|^{\frac{4}{3}} \leqslant 0\right\} \\ &= \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : |x_1| \leqslant c|x_2|\}, \end{aligned}$$

where $c = 1.3548\cdots > 1$ is a root of the equation $4c^{4/3} + (1 + c^2)^{2/3} - 8 = 0$. A switching signal can now be constructed by the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and the generalized rule $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$. Theorem 3.1 then guarantees the resulting switched system is (h_0, h)-globally uniformly asymptotically stable. Since h_0 and h are both chosen to be $|x|$, the (h_0, h)-stability concluded here is actually equivalent to Lyapunov stability.

Discussions: In the following, we discuss in more details how the minimal rule and the generalized rule help to stabilize the switched system in this particular example.

First, we show that a switching signal constructed using the minimal rule actually exhibits Zeno behavior. It is easy to check that the minimal rule implies that

$$\frac{dV(t,x(t))}{dt} \leqslant -\frac{1}{2}V^{\frac{2}{3}}(t,x(t)). \tag{4.3}$$

We suppose that $V(t_0, x_0) \neq 0$ (otherwise, the switched system admit a trivial solution without the minimal rule engaged). Integrating the differential inequality (4.3) gives

$$V^{\frac{1}{3}}(t,x(t)) \leqslant V^{\frac{1}{3}}(t_0,x_0) - \frac{1}{6}(t - t_0), \quad t \geqslant t_0, \tag{4.4}$$

as long as $V(t,x(t)) \neq 0$, which shows that $V(t,x(t))$ becomes 0 in finite time (so does $|x|$). Moreover, based on the explicit solutions, we know that a solution of subsystem (4.1) approaches the x_2-axis ($x_1 = 0$) and a solution of subsystem (4.2) approaches the x_1-axis ($x_2 = 0$), both in finite time. In neither case can the solution converge to the origin, provided that the solution does not start from a point not on the axes. Note that Ω_1' does not contain the x_2-axis except the origin and Ω_2' does not contain the x_1-axis except the origin. Following the minimal rule, a nontrivial solution eventually enters the common region $\Omega_1' \cap \Omega_2'$ and remains in the region for all future time. Therefore, in order that a solution converges to the origin in finite time, as implied by (4.4), a switching solution has to switch between the two subsystems infinitely many times in a finite time (i.e., exhibits Zeno behavior).

Second, we can show that, in this example, *a switching solution generated by the generalized rule still enjoys the property of converging to the origin in finite time*, while we know from Proposition 3.4 that the generalized rule can avoid Zeno behavior.

To show the above statement is true, we first observe that the x_1-axis is a stable manifold for subsystem (4.1) and the x_2-axis is a stable manifold for subsystem (4.2), which means that a solution of subsystem (4.1) starting on the x_1-axis and a solution of subsystem (4.2) starting on the x_2-axis will both converge to the origin in finite time.

During the minimal rule time, inequality (4.4) shows that, if $V(t_0,x(t_0))$ is sufficiently small, where t_0 is the starting time of any cycle with $(t_0,x(t_0)) \in \mathcal{D}_{k_0}$ for some $k_0 \in \mathbb{Z},\ V(t,x(t))$ would decrease faster than any given exponential rate and therefore it takes less than τ unit of time for $V(t,x(t))$ to reach the boundary of $\mathcal{D}_{k_0-2}$. Therefore, by the generalized rule, following a portion of minimal rule time, there must be a portion of wandering time, the starting time of which we denote by t_1. Without loss of generality, suppose that $x_1(t_1,x(t_1)) \neq 0,\ x_2(t_1,x(t_1)) \neq 0$, and $\sigma(t_1)=1$, i.e., the first mode is maintained during this portion of wandering time. The wandering time ends only if $\tau-(t_1-t_0)$ unit of time has elapsed or $V(t,x(t))$, starting from $V(t_1,x(t_1))=2^{k_0-1}$, reaches D_{k_0}, i.e., $V(t_2,x(t_2))=2^{k_0}$ for some $t_2>t_1$, whichever occurs first. If $|x_1(t_1,x(t_1))|$ is sufficiently small, we claim that:

Claim: $(t,x(t))$ will hit the x_2-axis first before $V(t,x(t))$ increases to 2^{k_0}.

For sufficiently small $|x_1(t_1,x(t_1))|$, the explicit solution of (4.1) implies that it will not take $\tau-(t_1-t_0)$ unit of time for $x_1(t,x(t))$ to become 0. Moreover, it can be easily computed that, once $x_1(\bar{t},x(\bar{t}))$ becomes 0 for some $\bar{t}>t_1$

$$V(\bar{t},x(\bar{t})) = \left[x_2^{\frac{2}{3}}(t_1,x(t_1)) + \frac{1}{2}x_1^{\frac{2}{3}}(t_1,x(t_1))\right]^3, \tag{4.5}$$

while

$$V(t_1,x(t_1)) = x_2^2(t_1,x(t_1)) + x_1^2(t_1,x(t_1)) = 2^{k_0-1}. \tag{4.6}$$

Following (4.1), we have

$$\frac{dV(t,x(t))}{dt} = -4x_1^{\frac{4}{3}}(t,x(t)) + 2x_2^{\frac{4}{3}}(t,x(t)), \quad t \in (t_1,\bar{t}),$$

which shows that the dynamic of $V(t,x(t))$ during the interval $[t_1,\bar{t}]$ is characterized by a possible initial decrease and a subsequent increase in the value of $V(t,x(t))$. Particularly, we know that the maximum of $V(t,x(t))$ during the interval $[t_1,\bar{t}]$ is attained either at $t=t_1$ or $t=\bar{t}$. Therefore, if we can check

$$V(\bar{t},x(\bar{t})) < 2V(t_1,x(t_1)) = 2^{k_0}, \tag{4.7}$$

then the claim must be true. In view of (4.5) and (4.6), inequality (4.7) would follow from

$$\left(a+\frac{1}{2}b\right)^3 < 2a^3+2b^3, \quad a>0,\ b>0, \tag{4.8}$$

which can be easily proved using Young's inequality about products.[2] Therefore, (4.8) implies that (4.7) is true and the claim is proved.

The following dynamic of the switched system by the generalized rule is simple. While $V(t,x(t))$ continues to rise until it hits 2^{k_0}, the minimal rule will take over afterwards. The solution will stay on the x_2-axis and follow the second mode for all future time. The explicit solution of (4.2) shows that x_2 converges to 0 in finite time.

Simulation: In Fig. 1, typical switching solutions with the same initial condition are shown. Both the minimal rule switching and generalized rule switchings with different cycle times are plotted. It can be observed that, in this particular example, generalized rule switchings can lead to convergence of the solution to the origin in finite time, by exiting the minimal rule switching and converging to one of the axes first, after only a finite number of switchings.

Remark 4.1. As shown in Fig. 1, if we choose the minimal cycle time τ sufficiently small, switching solutions generated by the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and the generalized rule $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$ are indistinguishable from a finite precision simulation. Therefore, in the following examples, simulations are performed for the minimal rule switchings only.

Example 4.2 (*Lyapunov Stability*). Consider two subsystems given by

$$x' = f_1(t,x) = \begin{pmatrix} 2x_1 + 2x_2^3 \\ -2x_2 + x_1x_2^2 \end{pmatrix} \tag{4.9}$$

and

[2] Applying Young's inequality, $xy \leqslant x^p/p + y^q/q$, where $x \geqslant 0,\ y \geqslant 0,\ 1/p+1/q=1$ with $p>0$ and $q>0$, to the products $(a^2)(3b/2)$ and $(a)(3b^2/4)$, one can prove (4.8).

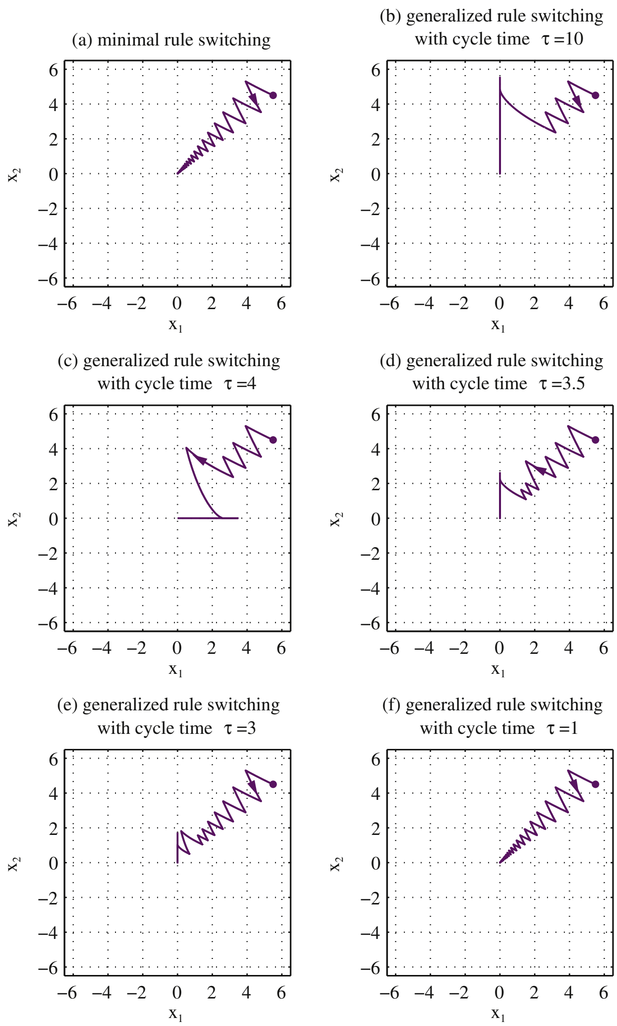

Fig. 1. Stabilized switching solutions with initial data [5.5 4.5] at $t_0 = 0$ for the switched system given by (4.1) and (4.2).

$$x' = f_2(t,x) = \begin{pmatrix} -3x_1 - x_2^3 \\ x_2 - 2x_1x_2^2 \end{pmatrix}. \tag{4.10}$$

By linearization near the origin, it can be shown that both subsystems are locally unstable. To apply the switching stabilization method of this paper, one can let

$$f(t,x) = \frac{1}{2}f_1(t,x) + \frac{1}{2}f_2(t,x) = \frac{1}{2}\begin{pmatrix} -x_1 + x_2^3 \\ -x_2 - x_1x_2^2 \end{pmatrix}$$

and $V(t,x) = x_1^2 + x_2^2$. Then

$$\nabla V(t,x) \cdot f(t,x) = -x_1^2 - x_2^2 = -V(t,x)$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, and the set Γ_0 in Assumption 3.1 is given by $\Gamma_0 = \mathbb{R}^+ \times \{0\}$. A solution for either (4.9) or (4.10), starting from $(t_0, 0)$, certainly stays 0 for all $t \geqslant t_0$. It is easy to see that Assumption 3.2 is satisfied with $h_0 = h = |x|$, $c_1 = c_2 = c_3 = 1$, and $p = 2$. By choosing $\zeta = 2$, one can define

$$\Delta_1' = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \nabla V(t,x) \cdot f_1(t,x) \leqslant -\frac{1}{2}|x|^2\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : 9x_1^2 - 7x_2^2 + 12x_1x_2^3 \leqslant 0\right\}$$

and

$$\Delta_2' = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \nabla V(t,x) \cdot f_2(t,x) \leqslant -\frac{1}{2}|x|^2\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : -11x_1^2 + 5x_2^2 - 12x_1x_2^3 \leqslant 0\right\}.$$

A switching signal can now be constructed by the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and the generalized rule $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$. Theorem 3.2 then guarantees the resulting switched system is (h_0, h)-globally uniformly exponentially stable. Since h_0 and h are both chosen to be $|x|$, the (h_0, h)-stability concluded here is actually equivalent to Lyapunov stability.

Simulation: Typical switching solutions with various initial conditions are shown in Fig. 2. It is observed that the switching solutions converge to the origin as expected.

Example 4.3 (*Partial Stability*). Consider two subsystems given by

$$x' = f_1(t,x) = \begin{pmatrix} -3x_1 + x_2 \\ x_2 \end{pmatrix} \tag{4.11}$$

and

$$x' = f_2(t,x) = \begin{pmatrix} 2x_1 - x_2 + x_1x_2e^{-t} \\ -x_1^2 - x_2e^t \end{pmatrix}. \tag{4.12}$$

By linearization near the origin, it can be shown that both subsystems are locally unstable. To apply the switching stabilization method of this paper, one can let

$$f(t,x) = \frac{1}{2}f_1(t,x) + \frac{1}{2}f_2(t,x) = \frac{1}{2}\begin{pmatrix} -x_1 + x_1x_2e^{-t} \\ x_2 - x_1^2 - x_2e^t \end{pmatrix}$$

and $V(t,x) = x_1^2 + x_2^2e^{-t}$. Then

$$\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f(t,x) = -x_1^2 - x_2^2$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, and the set Γ_0 in Assumption 3.1 is given by $\Gamma_0 = \mathbb{R}^+ \times \{0\}$. A solution for either of the two subsystems (4.11) and (4.12), starting from $(t_0, 0)$, certainly stays 0 for all $t \geqslant t_0$. It is easy to see that Assumption 3.2 is satisfied with $h_0 = |x|,\ h = |x_1|,\ p = 2$, and $c_1 = c_2 = c_3 = 1$. By choosing $\zeta = 2$, one can define

$$\Delta_1' = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_1(t,x) \leqslant -\frac{1}{2}|x|^2\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : -11x_1^2 + x_2^2 + 2x_2^2e^{-t} + 4x_1x_2 \leqslant 0\right\}$$

and

$$\Delta_2' = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_2(t,x) \leqslant -\frac{1}{2}|x|^2\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : 9x_1^2 - 3x_2^2 - 2x_2^2e^{-t} - 4x_1x_2 \leqslant 0\right\}.$$

A switching signal can now be constructed by the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and the generalized rule $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$. Theorem 3.2 then guarantees the resulting switched system is (h_0, h)-globally uniformly exponentially stable. Since $h_0 = |x|$ and $h = |x_1|$, the (h_0, h)-stability concluded here is equivalent to partial stability of the first component.

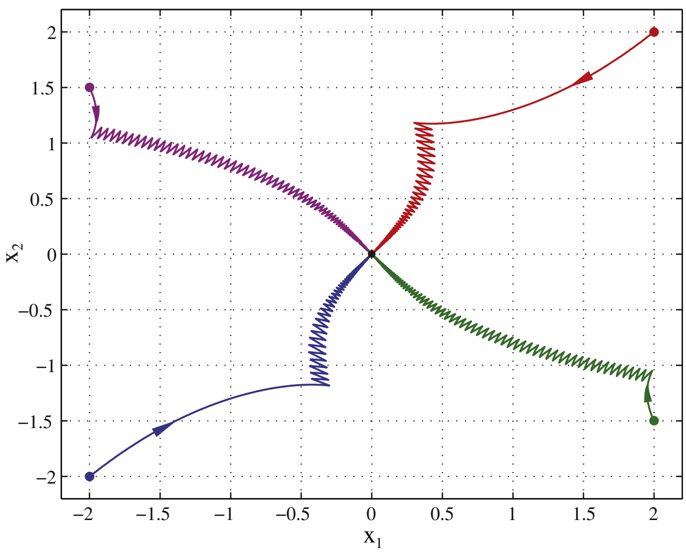

Fig. 2. Stabilized switching solutions with initial data [2 2], [−2 −2], [−2 1.5], and [2 −1.5] at $t_0 = 0$ for the switched system given by (4.9) and (4.10).

Example 4.4 (*Orbital Stability*). Consider two subsystems given by

$$x' = f_1(t,x) = \begin{pmatrix} -x_2 + 3(1 - x_1^2 - x_2^2)x_1 \\ x_1 - (1 - x_1^2 - x_2^2)x_2 \end{pmatrix} \tag{4.13}$$

and

$$x' = f_2(t,x) = \begin{pmatrix} -x_2 - (1 - x_1^2 - x_2^2)x_1 \\ x_1 + 3(1 - x_1^2 - x_2^2)x_2 \end{pmatrix}. \tag{4.14}$$

It is easy to see that $x = 0$ is a trivial solution for both subsystems. Moreover, $x = (\cos t, \sin t)$ is a periodic solution for both subsystems. By linearization near the origin, it can be shown that both subsystems are locally unstable. To apply the stabilization method of this paper to study the periodic solution, one can let

$$f(t,x) = \frac{1}{2}f_1(t,x) + \frac{1}{2}f_2(t,x) = \begin{pmatrix} -x_2 + (1 - x_1^2 - x_2^2)x_1 \\ x_1 + (1 - x_1^2 - x_2^2)x_2 \end{pmatrix}$$

and $V(t,x) = (1 - x_1^2 - x_2^2)^2$. Then

$$\nabla V(t,x) \cdot f(t,x) = -4(1 - x_1^2 - x_2^2)^2(x_1^2 + x_2^2) \tag{4.15}$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, and the set Γ_0 in Assumption 3.1 is given by $\Gamma_0 = \mathbb{R}^+ \times \{(x_1,x_2) : x_1^2 + x_2^2 = 1\}$. A solution for either of the two subsystems (4.13) and (4.14), starting from (t_0,x_0) with $x_{10}^2 + x_{20}^2 = 1$, will stay in Γ_0 for all $t \geqslant t_0$. To see this, we observe that $x = (\cos t, \sin t)$ is a periodic solution for both subsystems and it is unique with respect any initial condition on Γ_0 (the argument is that both vector fields are locally Lipschitz in x). If one chooses $h_0 = h = |1 - x_1^2 - x_2^2|$, then (h_0,h)-stability is equivalent to stability of the periodic solution $(\cos t, \sin t)$. Since $x = 0$ is a trivial solution of both subsystems, one cannot expect global attraction of the periodic solution. However, it can be shown that (4.15) implies Assumption 3.2, except in an arbitrarily small neighborhood of the origin. Actually, one has

$$\nabla V(t,x) \cdot f(t,x) \leqslant -4\varepsilon(1 - x_1^2 - x_2^2)^2 \tag{4.16}$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \setminus B(\varepsilon)$, where $\varepsilon > 0$ can be arbitrarily chosen. Moreover, by virtue of (4.15), any solution starting from $\mathbb{R}^n \setminus B(\varepsilon)$ remains in $\mathbb{R}^n \setminus B(\varepsilon)$ for all $t \geqslant t_0$. Similar argument as in the proof of Theorem 3.2 can show any solution starting from $\mathbb{R}^n \setminus B(\varepsilon)$ is attracted to the periodic solution $(\cos t, \sin t)$. It is easy to see that Assumption 3.2 is satisfied with $h_0 = h = |1 - x_1^2 - x_2^2|,\ p = 2,\ c_1 = c_2$, and $c_3 = 4\varepsilon$, for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \setminus B(\varepsilon)$. To construct a stabilizing switching signal, one can define, by choosing $\zeta = 2$

$$\Delta_1' = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \nabla V(t,x) \cdot f_1(t,x) \leqslant -2h_0^2 \cdot (x_1^2 + x_2^2)\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : |x_1| \geqslant \sqrt{\frac{3}{5}}|x_2| \quad \text{or} \quad x_1^2 + x_2^2 = 1\right\}$$

and

$$\Delta_1' = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \nabla V(t,x) \cdot f_2(t,x) \leqslant -2h_0^2 \cdot (x_1^2 + x_2^2)\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : |x_2| \geqslant \sqrt{\frac{3}{5}}|x_1| \quad \text{or} \quad x_1^2 + x_2^2 = 1\right\}.$$

A switching signal can now be constructed by the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and the generalized rule $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$. Theorem 3.2 then guarantees the resulting switched system is (h_0,h)-globally uniformly exponentially stable. Since $h_0 = h = |1 - x_1^2 - x_2^2|$, the (h_0,h)-stability concluded here is equivalent to stability of the periodic solution $(\cos t, \sin t)$.

Simulation: The phase portraits for both subsystems are shown in Fig. 3. It can be observed that the periodic solution is unstable for both subsystems. The switching domains Δ_1' and Δ_2' are shown in Fig. 4. Typical switching solutions with various initial conditions are shown in Fig. 5. It is observed that the switching solutions converge to the periodic solution as expected.

Example 4.5 (*Stability of Conditionally Invariant Set*). Consider two subsystems given by

$$x' = f_1(t,x) = \begin{pmatrix} -x_1 - 2x_2e^{-t} \\ 2x_1 \\ x_3 \sin t \end{pmatrix} \tag{4.17}$$

and

$$x' = f_2(t,x) = \begin{pmatrix} -x_1 \sin^2 x_3 \\ -x_2e^t \\ -x_1e^{-t} + x_2 \cos t \end{pmatrix}. \tag{4.18}$$

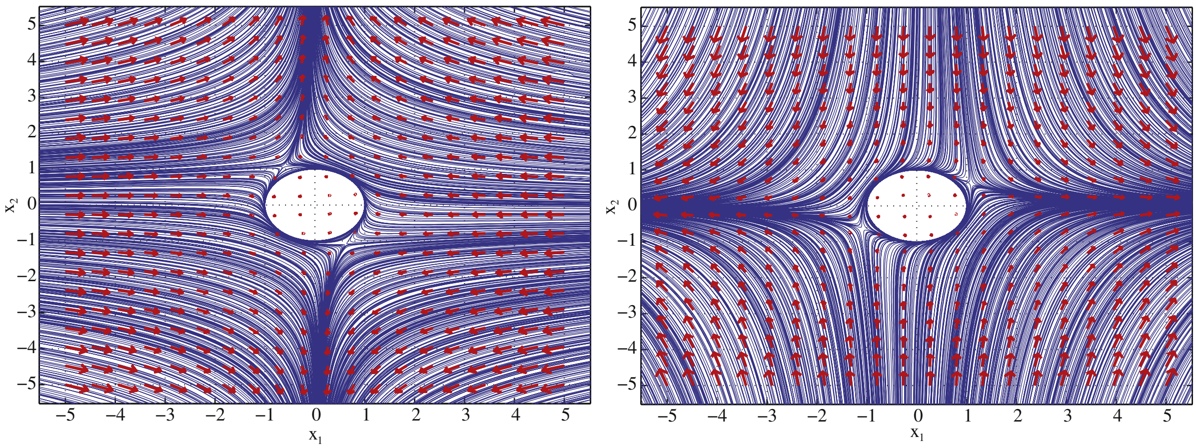

Fig. 3. Phase portraits for subsystems (4.13) (left) and (4.14) (right).

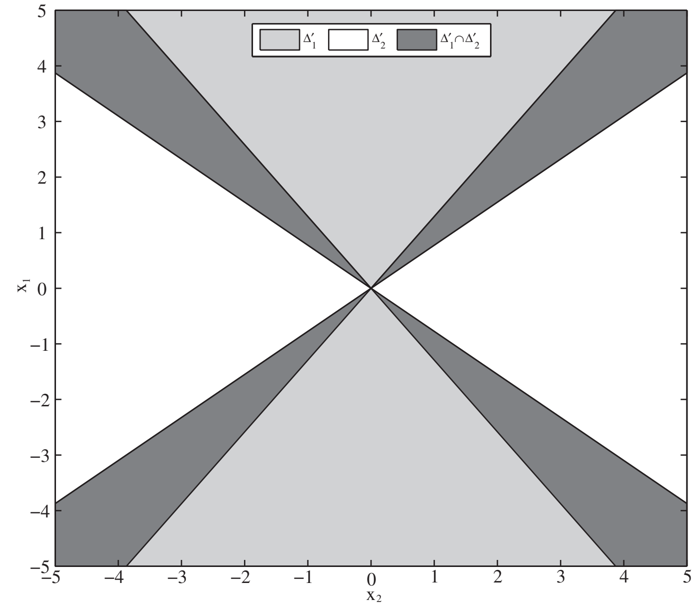

Fig. 4. Stabilizing switching domains for the switched system given by (4.13) and (4.14).

Let

$$f(t,x) = \frac{1}{2}f_1(t,x) + \frac{1}{2}f_2(t,x) = \frac{1}{2}\begin{pmatrix} -x_1\left(1+\sin^2 x_3\right) - 2x_2e^{-t} \\ 2x_1 - x_2e^{t} \\ -x_1e^{-t} + x_2\cos t + x_3\sin t \end{pmatrix}.$$

$V(t,x) = x_1^2 + x_2^2e^{-t},\ h(t,x) = d(x,B)$, and $h_0(t,x) = d(x,A)$, where $A = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$ and $B = \{x \in \mathbb{R}^3 : x_1 = 0\}$. Therefore, $h_0(t,x) = x_1^2 + x_2^2$ and $h(t,x) = x_1^2$. It is easy to see that $A \subset B$

$$h^2(t,x) \leqslant V(t,x) \leqslant h_0^2(t,x)$$

and

$$\frac{\partial V}{\partial t}(t,x) + \nabla V(t,x)\cdot f(t,x) \leqslant -h_0^2(t,x)$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and the set Γ_0 in Assumption 3.1 is given by $\Gamma_0 = \mathbb{R}^+ \times \{(x_1,x_2,x_3) : x_1 = x_2 = 0\}$. A solution for either of the two subsystems (4.17) and (4.18), starting from (t_0,x_0) with $x_{10} = x_{20} = 0$, will stay in Γ_0 for all $t \geqslant t_0$. Actually, for

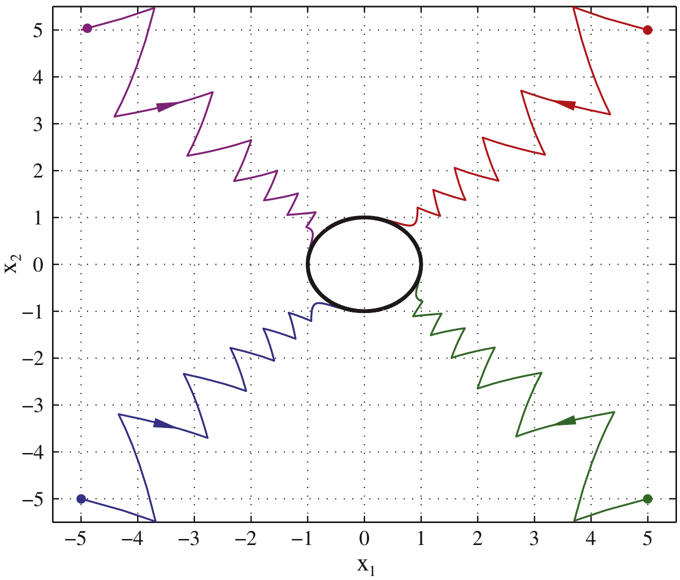

Fig. 5. Stabilized switching solutions with initial data $[5\ 5]$, $[5\ -5]$, $[-5\ 5]$, and $[-5\ -5]$ at $t_0 = 0$.

(4.17), the equations for the first two components and the third one are decoupled. It is easy to see that $x_1 = 0$ and $x_2 = 0$ is the unique solution with respect to $x_{10} = x_{20} = 0$. Therefore, a solution will stay in Γ_0 for all $t \geqslant t_0$. For (4.17), one can observe a unique solution $(0, 0, x_{30})$ for $x_{10} = x_{20} = 0$. Therefore, Assumption 3.2 is satisfied with $c_1 = c_2 = c_3 = 1$ and $p = 2$. By choosing $\zeta = 2$, one can define

$$\Delta_1' = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_1(t,x) \leqslant -\frac{1}{2} h_0^2(t,x)\right\} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : -3x_1^2 + x_2^2 - 2x_2^2 e^{-t} \leqslant 0\right\}$$

and

$$\Delta_2' = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) \cdot f_2(t,x) \leqslant -\frac{1}{2} h_0^2(t,x)\right\}$$
$$= \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : x_1^2 - 4x_1^2 \sin^2 x_3 - 3x_2^2 - 2x_2^2 e^{-t} \leqslant 0\right\}.$$

A switching signal can be constructed by the minimal rule $(\mathbf{R}_1)$–$(\mathbf{R}_3)$ and the generalized rule $(\mathbf{GR}_1)$–$(\mathbf{GR}_3)$. Theorem 3.2 guarantees that the resulting switched system is (h_0, h)-globally uniformly exponentially stable. Since $h(t,x) = d(x,B)$ and $h_0(t,x) = d(x,A)$, the (h_0, h)-stability concluded here is equivalent to stability of conditionally invariant set B with respect to A (see [6]).

5. Conclusions

In this paper, we have investigated the problem of switching stabilization for some general nonlinear systems. We propose two general rules to construct stabilizing switching signals, i.e., the minimal rule and the generalized rule. Not only have we showed that the resulting switched systems are globally uniformly asymptotically stable and globally uniformly exponentially stable, we have also rigorously proved that the switching signals generated by the generalized rule are well-defined in that the signals do not exhibit chattering and Zeno behavior. The stability analysis has been performed in terms of two measures so that the results can unify many different stability criteria, such as Lyapunov stability, partial stability, orbital stability, and stability of an invariant set. We have presented both numerical examples and simulations to illustrate the applications of the main results.

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