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# Generalized invariance principles for switched delay systems

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In this paper, the classical LaSalle's invariance principle for ordinary differential equations is extended to switched delay systems. Generalized invariance principles are established, using both multiple Lyapunov functionals and multiple Lyapunov–Razumikhin functions. As an important application, stability criteria for switched delay systems are obtained.

*Keywords*: switched systems; delay systems; hybrid systems, invariance principles; stability; multiple Lyapunov functionals; multiple Lyapunov–Razumikhin functions.

## 1. Introduction

In many applications, the physical processes are governed by more than one dynamics in which the dynamic changes among a family of choices depending on the time $t$ or the state $x$. Such processes are often described by switched systems or more generally hybrid systems, which have been studied extensively in recent years (see Goebel *et al.*, 2009; Liberzon, 2003; Shorten *et al.*, 2007; van der Schaft & Schumacher, 2000 and references therein). A good many of current studies on switched systems have been focused on finding conditions such that the systems are stable under arbitrary switching rules or under some kind of constrained switching rules (see Agrachev & Liberzon, 2001; Boscain, 2002; Chatterjee & Liberzon, 2006; Geromel & Colaneri, 2006; Liberzon, 2003; Mason *et al.*, 2006). Besides switching properties, real processes always involve time delay, i.e., the future state of the system depends not only on the present state but also on the past states. In many instances, time-delay systems (or functional differential equations) are more appropriate models to describe practical processes (see Hale & Lunel, 1993). However, the fact that the future state of a time-delay system depends not only on its current state but also on its past trajectories brings more difficulty to its stability analysis (see Chapter 5 of Hale & Lunel, 1993 and references therein).

Since the classical LaSalle's invariance principle (see LaSalle, 1960, 1968) has been proved to be a powerful tool for the stability analysis of autonomous ordinary differential equations, numerous extensions of the original invariance principle have been derived for various differential systems. In the 1960s, Hale (1965) extended the invariance principle to functional differential equations in which Lyapunov functionals were employed to discuss the extensions of LaSalle's invariance principles. In the 1980s, Haddock & Terjéki (1983) extended the work of Hale (1965) and established an invariance principle using Lyapunov–Razumikhin functions. On the other hand, recently, several invariance principles for switched systems are derived by various authors. In Hespanha (2004), under rather general switching, an extension of LaSalle's principle is obtained for switched linear systems; in Bacciotti & Mazzi (2005), a more traditional approach is taken and the results obtained there cover general non-linear switched systems, while a positive dwell-time condition is assumed. In Mancilla-Aguilar & Garcia (2006), the results in Bacciotti & Mazzi (2005) and Hespanha (2004) are extended and improved such that the results can deal with switched non-linear systems with average dwell-time switching. The recent work of Liu *et al.* (2009a) investigates invariance principles for general switched non-linear systems under weak dwell-time conditions and with impulse effects; the results are shown to improve those of Bacciotti & Mazzi (2005), Hespanha (2004) and Mancilla-Aguilar & Garcia (2006).

Although stability analysis of switched delay systems has attracted many researchers from various fields and stability criteria or stabilizing controllers have been obtained using techniques combining the classical Lyapunov functional method for functional differential equations and the new average dwell-time (or dwell-time) scheme for switched ordinary systems (see, e.g., Alwan & Liu, 2008; Hetel *et al.*, 2006; Kim *et al.*, 2006, 2008; Liu *et al.*, 2008a,b; Michel *et al.*, 2004; Sun *et al.*, 2006; Wang *et al.*, 2003; Xie & Wang, 2005; Yan & Özbay, 2008; Zhang *et al.*, 2007), the idea of invariance-like principles for switched delay systems has not yet been addressed and explored in the literature. In this paper, we aim to fill this gap by extending LaSalle's invariance principles to switched delay systems, i.e., switched systems generated by functional differential equations. By introducing the notions of weak $\tau$-invariance, $\tau$-persistent mode and $\tau$-persistent limit function, where $\tau > 0$ is a weak dwell-time for the class of switching signals considered in this paper, both results of Haddock & Terjéki (1983) and Hale (1965) are generalized to switched delay systems with weak dwell-time switchings. Invariance principles under weak dwell-time conditions for switched delay systems are established using both multiple Lyapunov functionals (MLFs) and multiple Lyapunov–Razumikhin functions (MLRFs). Several stability and instability criteria under weak dwell-time conditions for switched delay systems are then derived, as important applications of the invariance principles obtained.

The rest of this paper is organized as follows. In Section 2, the necessary notations and definitions are given to formulate a switched delay system. Section 3 presents the preliminary definitions and lemmas that are essential to prove weak invariance principles for switched delay systems under weak dwell-time conditions. The main results are shown in Section 4 in which Section 4.1 is devoted to prove a weak invariance principle using MLFs, Section 4.2 is on applying the weak invariance principle to derive asymptotic stability criteria for switched delay systems and Section 4.3 shows an instability test for switched delay systems based on the invariance principle obtained. Section 5 shows results parallel to those of Section 4, now using MLRFs. Section 6 demonstrates applications of the main results by several examples. The paper is summarized and concluded with Section 7.

## 2. Notations and definitions

Let $\mathbb{R}^n$ denote the $n$-dimensional real Euclidean space and $|x|$ the norm of the vector $x$ in $\mathbb{R}^n$. If $r \geqslant 0$ is given, let $\mathscr{C} := C([-r, 0], \mathbb{R}^n)$ denote the space of continuous functions with domain $[-r, 0]$ and range in $\mathbb{R}^n$. The norm in this space is defined by $\|\phi\| = \max_{-r\leqslant\theta\leqslant 0} |\phi(\theta)|$ for $\phi$ in $\mathscr{C}$. Suppose $x$ is a function from $\mathbb{R}$ to $\mathbb{R}^n$. For any $t \in \mathbb{R}$, let $x_t$ be an element of $\mathscr{C}$ defined by $x_t(\theta) := x(t+\theta)$, $-r \leqslant \theta \leqslant 0$.

Let $\mathscr{P}$ be a finite index set. By a 'switching signal', we mean a piecewise constant and right-continuous function $\sigma$ from $[0, \infty)$ to $\mathscr{P}$, with only finitely many discontinuities on each bounded subinterval of $[0, \infty)$. Such points of discontinuity are call 'switching times' of $\sigma$. Given an arbitrary switching signal $\sigma$, let $\varpi(\sigma)$ be the set of all switching times for $\sigma$. Without ambiguity, we may also write $\varpi(\sigma) = \{t_k\colon t_k < t_{k+1}, k = 1, 2, \ldots\}$. A 'switched delay system' can be written as

$$\dot{x} = f_\sigma(x_t), \quad t \geqslant 0, \ \sigma \in \mathscr{S}, \tag{2.1}$$

where $\{f_p\colon p \in \mathscr{P}\}$ is a family of functionals from $\mathscr{C}$ to $\mathbb{R}^n$ and $\mathscr{S}$ denotes a certain admissible set of switchings.

DEFINITION 2.1 For a switching signal $\sigma \in \mathscr{S}$ and a function $\phi \in \mathscr{C}$, we say that a function $x(t)$ from $[-r, \infty)$ to $\mathbb{R}^n$ is a 'solution' of (2.1) with initial condition $\phi \in \mathscr{C}$ and switching signal $\sigma \in \mathscr{S}$ if $x(t)$ is continuous and satisfies

$$\dot{x}(t) = f_{\sigma(t)}(x_t), \quad t \geqslant 0, \ x_0 = \phi, \tag{2.2}$$

where $\dot{x}$ denotes the right-hand derivative of $x$.

To emphasize the initial function $\phi$ and the switching signal $\sigma$, the solution can be written as $x(\phi, \sigma)$. To guarantee that (2.1) has a local solution for any $\phi \in \mathscr{C}$ and $\sigma \in \mathscr{S}$, it is assumed that the functionals $\{f_p\colon p \in \mathscr{P}\}$ are continuous and map bounded sets into bounded sets. We may also, without ambiguity, write the solution as $x(\phi)$ or $x$ if the switching signal $\sigma$ and the initial condition are not emphasized.

REMARK 2.1 It is easy to see that the functional differential equation (2.2) is equivalent to the integral equation

$$x(t) = \phi(0) + \int_0^t f_{\sigma(s)}(x_s)\mathrm{d}s, \quad t \geqslant 0.$$

Since $f_{\sigma(t)}(x_t)$ remains continuous on $[0, \infty)\backslash\varpi(\sigma)$, we can see that the solution $x(t)$ is continuously differentiable on $[0, \infty)\backslash\varpi(\sigma)$, although only right-hand derivative is involved in the original equation (2.1).

Moreover, assume $f_p(0) = 0$, for all $p \in \mathscr{P}$, such that $x = 0$ is a trivial solution of system (2.1) for any $\sigma \in \mathscr{S}$.

DEFINITION 2.2 The trivial solution of (2.1) is said to be

(I) stable (with respect to $\mathscr{S}$) if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that, for all $\sigma \in \mathscr{S}$, $x(\phi, \sigma)(t)$ exists and $|x(\phi, \sigma)(t)| < \varepsilon$ for all $t \geqslant 0$;

(II) asymptotically stable (with respect to $\mathscr{S}$) if (I) is satisfied and there exists a $\rho > 0$ such that $\|\phi\| < \rho$ implies $x(\phi, \sigma)(t) \to 0$ as $t \to \infty$;

(III) globally asymptotically stable (with respect to $\mathscr{S}$) if (II) is satisfied with an arbitrary $\rho > 0$ and

(IV) unstable (with respect to $\mathscr{S}$) if (I) fails.

DEFINITION 2.3 A function $\chi$ is said to be a 'solution piece' of (2.1.$p$), for some $p \in \mathscr{P}$, on an interval $[\alpha, \beta]$ if $\chi$ is defined on $[\alpha - r, \beta]$ and satisfies

(i) $\chi$ is continuous on $[\alpha - r, \beta]$;

(ii) $\chi$ is differentiable on $[\alpha, \beta]$ (for the endpoints $\alpha$ and $\beta$, only one-side derivatives are required to exist); and

(iii) $\chi$ satisfies $\dot{\chi}(t) = f_p(\chi_t)$ for all $t \in [\alpha, \beta]$.

DEFINITION 2.4 Let $V$ be a continuous functional from $\mathscr{C}$ to $\mathbb{R}$, $\phi$ a function in $\mathscr{C}$ and $\chi$ a solution piece of (2.1.$p$) on $[\alpha, \beta]$, with $\alpha \leqslant 0 \leqslant \beta$ and $\beta - \alpha > 0$, and $\chi_0 = \phi$. If $\beta > 0$, define the upper right-hand derivative of $V$ with respect to the $p$th mode of (2.1) to be

$$\underbrace{D^+V(\phi)}_{(2.1.p)} := \limsup_{h \to 0^+} \frac{1}{h}[V(\chi_h(\phi)) - V(\phi)]. \tag{2.3}$$

If $\alpha < 0$, define the upper left-hand derivative of $V$ with respect to the $p$th mode of (2.1) to be

$$\underbrace{D^-V(\phi)}_{(2.1.p)} := \limsup_{h \to 0^-} \frac{1}{h}[V(\chi_h(\phi)) - V(\phi)]. \tag{2.4}$$

The following proposition gives an alternative way to evaluate $D^+V$ and $D^-V$ without explicit reference to solution pieces, provided that the functional $V$ is locally Lipschitz, i.e., $V$ is Lipschitz continuous on each compact set of $\mathscr{C}$.

PROPOSITION 2.1 If $V$ is a locally Lipschitz functional from $\mathscr{C}$ to $\mathbb{R}$, then

$$\underbrace{D^+V(\phi)}_{(2.1.p)} = \limsup_{h \to 0^+} \frac{1}{h}[V(\phi_{[h]}) - V(\phi)], \tag{2.5}$$

$$\underbrace{D^-V(\phi)}_{(2.1.p)} = \limsup_{h \to 0^-} \frac{1}{h}[V(\phi_{[h]}) - V(\phi)], \tag{2.6}$$

where $\phi_{[h]}$ is defined as follows:

(1) for $r > h > 0$,

$$\phi_{[h]}(s) = \begin{cases} \phi(s+h), & s \in [-r, -h], \\ \phi(0) + (s+h) f(\phi), & s \in (-h, 0]. \end{cases} \tag{2.7}$$

(2) for $-r < h < 0$,

$$\phi_{[h]}(s) = \begin{cases} \phi(s+h), & s \in [-r-h, 0], \\ \phi(-r) + (s+r+h) f(\phi), & s \in [-r, -r-h). \end{cases} \tag{2.8}$$

*Proof.* See Ballinger (1999, p. 53) for the proof for (2.5). The proof for (2.6) is similar. □

DEFINITION 2.5 A family of functionals $\{V_p\colon p \in \mathscr{P}\}$ are called MLFs for system (2.1) on a set $G \subset \mathscr{C}$ if $V_p$ is continuous on $\overline{G}$, the closure of $G$ and $\underbrace{D^+V_p(\phi)}_{(2.1.p)} \leqslant 0$ for all $\phi \in G$ and $p \in \mathscr{P}$.

The following assumption imposes a condition on the evolution of the functionals along a solution at switching instants. This type of condition is typically encountered in results involving MLFs (see Liberzon, 2003). A weaker version is stated here, which only stipulates conditions on a single functional $V_p$.

ASSUMPTION 2.1 For some $p \in \mathscr{P}$ and every pair of switching instants $t_j < t_k$ such that $\sigma(t_j) = \sigma(t_k) = p$, we have

$$V_p(x_{t_k}(\phi, \sigma)) \leqslant V_p(x_{t_{j+1}}(\phi, \sigma)). \tag{2.9}$$

REMARK 2.2 In other words, if Assumption 2.1 is satisfied for some $p \in \mathscr{P}$, then the value of $V_p$ at the beginning of each interval on which $\sigma = p$ does not exceed the value of $V_p$ at the end of previous such intervals (if one exists). This condition, together with the fact that $\{V_p\colon p \in \mathscr{P}\}$ is a family of MLFs for system (2.1), ensures that $V_p(x_t(\phi))$ is non-increasing on the union of all the intervals where the $p$th subsystem is activated.

REMARK 2.3 It should be remarked that Assumption 2.1 imposes conditions on the switching signal $\sigma$ (hence $\mathscr{S}$), the initial condition $\phi$ and the solution $x$ as a whole. As mentioned in Hespanha (2004), in hybrid control systems, the switching signal is often generated by a supervisory logic that guarantees, by construction, that assumptions such as Assumption 2.1 hold.

REMARK 2.4 Assumption 2.1 is trivially satisfied in the case of common Lyapunov functional (CLF), i.e., the family $\{V_p\colon p \in \mathscr{P}\}$ are identical for all $p \in \mathscr{P}$.

## 3. Preliminary results

In this section, we formulate some preliminaries that are necessary to establish weak invariance principles along the lines of Hale (1965) and LaSalle (1960), as well as Bacciotti & Mazzi (2005), Liu *et al.* (2009a) and Mancilla-Aguilar & Garcia (2006). Especially, the notions of $\tau$-persistent mode, $\tau$-persistent limit and weak $\tau$-invariance are introduced for switched delay systems and some preliminary results are established.

DEFINITION 3.1 (Liu *et al.*, 2009a). A switching signal $\sigma$ is said to have 'weak dwell-time' $\tau > 0$ if any one of the following equivalent statements is satisfied:

(1) for every $T \geqslant 0$, we can find a positive integer $m$ such that $t_{m+1} - t_m \geqslant \tau$ and $t_m \geqslant T$, where $t_m$ and $t_{m+1}$ belong to $\varpi(\sigma) = \{t_k\colon t_k < t_{k+1}, k = 1, 2, \ldots\}$ (same for (2) below);

(2) if there exists some $p \in \mathscr{P}$ such that, for every $T \geqslant 0$, we can find a positive integer $m$ such that $t_{m+1} - t_m \geqslant \tau$ with $t_m \geqslant T$ and $\sigma(t_m) = p$ (the $p$th mode is called a '$\tau$-persistent mode' of $\sigma$); or

(3) if there exists some $p \in \mathscr{P}$ such that the union of all the intervals of length greater than $\tau$ on which $\sigma = p$, denoted by $\mathscr{I}_p$, has an infinite Lebesgue measure (we call $\mathscr{I}_p$ a $\tau$-'persistent domain' for $\sigma$).

REMARK 3.1 In Hespanha (2004), the author shows by an example that LaSalle's invariance principle for ordinary differential equations cannot be applied directly to switched systems. One has to impose some restrictions on the switching property of the systems. In Bacciotti & Mazzi (2005), invariance principles are established for switched systems with non-vanishing dwell-time. Along this line, we are interested in establishing invariance principles for switched delay systems. In this paper, following the treatment in Liu *et al.* (2009a) for impulsive switched systems, only a rather mild condition, i.e., the weak dwell-time condition as formulated in Definition 3.1, is assumed on the switching signals.

Let $\mathscr{S}(\tau)$ denote the set of all switching signals with weak dwell-time $\tau$. For $\phi \in \mathscr{C}$ and $\sigma \in \mathscr{S}(\tau)$, let $x(\phi, \sigma)$ be a solution of system (2.1) with initial condition $\phi$ and switching signal $\sigma$. The following definitions and lemmas are related to this specific solution $x(\phi, \sigma)$.

DEFINITION 3.2 Given some $p \in \mathscr{P}$, an element $\psi$ of $\mathscr{C}$ is called a '$\tau$-persistent limit function' of $x(\phi, \sigma)$ in the $p$th mode if $p$ is a $\tau$-persistent mode of $\sigma$ and there exists a sequence of $s_n \in \mathscr{I}_p$, with $s_n \to \infty$ as $n \to \infty$, such that $x_{s_n}(\phi, \sigma) \to \psi$ as $n \to \infty$. The set of all such functions in $\mathscr{C}$ is called the '$\tau$-persistent limit set' of $x(\phi, \sigma)$ in the $p$th mode and is denoted by $\Omega_p(x(\phi, \sigma))$.

DEFINITION 3.3 The solution $x(\phi, \sigma)$ is said to 'weakly approach' a set $M \subset \mathscr{C}$ in the $p$th mode as $t \to \infty$ if the $p$ is a $\tau$-persistent mode of $\sigma$ and

$$\lim_{\substack{t \to \infty \\ t \in \mathscr{I}_p}} \operatorname{dist}(x_t(\phi, \sigma), M) = 0,$$

where $\operatorname{dist}(\psi, M)$ for $\psi \in \mathscr{C}$ is defined by $\operatorname{dist}(\psi, M) = \inf_{\chi \in M} \|\psi - \chi\|$.

DEFINITION 3.4 For some $\tau > 0$, a set $M \subset \mathscr{C}$ is called a 'weakly $\tau$-invariant set' with respect to the $p$th mode of (2.1), if, for any $\psi$ in $M$, there exists a solution piece $\chi$ of (2.1.$p$) on $[\alpha, \beta]$, with $\alpha \leqslant 0 \leqslant \beta$ and $\beta - \alpha \geqslant \tau$, such that $\chi_0 = \psi$ and $\chi_t \in M$ for all $t \in [\alpha, \beta]$.

LEMMA 3.1 If $p \in \mathscr{P}$ is a $\tau$-persistent mode of $\sigma$, $\mathscr{I}_p$ is the corresponding $\tau$-persistent domain, and the solution $x(\phi, \sigma)$ is bounded in the $p$th, i.e., there exists a positive real number $b$ such that $\|x_t(\phi, \sigma)\| \leqslant b$ for all $t \in \mathscr{I}_p$, then the family of functions $\{x_t(\phi, \sigma): t \in \mathscr{I}_p, t \geqslant r\}$ belongs to a compact subset of $\mathscr{C}$.

*Proof.* Define

$$\mathscr{B} := \{\psi \in \mathscr{C}: \|\psi\| \leqslant b,\ \|\dot{\psi}\| \leqslant b'\},$$

where $b$ is the bound specified in the lemma condition and $b'$ is a positive real number such that $f_p$ maps the bounded set $\{\psi \in \mathscr{C}: \|\psi\| \leqslant b\}$ into the bounded set $\{x \in \mathbb{R}^n: |x| \leqslant b'\}$ for all $p \in \mathscr{P}$. It is clear that the family $\{x_t(\phi, \sigma): t \in \mathscr{I}_p, t \geqslant r\}$ is contained in $\mathscr{B}$. Moreover, by Arzela–Ascoli's compactness criterion, $\mathscr{B}$ is a compact subset of $\mathscr{C}$. □

LEMMA 3.2 If $p \in \mathscr{P}$ is a $\tau$-persistent mode of $\sigma$ and the solution $x(\phi, \sigma)$ is bounded in the $p$th mode, then $\Omega_p(x(\phi, \sigma))$ is a non-empty, compact and weakly $\tau$-invariant set w.r.t. the $p$th mode of (2.1). Moreover, $x(\phi, \sigma)$ weakly approaches $\Omega_p(x(\phi, \sigma))$ in the $p$th mode as $t \to \infty$.

*Proof.* From Lemma 3.1, the family of functions $\{x_t(\phi, \sigma): t \in \mathscr{I}_p, t \geqslant r\}$ belongs to a compact subset $\mathscr{B}$ of $\mathscr{C}$ and $\mathscr{B}$ could be defined as in the proof of Lemma 3.1. Since $p$ is a persistent mode, there must be a sequence $\{s_n\}$ in $\mathscr{I}_p$ such that $x_{s_n}(\phi, \sigma)$ has a limit in $\mathscr{B}$. This shows $\Omega_p(x(\phi, \sigma))$ is not empty. Clearly, $\Omega_p(x(\phi, \sigma))$ is a subset of the compact set $\mathscr{B}$. Thus to show $\Omega_p(x(\phi, \sigma))$ is compact, it suffices to show $\Omega_p(x(\phi, \sigma))$ is closed.

To show that $\Omega_p(x(\phi, \sigma))$ is closed, suppose $\psi_n \in \Omega_p(x(\phi, \sigma))$ approaches $\psi$ as $n \to \infty$. Since $\psi_n \in \Omega_p(x(\phi, \sigma))$ for each $n$, by Definition 3.2, one can choose $s_n \in \mathscr{I}_p$ large enough such that $\|x_{s_n}(\phi, \sigma) - \psi_n\| < 1/n$ for each $n$. Now given any $\varepsilon > 0$, choose $n$ sufficiently large such that $\|\psi_n - \psi\| < \varepsilon/2$ and $\|x_{t_n}(\phi, \sigma) - \psi_n\| < \varepsilon/2$. Then $\|x_{t_n}(\phi, \sigma) - \psi\| < \varepsilon$ for $n$ large enough, which shows $\psi \in \Omega_p(x(\phi, \sigma))$ and therefore $\Omega_p(x(\phi, \sigma))$ is closed. It follows then $\Omega_p(x(\phi, \sigma))$ is compact.

The last assertion of the lemma can be shown by contradiction. Suppose that there exists an increasing sequence of $s_n \in \mathscr{I}_p$, with $s_n \to \infty$ as $n \to \infty$, and a $\delta > 0$ such that $\|x_{s_n}(\phi, \sigma) - \psi\| \geqslant \delta$ for all $\psi \in \Omega_p(x(\phi, \sigma))$. Now since $x_{s_n}(\phi, \sigma)$ belongs to $\mathscr{B}$, which is compact, there must exist a subsequence of $x_{s_n}(\phi, \sigma)$ that converges to some $\psi \in \Omega_p(x(\phi, \sigma))$. This contradicts with the inequality above and shows the last assertion of the lemma holds.

Finally, we show that $\Omega_p(x(\phi,\sigma))$ is weakly $\tau$-invariant with respect to the $p$th mode of (2.1), i.e., for any $\psi$ in $\Omega_p(x(\phi,\sigma))$, there exists a solution piece $\chi$ of (2.1.$p$) on $[\alpha-r,\beta]$, with $\alpha\leqslant 0\leqslant\beta$ and $\beta-\alpha\geqslant\tau$, such that $\chi_0=\psi$ and $\chi_t\in\Omega_p(x(\phi,\sigma))$, $\forall\, t\in[\alpha,\beta]$.

Since $\psi\in\Omega_p(x(\phi,\sigma))$, there exists an increasing sequence of $s_n\in\mathscr{I}_p$ such that $s_n\to\infty$ and $x_{s_n}(\phi,\sigma)\to\psi$ as $n\to\infty$. Moreover, we can pick $s_n$ so that there exists a sequence of intervals $[\tau_{2n-1},\tau_{2n}]$ which verifies that, for all $n$, $s_n\in[\tau_{2n-1},\tau_{2n}]\subset\mathscr{I}_p$. By this choice, $x(\phi,\sigma)$ satisfies the $p$th subsystem on $[\tau_{2n-1},\tau_{2n}]$, i.e.,

$$\dot{x}(\phi,\sigma)(t)=f_p(x_t(\phi,\sigma)),\quad \forall\, t\in[\tau_{2n-1},\tau_{2n}].$$

Moreover, $x(\phi,\sigma)(t)$ is continuously differentiable on $[\tau_{2n-1},\tau_{2n}]$ and continuous on $[\tau_{2n-1}-r,\tau_{2n}]$ according to Remark 2.1. Putting

$$\alpha_n=\tau_{2n-1}-s_n \quad\text{and}\quad \beta_n=\tau_{2n}-s_n, \tag{3.1}$$

then $\beta_n-\alpha_n\geqslant\tau$ and $\alpha_n\leqslant 0\leqslant\beta_n$. Define a sequence of functions $\chi^n$ as

$$\chi^n(t):=x(\phi,\sigma)(t+s_n),\quad t\in[\alpha_n-r,\beta_n].$$

It follows that $\chi_n$ satisfies

$$\dot{\chi}^n(t)=f_p(\chi_t^n),\quad \chi_t^n=x_{t+s_n}(\phi),\quad \forall\, t\in[\alpha_n,\beta_n]. \tag{3.2}$$

It is clear that $\chi^n$ is continuously differentiable on $[\alpha_n,\beta_n]$ and continuous on $[\alpha_n-r,\beta_n]$.

To show convergence of (some subsequence of) $\chi^n$, we need to consider $\chi^n$ as a sequence of functions defined on a common interval $[\alpha-r,\beta]$. Moreover, to show weak $\tau$-invariance, we require $\beta-\alpha\geqslant\tau$ and $\alpha\leqslant 0\leqslant\beta$. Consider two cases:

1. If either $\limsup_{n\to\infty}\beta_n=\infty$ or $\liminf_{n\to\infty}\alpha_n=-\infty$, we can choose $[\alpha,\beta]=[0,\tau]$ or $[\alpha,\beta]=[-\tau,0]$, respectively, and an appropriate subsequence of $\chi^n$, which we can keep the same designation, such that (3.2) is satisfied on $[\alpha,\beta]$, while $\chi^n$ is defined and continuous on $[\alpha-r,\beta]$ and continuously differentiable on $[\alpha,\beta]$. It is clear that $\beta-\alpha\geqslant\tau$ and $\alpha\leqslant 0\leqslant\beta$.
2. If both $\alpha_n$ and $\beta_n$ are bounded sequences. Choose subsequences of $\alpha_n$ and $\beta_n$, with the same designation, such that $\alpha_n\to\alpha$ and $\beta_n\to\beta$ as $n\to\infty$. It is clear that $\beta-\alpha\geqslant\tau$ and $\alpha\leqslant 0\leqslant\beta$. However, $[\alpha-r,\beta]$ might not be a common subinterval of $[\alpha_n-r,\beta_n]$. We need to extend $\chi^n$ to $[\alpha,\beta]$ if $[\alpha-r,\beta]\not\subset[\alpha_n-r,\beta_n]$ for some $n$. If $\alpha<\alpha_n$, define

$$\chi^n(t):=\chi^n(\alpha_n-r),\quad t\in[\alpha-r,\alpha_n-r).$$

   If $\beta>\beta_n$, define

$$\chi^n(t):=\chi^n(\beta_n),\quad t\in(\beta_n,\beta].$$

   Note that (3.2) is only guaranteed for $t\in[\alpha_n,\beta_n]$, even if $\chi^n$ is now defined on $[\alpha-r,\beta]$ for all $n$. However, we do have $\alpha_n\to\alpha$ and $\beta_n\to\beta$ as $n\to\infty$.

In both cases, what we have obtained is a sequence of functions $\chi^n$ defined on a common interval $[\alpha-r,\beta]$, with $\beta-\alpha\geqslant\tau$ and $\alpha\leqslant 0\leqslant\beta$.

We proceed to show that $\chi^n$, as a sequence of functions defined on $[\alpha - r, \beta]$, has a subsequence that uniformly converges to a function $\chi$ on $[\alpha - r, \beta]$. Since $x_t(\phi, \sigma)$ is bounded by some $b > 0$ for all $t \geqslant 0$, it follows that $\chi^n$ is uniformly bounded by $b$ on $[\alpha - r, \beta]$. Moreover, still let $b'$ be the positive real number such that $f_p$ maps the bounded set $\{\psi \in \mathscr{C}: \|\psi\| \leqslant b\}$ into the bounded set $\{x \in \mathbb{R}^n: |x| \leqslant b'\}$ for each $p \in \mathscr{P}$. Define

$$\mathscr{B}' := \{\psi \in C([\alpha - r, \beta], \mathbb{R}^n): \|\psi\| \leqslant b, \|\dot{\psi}\| \leqslant b'\},$$

where $C([\alpha - r, \beta], \mathbb{R}^n)$ denotes the space of continuous functions with domain $[\alpha - r, \beta]$ and range in $\mathbb{R}^n$. By Arzela–Ascoli's compactness criterion, $\mathscr{B}'$ is a compact set. It is clear that, for both cases, the sequence of $\chi^n$ belongs to $\mathscr{B}'$ and therefore $\chi^n$ has a subsequence, still designated by $\chi^n$, converges to some function, say $\chi$, on $[\alpha - r, \beta]$ in the uniform norm. Hence, $\chi$ is continuous on $[\alpha - r, \beta]$. Moreover, for each $t \in (\alpha, \beta)$, we have $\dot{\chi}(t) = f_p(\chi_t)$ (note that, in the second case, $t \in (\alpha, \beta)$ implies that $t \in (\alpha_n, \beta_n)$ for $n$ sufficiently large). Since $\dot{\chi}(t) = f_p(\chi_t)$ is uniformly continuous on $(\alpha, \beta)$, we have that $\lim_{t\to\alpha^+} \dot{\chi}(t)$ and $\lim_{t\to\beta^-} \dot{\chi}(t)$ exist and equal $\dot{\chi}(\alpha^+)$ and $\dot{\chi}(\beta^-)$ (the right-hand derivative at $t = \alpha$ and left-hand derivative at $t = \beta$), respectively. Hence, by continuity, $\dot{\chi}(t) = f_p(\chi_t)$ for all $t \in [\alpha, \beta]$, and $\chi$ is a solution piece of (2.1.$p$) on $[\alpha, \beta]$. Moreover, $\chi_0 = \lim_{n\to\infty} \chi_0^n = \lim_{n\to\infty} x_{s_n}(\phi, \sigma) = \psi$. Finally, we have $\chi_t = \lim_{n\to\infty} \chi_t^n = \lim_{n\to\infty} x_{t+s_n}(\phi)$ for any fixed $t$ in $(\alpha, \beta)$. For a fixed $t \in (\alpha, \beta)$, put $s_n' = t + s_n$. According to (3.1) and the fact that $\alpha_n \to \alpha$ and $\beta_n \to \beta$, as $n \to \infty$, $s_n' \in [\tau_{2n-1}, \tau_{2n}]$ for $n$ sufficiently large. Since $\sigma = p$ on $[\tau_{2n-1}, \tau_{2n}]$ and $\tau_{2n} - \tau_{2n-1} \geqslant \tau$, it follows that $\chi_t \in \Omega_p(x(\phi, \sigma))$ for all $t \in (\alpha, \beta)$. Since $\Omega_p(x(\phi, \sigma))$ is closed and $\chi_t$ is continuous on $[\alpha, \beta]$, $\chi_\alpha$ and $\chi_\beta$ belong to $\Omega_p(x(\phi, \sigma))$. Therefore, $\Omega_p(x(\phi, \sigma))$ is shown to be weakly $\tau$-invariant w.r.t. the $p$th mode of (2.1) and the proof is complete. □

## 4. Multiple Lyapunov functionals

### 4.1 *A weak invariance principle*

In this section, we apply MLFs to establish a weak invariance principle for switched delay systems.

Let $\{V_p: p \in \mathscr{P}\}$ be a family of MLFs for (2.1) on $G \subset \mathscr{C}$. For some $\tau > 0$, define $E_p$ to be the set of all $\phi \in \overline{G}$ such that there exists a solution piece $\chi$ of (2.1.$p$) on some interval $[\alpha, \beta]$, with $\beta - \alpha > 0$ and $\alpha \leqslant 0 \leqslant \beta$, satisfying $\chi_0 = \phi$ and $V_p(\chi_t) \equiv c$, some constant, on $[\alpha, \beta]$. Let $M_p(\tau)$ be the largest set in $E_p$ that is weakly $\tau$-invariant with respect to the $p$th mode of (2.1).

THEOREM 4.1 Let $\{V_p: p \in \mathscr{P}\}$ be a family of MLFs for (2.1) on $G$, $x(\phi, \sigma)$ be a solution of (2.1) such that $x_t(\phi, \sigma)$ remains in $G$ for $t \geqslant 0$, $p \in \mathscr{P}$ be a $\tau$-persistent mode of $\sigma$, and $x(\phi, \sigma)$ is bounded in the $p$th mode. Suppose, in addition, Assumption 2.1 is satisfied for this $p$. Then $x(\phi, \sigma)$ weakly approaches $M_p(\tau) \cap V_p^{-1}(c)$, for some $c$, in the $p$th mode as $t \to \infty$.

*Proof.* By Lemma 3.2, $x_t(\phi, \sigma)$ has a non-empty $\omega$-limit set $\Omega_p(x(\phi, \sigma))$. We proceed to show that $\Omega_p(x(\phi, \sigma)) \subset E_p$. Let $\mathscr{I}_p$ be the $\tau$-persistent domain of the $p$th mode of $\sigma$. The conditions on $V_p$ imply that $V_p(x_t(\phi, \sigma))$ is non-increasing on $\mathscr{I}_p$. Moreover, $V_p(x_t(\phi, \sigma))$ is bounded from below on $\mathscr{I}_p$ since $\{x_t(\phi, \sigma): t \in \mathscr{I}_p\}$ belongs to a compact set of $\mathscr{C}$ according to Lemma 3.1. Therefore, as $t \to \infty$ in $\mathscr{I}_p$, $V_p(x_t(\phi, \sigma))$ yields a limit as

$$\lim_{\substack{t\to\infty \\ t\in\mathscr{I}_p}} V_p(x_t(\phi, \sigma)) = c.$$

For any $\psi \in \Omega_p(x(\phi,\sigma))$, there exists a sequence $s_n \in \mathscr{I}_p$ such that $s_n \to \infty$ and $x_t(\phi,\sigma) \to \psi$ as $n \to \infty$. It follows by continuity of $V_p$ that

$$V_p(\psi) = \lim_{n\to\infty} V_p(x_{s_n}(\phi,\sigma)) = \lim_{\substack{t\to\infty \\ t\in\mathscr{I}_p}} V_p(x_t(\phi,\sigma)) = c.$$

Hence, $V_p(\psi) = c$ for all $\psi \in \Omega_p(x(\phi,\sigma))$ and $\Omega_p(x(\phi,\sigma)) \subset V_p^{-1}(c)$. By Lemma 3.2, $\Omega_p(x(\phi,\sigma))$ is weakly $\tau$-invariant w.r.t. the $p$th mode of (2.1), i.e., for each $\psi \in \Omega_p(x(\phi,\sigma))$, there exists a solution piece $\chi$ of (2.1.$p$) on some interval $[\alpha,\beta]$ with $\alpha \leqslant 0 \leqslant \beta$ and $\beta - \alpha \geqslant \tau$ such that $\chi_0 = \psi$ and $\chi_t \in \Omega_p(x(\phi,\sigma))$ for all $t \in [\alpha,\beta]$. Therefore, $V_p(\chi_t) = c$ for all $t \in [\alpha,\beta]$. It follows that $\Omega_p(x(\phi,\sigma)) \subset E_p$. By the definition of $M_p(\tau)$ and because $\Omega_p(x(\phi,\sigma))$ is weakly $\tau$-invariant w.r.t. the $p$th mode of (2.1), we have $\Omega_p(x(\phi,\sigma)) \subset M_p(\tau) \subset E_p$. From Lemma 3.2, $x(\phi,\sigma)$ weakly approaches $\Omega_p(x(\phi,\sigma))$ in the $p$th mode and therefore it weakly approaches $M_p(\tau) \cap V_p^{-1}(c)$ in the $p$th mode. This completes the proof. □

The definition we choose for $E_p$ gives more precise characterization of the set $M_p$. The following proposition gives a sometimes more convenient way for locating $E_p$ (hence $M_p$).

PROPOSITION 4.1 For $\phi \in E_p$, we have either $\underbrace{D^+V_p(\phi)}_{(2.1.p)} = 0$ or $\underbrace{D^-V_p(\phi)}_{(2.1.p)} = 0$, i.e., $E_p$ is a subset of

$$E'_p := \{\phi \in \overline{G}\colon \underbrace{D^+V_p(\phi)}_{(2.1.p)} = 0 \text{ or } \underbrace{D^-V_p(\phi)}_{(2.1.p)} = 0\}.$$

### 4.2 *Stability criteria*

Before we apply the invariance principle obtained in Theorem 4.1 to derive some stability results, we need to introduce a local stability result for switched delay systems by MLFs. In this section and later on, we use the notation $\mathscr{B}_r$ to denote the set $\{\phi \in \mathscr{C}\colon \|\phi\| < r\}$ for any $r > 0$ and $\mathscr{C}_0$ the set $\{\phi \in \mathscr{C}\colon \phi(0) = 0\}$.

PROPOSITION 4.2 Suppose

(i) $\{V_p\colon p \in \mathscr{P}\}$ is a family of MLFs for (2.1) on $\mathscr{B}_\rho$ for some $\rho > 0$ satisfying

$$u(|\phi(0)|) \leqslant V_p(\phi) \leqslant \upsilon(\|\phi\|), \quad \forall \phi \in \mathscr{B}_\rho, \quad p \in \mathscr{P},$$

where $u$ and $\upsilon$ are two continuous and strictly increasing functions from $[0,\infty)$ to $[0,\infty)$ with $u(0) = \upsilon(0) = 0$; and

(ii) Assumption 2.1 is satisfied for all $p \in p$.

Then the trivial solution $x = 0$ of system (2.1) is stable.

*Proof.* Given any $\varepsilon \in (0,\rho)$, we let $\delta_0 = \varepsilon$ and define $\delta_1, \delta_2, \ldots, \delta_N$ recursively such that $\delta_{j+1} < \delta_j$ and $\upsilon(\delta_{j+1}) < u(\delta_j)$ for $j = 0, 1, 2, \ldots, N-1$, where $N$ is the cardinality of $\mathscr{P}$. Stability follows from the following claim.

*Claim.* $\|\phi\| < \delta_N$ implies $|x(\phi,\sigma)(t)| < \varepsilon$ for all $t \geqslant 0$.

*Proof of the Claim.* Assume $t_1$ is the first switching instant and the $i$th subsystem is activated on $[t_0, t_1]$. Then conditions on $V_i$ imply that

$$u(|x(\phi,\sigma)(t)|) \leqslant V_i(x_t(\phi,\sigma)) \leqslant V_i(\phi)$$
$$\leqslant v(\|\phi\|) \leqslant v(\delta_N) < u(\delta_{N-1}) \leqslant u(\varepsilon), \quad \forall t \in [0, t_1].$$

This gives $|x(\phi,\sigma)(t)| < \delta_{N-1} \leqslant \varepsilon$, $t \in [0, t_1]$. Since $\|\phi\| < \delta_N < \delta_{N-1} \leqslant \varepsilon$, it follows that $\|x_t(\phi,\sigma)\| < \delta_{N-1} \leqslant \varepsilon$. The claim is proved on the first dwell interval $[0, t_1]$. Particularly, we have $\|x_{t_1}(\phi,\sigma)\| < \delta_{N-1}$. Now assume $t_2$ is the second switching instant and $j$th subsystem is activated on $[t_1, t_2]$. Assume with out loss of generality that $N \geqslant 2$ and $j \neq i$. Then conditions on $V_i$ imply that

$$u(|x(\phi,\sigma)(t)|) \leqslant V_j(x_t(\phi,\sigma)) \leqslant V_j(x_{t_1}(\phi,\sigma))$$
$$\leqslant v(\|\phi\|) \leqslant v(\delta_{N-1}) < u(\delta_{N-2}) \leqslant u(\varepsilon), \quad \forall t \in [t_1, t_2].$$

This gives $|x(\phi,\sigma)(t)| < \delta_{N-2} \leqslant \varepsilon$, $t \in [t_1, t_2]$. Since $\|\phi\| < \delta_{N-1} < \delta_{N-2} \leqslant \varepsilon$, it follows that $\|x_t(\phi,\sigma)\| < \delta_{N-2} \leqslant \varepsilon$. The claim is proved on the first dwell interval $[t_1, t_2]$. Particularly, we have $\|x_{t_2}(\phi,\sigma)\| < \delta_{N-2}$. The remaining of this proof is to repeat the above procedure over all switching instants. At each switching time, we may either encounter the case where a subsystem among the family is activated for the first time or a subsystem activated once before is active again. In the former case, we can use the previous argument and show $|x(\phi,\sigma)(t)| < \delta_{N-m} \leqslant \varepsilon$ on the current dwell interval, where $m \leqslant N$ is such that $|x(\phi,\sigma)(t)| < \delta_{N-m+1} \leqslant \varepsilon$ holds on the previous dwell interval. In the latter case, Assumption 2.1 ensures that $|x(\phi,\sigma)(t)| < \delta_{N-n} \leqslant \varepsilon$ is conserved for the current dwell interval, where $n$ is the number of subsystems that have been activated up to (and including) the current instant (not counting multiplicity). Since there exist only $N$ subsystems in the family, the latter case can occur at most $N$ times. Therefore, $|x(\phi,\sigma)(t)| < \delta_0 \leqslant \varepsilon$ is guaranteed for all $t \geqslant 0$. This proves the claim and completes the proof. □

REMARK 4.1 It is clear from the proof of Proposition 4.2 that, as far as only local stability is concerned, we can actually replace Assumption 2.1 by the following weaker one: for some $p \in \mathscr{P}$ and every pair of switching instants $t_j < t_k$ such that $\sigma(t_j) = \sigma(t_k) = p$ and $t_j$ is the first time the $p$th mode is activated, we have

$$V_p(x_{t_k}(\phi,\sigma)) \leqslant V_p(x_{t_j}(\phi,\sigma)). \tag{4.1}$$

Inequality (4.1) is weaker than (2.9) since it only requires that the value of $V_p$ at the beginning of each interval on which $\sigma = p$ doest not exceed the value of $V_p$ at the very first time the $p$th mode is activated. This observation is in accordance with the case of general switched non-linear systems not involving time delays (see Theorem 3.1 and Remark 3.2 of Liu *et al.*, 2009b).

For $\tau > 0$ and $p \in \mathscr{P}$, let $\mathscr{S}(\tau, p)$ denote the set of all switching signals in $\mathscr{S}$ that have $\tau$ as weak dwell-time and $p$ as the corresponding $\tau$-persistent mode. It is clear that

$$\mathscr{S}(\tau) = \bigcup_{p\in\mathscr{P}} \mathscr{S}(\tau, p),$$

where $\mathscr{S}(\tau)$ represent all switching signals in $\mathscr{S}$ with weak dwell-time $\tau > 0$.

THEOREM 4.2 Suppose the conditions of Proposition 4.2 hold. Then the trivial solution $x = 0$ of system (2.1) is asymptotically stable w.r.t. $\mathscr{S}(\tau, p)$, provided that $\tau \geqslant r$ and $M_p(\tau) \subset \mathscr{C}_0$. If $\tau \geqslant r$ and $\bigcup_{p\in\mathscr{P}} M_p(\tau) \subset \mathscr{C}_0$, then the trivial solution $x = 0$ of system (2.1) is asymptotically stable w.r.t. $\mathscr{S}(\tau)$.

*Proof.* Local stability of $x = 0$ follows from Proposition 4.2. Given some $\varepsilon_0 > 0$, let $\delta(\varepsilon_0)$ be the local stability constant found in the proof for Proposition 4.2 ($\delta_N(\varepsilon_0)$ there). Then any solution enters $\mathscr{B}_{\delta(\varepsilon_0)}$ will remain in $\mathscr{B}_{\delta(\varepsilon_0)}$ for all future time. We claim that $\mathscr{B}_{\delta(\varepsilon_0)}$ is a domain of attraction for (2.1), i.e., any solution starting from $\mathscr{B}_{\delta(\varepsilon_0)}$ converges to 0. Actually, since $\|x_t(\phi,\sigma)\| < \varepsilon_0$, $x(\phi,\sigma)$ is a bounded solution. According to Theorem 4.1, $x(\phi,\sigma)$ weakly approaches $M_p(\tau) \subset \mathscr{C}_0$ in the $p$th mode, which shows

$$\lim_{\substack{t\to\infty \\ t\in\mathscr{I}_p}} \operatorname{dist}(x_t(\phi,\sigma), M_p(\tau)) = 0. \tag{4.2}$$

To show that $x_t(\phi,\sigma) \to 0$ as $t \to \infty$, choose an arbitrary $\varepsilon \in (0, \varepsilon_0)$ and let $\delta(\varepsilon)$ be the local stability constant found in the proof for Proposition 4.2 ($\delta_N$ there). Then any solution enters $\mathscr{B}_{\delta(\varepsilon)}$ will remain in $\mathscr{B}_\varepsilon$ for all future time. Now (4.2) implies that there exists some $T_1 \geqslant 0$ such that

$$\operatorname{dist}(x_t(\phi,\sigma), M_p(\tau)) < \frac{\delta(\varepsilon)}{2}, \quad \forall\, t \geqslant T_1, \ \ t \in \mathscr{I}_p.$$

Particularly, there exists an interval $[T_2, T_3] \subset \mathscr{I}_p$ with $T_2 \geqslant T_1$ and $T_3 - T_2 \geqslant \tau \geqslant r$ such that

$$\operatorname{dist}(x_t(\phi,\sigma), M_p(\tau)) < \frac{\delta(\varepsilon)}{2}, \quad \forall\, t \in [T_2, T_3],$$

which implies that, for each $t \in [T_2, T_3]$, there exists some $\psi_t \in M_p(\tau)$ such that

$$\|x_t - \psi_t\| < \delta(\varepsilon). \tag{4.3}$$

Since $M_p(\tau) \subset \mathscr{C}_0$, it follows from (4.3) that

$$|x(t)| < \delta(\varepsilon), \quad \forall\, t \in [T_2, T_3].$$

Since $T_3 - T_2 \geqslant r$, this shows $\|x_{T_3}(\phi,\sigma)\| < \delta(\varepsilon)$ and hence $|x(\phi,\sigma)(t)| < \varepsilon$ for all $t \geqslant T_3$, i.e., we have shown $x(\phi,\sigma)(t) \to 0$ as $t \to \infty$. Hence, the trivial solution is asymptotically stable with respect to $S(\tau,\rho)$. The last assertion of the theorem follows from the fact that $\mathscr{S}(\tau) = \bigcup_{p\in\mathscr{P}} \mathscr{S}(\tau, p)$. □

THEOREM 4.3 Suppose

(i) $\{V_p\colon p \in \mathscr{P}\}$ is a family of MLFs for (2.1) on $\mathscr{C}$ satisfying

$$u(|\phi(0)|) \leqslant V_p(\phi) \leqslant \upsilon(\|\phi\|), \quad \forall\, \phi \in \mathscr{C}, \ \ p \in \mathscr{P},$$

where $u$ and $\upsilon$ are two continuous and strictly increasing functions from $[0,\infty)$ to $[0,\infty)$ with $u(0) = \upsilon(0) = 0$ and $u(s) \to \infty$ as $s \to \infty$; and

(ii) Assumption 2.1 is satisfied for all $p \in \mathscr{P}$.

Then the trivial solution $x = 0$ of system (2.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau, p)$, provided that $\tau \geqslant r$ and $M_p(\tau) \subset \mathscr{C}_0$. If $\tau \geqslant r$ and $\bigcup_{p\in\mathscr{P}} M_p(\tau) \subset \mathscr{C}_0$, then the trivial solution $x = 0$ of system (2.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau)$.

*Proof.* Local stability remains the same. To show global attraction, we only have to show that the constant $\delta(\varepsilon_0)$ in the proof for Theorem 4.2 can be chosen to be arbitrarily large, provided that $\varepsilon_0$ is given sufficiently large. Now given any $\varepsilon_0 > 0$, we let $\delta_0 = \varepsilon_0$ and define $\delta_1, \delta_2, \ldots, \delta_N$ recursively such that $v(\delta_{j+1}) = u(\delta_j/2)$ for $j = 0, 1, 2, \ldots, N-1$, where $N$ is the cardinality of $\mathscr{P}$. Since $u(\delta_j/2) < u(\delta_j)$, it is clear that this choice of $\delta_1, \delta_2, \ldots, \delta_N$ is in accordance with that in the proof of Proposition 4.2. Moreover, as $\varepsilon_0 \to \infty$, so is $\delta_1, \delta_2, \ldots, \delta_N$. Therefore $\delta(\varepsilon_0) = \delta_N(\varepsilon_0)$ can be arbitrarily large if $\varepsilon_0$ is given sufficiently large. This completes the proof. □

REMARK 4.2 It is clear from the proof of Theorem 4.2 that if we have $M_p(\tau) = \{0\}$ or $\bigcup_{p\in\mathscr{P}} M_p(\tau) = \{0\}$, then the conclusions of Theorems 4.2 and 4.3 hold without the restriction $\tau \geqslant r$. However, as shown by examples later in Section 6, if we want to show $M_p(\tau) = \{0\}$ or $\bigcup_{p\in\mathscr{P}} M_p(\tau) = \{0\}$, it is often (if not always) required that $\tau \geqslant r$.

The following corollary follows immediately from Theorem 4.3.

COROLLARY 4.1 Suppose the conditions of Theorem 4.3 are satisfied. If for some $p \in \mathscr{P}$, $\underbrace{D^+V_p(\phi)}_{(2.1.p)} < 0$ and $\underbrace{D^-V_p(\phi)}_{(2.1.p)} < 0$ for all $\phi \in \mathscr{C}$ such that $|\phi(0)| \neq 0$, then the trivial solution $x = 0$ of system (2.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau, p)$ for any $\tau \geqslant r$. If $\underbrace{D^+V_p(\phi)}_{(2.1.p)} < 0$ and $\underbrace{D^-V_p(\phi)}_{(2.1.p)} < 0$ for all $\phi \in \mathscr{C}$ such that $|\phi(0)| \neq 0$ and all $p \in \mathscr{P}$, then the trivial solution $x = 0$ of system (2.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau)$ for any $\tau \geqslant r$.

### 4.3 *Instability*

The following theorem gives a test for instability of switched delay systems based on the invariance principle given by Theorem 4.1.

THEOREM 4.4 Let $G$ be a subset of $\mathscr{C}$ such that $0 \in \overline{G}$. Assume that there exist a family of MLFs $\{V_p\colon p \in \mathscr{P}\}$ for (2.1) on $G$ such that, for some $p \in \mathscr{P}$,

(i) $M_p(\tau) \cap G$ is either empty or zero,
(ii) $V_p(\phi) < \varepsilon$ on $G$ when $\phi \neq 0$,
(iii) $V_p(0) = \varepsilon$ and $V_p(\phi) = \varepsilon$ when $\phi = \partial G$, where $\partial G$ is the boundary of $G$.

Suppose, in addition, Assumption 2.1 is satisfied for this $p$. Then given $\phi \in G \cap \mathscr{B}_\rho$, for any $\rho > 0$, and $\phi \neq 0$, the solution $x_t(\phi, \sigma)$ intersects $\partial\mathscr{B}_\rho$ in finite time, provided that $\sigma \in \mathscr{S}(\tau, p)$ and $0 \in \mathscr{I}_p$.

*Proof.* For any $\rho > 0$, let $\phi \in G \cap \mathscr{B}_\rho$ and $\phi \neq 0$. Conditions on $V_p$ imply that $V_p$ is non-increasing on $\mathscr{I}_p$. Moreover, $0 \in \mathscr{I}_p$ and hence

$$V_p(x_t(\phi, \sigma)) \leqslant V_p(\phi) < \varepsilon, \tag{4.4}$$

as long as $t \in \mathscr{I}_p$ and $x_t(\phi, \sigma)$ remains in $G \cap \mathscr{B}_\rho$. If $x_t(\phi, \sigma)$ remains in the bounded set $G \cap \mathscr{B}_\rho$ for all $t \in \mathscr{I}_p$, then, by Theorem 4.1, $x(\phi, \sigma)$ weakly approaches $M_p(\tau) = \{0\}$ in the $p$th mode. However, $V_p(0) = \varepsilon$. This contradicts (4.4). Therefore, there exists some $s > 0$ (not necessarily $s \in \mathscr{I}_p$) such that $x_s(\phi, \sigma) \in \partial(G \cap \mathscr{B}_\rho)$. Since $V_p(\phi) = \varepsilon$ on $\partial G$, we must have $x_s(\phi, \sigma) \in \partial\mathscr{B}_\rho$ and the theorem is proved. □

## 5. Multiple Lyapunov–Razumikhin functions

As pointed out in the introduction, Haddock & Terjéki (1983) extended the work of Hale (1965) by using Lyapunov–Razumikhin function. One of the major advantages of Haddock and Terjéki's results is that a Lyapunov–Razumikhin function can be easier to find than a Lyapunov functional from a practical point of view. In this section, we incorporate the idea of Haddock & Terjéki (1983) and establish an invariance principle for switched delay systems using MLRFs. The results in this section are based on the results established in the previous sections using MLFs and, therefore, can be seen as a unification of the results of Haddock & Terjéki (1983) and Hale (1965) now in the hybrid switching setting.

Some notations for using Lyapunov–Razumikhin functions are introduced in the following, which are parallel to those for MLFs.

DEFINITION 5.1 Let $V$ be a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$, $\phi$ a function in $\mathscr{C}$ and $\chi$ a solution piece of (2.1.$p$) on $[\alpha, \beta]$, with $\alpha \leqslant 0 \leqslant \beta$ and $\beta - \alpha > 0$, and $\chi_0 = \phi$. If $\beta > 0$, define the upper right-hand derivative of $V$ with respect to the $p$th mode of (2.1) to be

$$\underbrace{D^+V(\phi(0))}_{(2.1.p)} := \limsup_{h\to 0^+} \frac{1}{h}[V(\chi_h(\phi)(0)) - V(\phi(0))]. \tag{5.1}$$

If $\alpha < 0$, define the upper left-hand derivative of $V$ with respect to the $p$th mode of (2.1) to be

$$\underbrace{D^-V(\phi(0))}_{(2.1.p)} := \limsup_{h\to 0^-} \frac{1}{h}[V(\chi_h(\phi)(0)) - V(\phi(0))]. \tag{5.2}$$

REMARK 5.1 If $V(x)$ is continuously differentiable, then (5.1) and (5.2) reduce to

$$\underbrace{D^+V(\phi(0))}_{(2.1.p)} = \underbrace{D^-V(\phi(0))}_{(2.1.p)} = \underbrace{DV(\phi(0))}_{(2.1.p)} := \nabla V(\phi(0)) \cdot f_p(\phi),$$

where $\nabla$ is the gradient.

DEFINITION 5.2 A family of continuous functions $\{V_p\colon p \in \mathscr{P}\}$ from $\mathbb{R}^n$ to $\mathbb{R}$ are called MLRFs for system (2.1) on a set $G \subset \mathscr{C}$ if, for each $p \in \mathscr{P}$, $V_p$ is continuous on $\mathbb{R}^n$ and, for all $\phi \in G$ such that $V_p(\phi(0)) = \max_{-r\leqslant s\leqslant 0} V_p(\phi(s))$, we have $\underbrace{D^+V_p(\phi(0))}_{(2.1.p)} \leqslant 0$ .

The following proposition bridges the notions of MLFs and MLRFs.

PROPOSITION 5.1 Let $\{V_p\colon p \in \mathscr{P}\}$ be a family of MLRFs for (2.1) on $G$. Then the family of functionals $\{\overline{V}_p\colon p \in \mathscr{P}\}$, defined by

$$\overline{V}_p(\phi) := \max_{-r\leqslant s\leqslant 0} V_p(\phi(s)), \tag{5.3}$$

is a family of MLFs for (2.1) on $G$.

*Proof.* It can be shown that

$$\underbrace{D^+\overline{V}_p(\phi)}_{(2.1.p)} \leqslant 0$$

for each $\phi \in G$ and $p \in \mathscr{P}$ (see, e.g., Hale & Lunel, 1993, p. 152). Therefore, $\{\overline{V}_p\colon p \in \mathscr{P}\}$ form a family of MLFs for (2.1) on $G$. □

Let $\{V_p\colon p \in \mathscr{P}\}$ be a family of MLRFs for (2.1) on $G \subset \mathscr{C}$. The notation $\overline{V}_p$ refers exclusively to the one defined in (5.3). For some $\tau > 0$, define $E_p$ to be the set of all $\phi \in \overline{G}$ such that there exists a solution piece $\chi$ of (2.1.$p$) on some interval $[\alpha, \beta]$, with $\beta - \alpha > 0$ and $\alpha \leqslant 0 \leqslant \beta$, satisfying $\chi_0 = \phi$ and $\overline{V}_p(\chi_t) \equiv c$, some constant, on $[\alpha, \beta]$. Let $M_p(\tau)$ be the largest set in $E_p$ that is weakly $\tau$-invariant with respect to the $p$th mode of (2.1).

The following results follow immediately from those in Section 4.

THEOREM 5.1 (Weak invariance). Let $\{V_p\colon p \in \mathscr{P}\}$ be a family of MLRFs for (2.1) on $G$, $x(\phi, \sigma)$ be a solution of (2.1) such that $x_t(\phi, \sigma)$ remains in $G$ for $t \geqslant 0$, $p \in \mathscr{P}$ be a $\tau$-persistent mode of $\sigma$, and $x(\phi, \sigma)$ is bounded in the $p$th mode. Suppose, in addition, Assumption 2.1 is satisfied for $\overline{V}_p$ with this $p$. Then $x(\phi, \sigma)$ weakly approaches $M_p$ in the $p$th mode as $t \to \infty$.

PROPOSITION 5.2 (Local stability). Suppose

(i) $\{V_p\colon p \in \mathscr{P}\}$ is a family of MLRFs for (2.1) on $\mathscr{B}_\rho$ for some $\rho > 0$ satisfying

$$u(|x|) \leqslant V_p(x) \leqslant v(|x|), \quad \forall\, |x| < \rho, \quad p \in \mathscr{P},$$

where $u$ and $v$ are two strictly increasing functions from $[0, \infty)$ to $[0, \infty)$ with $u(0) = v(0) = 0$ and

(ii) Assumption 2.1 is satisfied for $\overline{V}_p$ with all $p \in p$.

Then the trivial solution $x = 0$ of system (2.1) is stable.

*Proof.* It is easy to see that

$$u(|\phi(0)|) \leqslant \overline{V}_p(\phi) \leqslant v(\|\phi\|), \quad \forall\, \phi \in \mathscr{B}_\rho, \quad p \in \mathscr{P}.$$

The conclusion follows from Proposition 4.2. □

THEOREM 5.2 (Asymptotic stability). Suppose the conditions of Proposition 5.2 hold. Then the trivial solution $x = 0$ of system (2.1) is asymptotically stable w.r.t. $\mathscr{S}(\tau, p)$, provided that $\tau \geqslant r$ and $M_p(\tau) \subset \mathscr{C}_0$. If $\tau \geqslant r$ and $\bigcup_{p\in\mathscr{P}} M_p(\tau) \subset \mathscr{C}_0$, then the trivial solution $x = 0$ of system (2.1) is asymptotically stable w.r.t. $\mathscr{S}(\tau)$.

THEOREM 5.3 (Global asymptotic stability). Suppose

(i) $\{V_p\colon p \in \mathscr{P}\}$ is a family of MLRFs for (2.1) on $\mathscr{C}$ satisfying

$$u(|x|) \leqslant V_p(x) \leqslant v(|x|), \quad \forall\, x \in \mathbb{R}^n, \quad p \in \mathscr{P},$$

where $u$ and $v$ are two strictly increasing functions from $[0, \infty)$ to $[0, \infty)$ with $u(0) = v(0) = 0$ and $u(s) \to \infty$ as $s \to \infty$ and

(ii) Assumption 2.1 is satisfied for $\overline{V}_p$ with all $p \in \mathscr{P}$.

Then the trivial solution $x = 0$ of system (2.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau, p)$, provided that $\tau \geqslant r$ and $M_p(\tau) \subset \mathscr{C}_0$. If $\bigcup_{p\in\mathscr{P}} M_p(\tau) \subset \mathscr{C}_0$, the trivial solution $x = 0$ of system (2.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau)$.

COROLLARY 5.1 (Global asymptotic stability). Suppose the conditions of Theorem 5.3 are satisfied. If for some $p \in \mathscr{P}$, $\underbrace{D^- V_p(\phi(0))}_{(2.1.p)} < 0$ for all $\phi \in \mathscr{C}$ such that $|\phi(0)| \neq 0$ and $V_p(\phi(0)) = \max_{-r \leqslant s \leqslant 0} V_p(\phi(s))$, then the trivial solution $x = 0$ of system (2.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau, p)$ for any $\tau > r$. If $\underbrace{D^- V_p(\phi(0))}_{(2.1.p)} < 0$ for all $p \in \mathscr{P}$ and all $\phi \in \mathscr{C}$ such that $|\phi(0)| \neq 0$ and $V_p(\phi(0)) = \max_{-r \leqslant s \leqslant 0} V_p(\phi(s))$, then the trivial solution $x = 0$ of system (2.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau)$ for any $\tau > r$.

REMARK 5.2 It is pointed out that, compared to Corollary 4.1, Corollary 5.1 requires a slightly stronger weak dwell-time condition $\tau > r$ and a Razumikhin-type condition only on the upper left-hand derivative of $V_p$.

*Proof.* To show that $M_p(\tau) = \{0\}$, we pick any $\psi \in M_p$ and assume that $\psi \neq 0$. There exists a solution piece $\chi$ of (2.1.$p$) on $[\alpha, \beta]$ with $\beta - \alpha \geqslant \tau > r$ such that $\chi_0 = \psi$ and $\chi_t \in M_p(\tau)$ for all $t \in [\alpha, \beta]$. By the definition of $M_p(\tau)$, $\overline{V}_p(\chi_t)$ is constant on $[\alpha, \beta]$. Then there must exist some $\theta \in (\alpha, \beta]$ such that $V_p(\chi_\theta(0)) = \max_{-r \leqslant s \leqslant 0} V_p(\chi_\theta(s))$. By the conditions on $V_p$, this would imply $\underbrace{D^- V_p(\chi_\theta(0))}_{(2.1.p)} < 0$. However, as $V_p(\chi_t(0))$ attains a local maximum at $t = \theta$, we have $\underbrace{D^- V_p(\chi_\theta(0))}_{(2.1.p)} \geqslant 0$, which is a contradiction. Therefore, $M_p = \{0\}$ and the conclusions of the corollary follow from Theorem 5.3. □

THEOREM 5.4 (Instability). Let $G$ be a subset of $\mathscr{C}$ such that $0 \in \overline{G}$. Assume that there exist a family of MLRFs $\{V_p \colon p \in \mathscr{P}\}$ for (2.1) on $G$ such that, for some $p \in \mathscr{P}$,

(i) $M_p(\tau) \cap G$ is either empty or zero,

(ii) $\overline{V}_p(\phi) < \varepsilon$ on $G$ when $\phi \neq 0$ and

(iii) $V_p(0) = \varepsilon$ and $\overline{V}_p(\phi) = \varepsilon$ when $\phi = \partial G$, where $\partial G$ is the boundary of $G$.

Suppose, in addition, Assumption 2.1 is satisfied for $\overline{V}_p$ with this $p$. Then given $\phi \in G \cap \mathscr{B}_p$, for any $\rho > 0$, and $\phi \neq 0$, the solution $x_t(\phi, \sigma)$ intersects $\partial \mathscr{B}_\rho$ in finite time, provided that $\sigma \in \mathscr{S}(\tau, p)$ and $0 \in \mathscr{S}(\tau, p)$.

## 6. Examples

EXAMPLE 6.1 [1]Consider the system in $R^1$

$$\dot{x}(t) = a_\sigma x^3(t) + b_\sigma x^3(t - r), \tag{6.1}$$

where $\sigma \colon [0, \infty) \to \mathscr{P}$ is a switching signal. Assume $a_p \neq 0$ for all $p \in \mathscr{P}$.

### 6.1 *MLF approach*

Define

$$V_p(\phi) = -\frac{\phi^4(0)}{2a_p} + \int_{-r}^{0} \phi^6(s)\mathrm{d}s$$

[1]The example is modified from Example 3.2 in Chapter 5 of Hale & Lunel (1993), which was originally due to LaSalle (1976).

for each $p \in \mathscr{P}$. Let $\chi$ be a solution piece of (2.1.$p$) on $[\alpha, \beta]$ with $\chi_0 = \phi$. Then

$$V_p(\chi_t) = -\frac{\chi^4(t)}{2a_p} + \int_{t-r}^{t} \chi^6(s)\mathrm{d}s, \quad \forall t \in [\alpha, \beta].$$

Differentiating $V_p(\chi_t)$ at $t = 0$ (assume $\alpha < 0$ and $\beta > 0$) gives

$$\underbrace{D^+V_p(\phi)}_{(6.1.p)} = \underbrace{D^-V_p(\phi)}_{(6.1.p)} = -\left[\phi^6(0) + \frac{2b_p}{a_p}\phi^3(0)\phi^3(-r) + \phi^6(-r)\right]. \tag{6.2}$$

Hence $\{V_p\colon p \in \mathscr{P}\}$ forms a family of MLFs for (6.1) on $\mathscr{C}$ if $|b_p| \leqslant |a_p|$ for all $p \in \mathscr{P}$. Moreover,

(1) if $a_p < 0$ and $|b_p| \leqslant |a_p|$ for all $p \in \mathscr{P}$, then condition (i) of Theorem 4.3 is satisfied. Assume that Assumption 2.1 is satisfied for all $p \in \mathscr{P}$.

   (i) If $|b_p| < |a_p|$ for some $p \in \mathscr{P}$, then, in view of Proposition 4.1 and (6.2), $E_p \subset \{\phi \in \mathscr{C}\colon \phi(0) = \phi(-r) = 0\} \subset \mathscr{C}_0$. Theorem 4.3 implies that the solution $x = 0$ of (6.1) is globally asymptotically w.r.t. $\mathscr{S}(\tau, p)$ for $\tau \geqslant r$.[2]

   (ii) If $b_p = a_p$ for some $p \in \mathscr{P}$, then, in view of Proposition 4.1 and (6.2), $E_p \subset \{\phi \in \mathscr{C}\colon \phi(0) = -\phi(-r)\}$. Choose any $\psi \in M_p(\tau)$. Since $M_p(\tau)$ is weakly $\tau$-invariant w.r.t. the $p$th mode of (6.1) and the $M_p(\tau) \subset E_p$. There exists a solution piece $\chi$ of (2.1.$p$) on $[\alpha, \beta]$, with $0 \in [\alpha, \beta]$ and $\beta - \alpha \geqslant \tau$, such that $\chi_0 = \psi$ and $\chi_t \in M_p(\tau) \subset E_p \subset \{\phi \in \mathscr{C}\colon \phi(0) = -\phi(-r)\}$ for all $t \in [\alpha, \beta]$, i.e., $\chi(t) = -\chi(t - r)$ for all $t \in [\alpha, \beta]$, which implies $\dot{x}(t) \equiv 0$ on $[\alpha, \beta]$ and hence $\chi(t) \equiv c$, some constant, on $[\alpha, \beta]$. If $\tau \geqslant r$, $\chi(t) = -\chi(t - r)$ implies $c = 0$ and hence $\psi = 0$. Therefore, $M_p(\tau) = \{0\}$. Theorem 4.3 implies that $x = 0$ is globally asymptotically stable w.r.t. $\mathscr{S}(\tau, p)$ for $\tau \geqslant r$.

   (iii) If, for each $p \in \mathscr{P}$, either $|b_p| < |a_p|$ or $b_p = a_p$, then, summarizing (i) and (ii) above, the solution $x = 0$ of (6.1) is globally asymptotically w.r.t. $\mathscr{S}(\tau)$ for $\tau \geqslant r$.

(2) If $a_p > 0$ and $|b_p| < a_p$ (or $b_p = a_p$) for some $p \in \mathscr{P}$, then the set $G = \{\phi \in \mathscr{C}\colon V_p(\phi) < 0\}$ is non-empty. As before, if $\tau \geqslant r$, we can show that $M_p(\tau) = \{0\}$. Choosing $\varepsilon = 0$ in Theorem 4.4 shows that $x = 0$ is unstable w.r.t. $\mathscr{S}(\tau, p)$ for $\tau \geqslant r$, provided that Assumption 2.1 is satisfied for this $p$.

### 6.2 *CLF approach*

Define

$$V(\phi) = -\frac{\phi^4(0)}{2a} + \int_{-r}^{0} \phi^6(s)\mathrm{d}s.$$

Then

$$\underbrace{D^+V(\phi)}_{(6.1.p)} = \underbrace{D^-V(\phi)}_{(6.1.p)} = -\left[\left(\frac{2a_p}{a} - 1\right)\phi^6(0) + \frac{2b_p}{a}\phi^3(0)\phi^3(-r) + \phi^6(-r)\right]. \tag{6.3}$$

[2] With $\tau \geqslant r$, we can also directly show that $M_p(\tau) = \{0\}$ as in case (ii) below.

If $|b_p| \leqslant |a|$ and $a_p \geqslant a$ for all $p \in \mathscr{P}$, then $V$ is a CLF for (6.1) on $\mathscr{C}$. Assumption 2.1 is trivially satisfied if we choose $V_p = V$ for each $p \in \mathscr{P}$. Using the same argument as in the case of MLFs, we can show that, if $a_p < 0$ for all $p \in \mathscr{P}$ and, for each $p \in \mathscr{P}$, either $|b_p| < |a|$ or $b_p = a$, where $a = \max\{a_p \colon p \in \mathscr{P}\}$, then the trivial solution $x = 0$ of (6.1) is globally asymptotically w.r.t. $\mathscr{S}(\tau)$ for $\tau \geqslant r$.

REMARK 6.1 As we can see, getting rid of Assumption 2.1 by a CLF approach, more severe conditions on the coefficients $a_p$ and $b_p$ are required.

### 6.3 *CLRF approach*

(1) If $a_p < 0$ and $|b_p| \leqslant |a_p|$ for all $p \in \mathscr{P}$, define

$$V_p(x) = V(x) = \frac{x^2}{2}$$

for all $p \in \mathscr{P}$. Then

$$\underbrace{D^+V(\phi(0))}_{(6.1.p)} = \underbrace{D^-V(\phi(0))}_{(6.1.p)} = a_p\phi^4(0) + b_p\phi(0)\phi^3(-r),$$

and, clearly, $V$ is a CLRF for (6.1) on $\mathscr{C}$. Assumption 2.1 is trivially satisfied for all $p \in \mathscr{P}$. If $|b_p| < |a_p|$ for some $p \in \mathscr{P}$, then by Corollary 5.1, the trivial solution $x = 0$ of (6.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau, p)$ for any $\tau > r$. If $|b_p| < |a_p|$ for all $p \in \mathscr{P}$, then the trivial solution $x = 0$ of (6.1) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau)$ for any $\tau > r$.

(2) If $a_p > 0$ and $|b_p| < a_p$ (or $b_p = a_p$) for some $p \in \mathscr{P}$, define

$$V_p(x) = -\frac{x^2}{2},$$

and $V_{p'}(x) \equiv 0$ for all $p' \in \mathscr{P}$ such that $p' \neq p$. Then

$$\underbrace{D^+V(\phi(0))}_{(6.1.p)} = \underbrace{D^-V(\phi(0))}_{(6.1.p)} = -a_p\phi^4(0) - b_p\phi(0)\phi^3(-r),$$

and, clearly, $\{V_p \colon p \in \mathscr{P}\}$ is a family of MLRFs for (6.1) on $\mathscr{C}$. The set $G = \{\phi \in \mathscr{C} \colon \overline{V}_p(\phi) < 0\}$ is non-empty. If $\tau > r$, as in the proof for Corollary 5.1, we can show that $M_p(\tau) = \{0\}$. Choosing $\varepsilon = 0$ in Theorem 5.4 shows that $x = 0$ is unstable w.r.t. $\mathscr{S}(\tau, p)$ for $\tau > r$, provided that Assumption 2.1 is satisfied for $\overline{V}_p$ with this $p$.

REMARK 6.2 By using a CLRF, not only do we get rid of Assumption 2.1, we also get less conservative stability conditions on the coefficients $a_p$ and $b_p$ compared to the CLF approach.

EXAMPLE 6.2 [3]Consider the system

$$A_\sigma \ddot{x}(t) + B_\sigma x(t) = \int_0^r F_\sigma(\theta)x(t-\theta)\mathrm{d}\theta, \tag{6.4}$$

[3]This example is modified from Example 3.11 in Chapter 5 of Hale & Lunel (1993).

where $\sigma\colon [0,\infty) \to \mathscr{P}$ is a switching signal. For each $p \in \mathscr{P}$, $A_p$, $B_p$ and $F_p$ are symmetric $n \times n$ matrices and $F_p$ is continuously differentiable. Let

$$H_p = B_p - \int_0^r F_p(\theta)\mathrm{d}\theta, \quad p \in \mathscr{P},$$

and write system (6.4) as

$$\begin{cases} \dot{x}(t) = y(t), \\ A_\sigma \dot{y}(t) = -H_\sigma x(t) + \int_0^r F_\sigma(\theta)[x(t-\theta) - x(t)]\mathrm{d}\theta. \end{cases} \tag{6.5}$$

The stability analysis of (6.5) is summarized in the following theorem, which generalizes Theorem 3.5 in Chapter 5 of Hale & Lunel (1993) to the hybrid setting.

THEOREM 6.1

(i) If $A_p > 0$, $H_p > 0$, $F_p(\theta) \geqslant 0$ on $[0,r]$, $\dot{F}_p(\theta) \leqslant 0$ on $[0,r]$, for all $p \in \mathscr{P}$, and there exists a $\theta_0$ in $[0,r]$ and some $p_0 \in \mathscr{P}$ such that $\dot{F}_{p_0}(\theta_0) < 0$, then the trivial solution $(0,0)$ of (6.5) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau, p_0)$ with $\tau \geqslant r$. If, for each $p \in \mathscr{P}$, there exists a $\theta_0$ in $[0,r]$ such that $\dot{F}_p(\theta_0) < 0$, then the trivial solution $(0,0)$ of (6.5) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau)$ with $\tau \geqslant r$.

(ii) If $A_p > 0$, $H_p > 0$, $\dot{F}_p \equiv 0$ and $F_p > 0$ for all $p \in \mathscr{P}$, then all solutions of (6.5) are bounded and the $\tau$-persistent limit set of any solution in the $p_0$th mode must be generated by $r$-periodic solution pieces of the ordinary system

$$\dot{x} = y, \quad A_{p_0}\dot{y} = -B_{p_0}x. \tag{6.6}$$

(iii) If $A_p > 0$, $H_p < 0$, $F_p(r) \geqslant 0$, $\dot{F}_p \leqslant 0$ on $[0,r]$, for all $p \in \mathscr{P}$, and there exists a $\theta_0$ and some $p_0 \in \mathscr{P}$ such that $\dot{F}_{p_0}(\theta_0) < 0$, then the trivial solution $(0,0)$ of (6.5) is unstable w.r.t. $\sigma \in \mathscr{S}(r, p_0)$.

*Proof.* Let $\phi$, $\psi$ be the initial values for $x$, $y$ in (6.5) and define, for each $p \in \mathscr{P}$,

$$V_p(\phi,\psi) = \frac{1}{2}\phi(0)^T H_p \phi(0) + \frac{1}{2}\psi(0)^T A_p \psi(0) + \frac{1}{2}\int_0^r [\phi(-\theta) - \phi(0)]^T F_p(\theta)[\phi(-\theta) - \phi(0)]\mathrm{d}\theta.$$

It is easy to show that

$$\begin{aligned} \underbrace{D^+V_p(\phi,\psi)}_{(6.5.p)} = \underbrace{D^-V_p(\phi,\psi)}_{(6.5.p)} &= -\frac{1}{2}[\phi(-r) - \phi(0)]^T F_p(r)[\phi(-r) - \phi(0)] \\ &\quad + \frac{1}{2}\int_0^r [\phi(-\theta) - \phi(0)]^T \dot{F}_p(\theta)[\phi(-\theta) - \phi(0)]\mathrm{d}\theta \\ &\leqslant 0, \quad \forall p \in \mathscr{P}, \end{aligned} \tag{6.7}$$

if the conditions in (i), (ii) or (iii) are satisfied. If either (i) or (iii) is satisfied, then there exists some subinterval of $[0, r]$, say $I_{\theta_0}$, such that $\dot{F}_{p_0}(\theta) < 0$ for $\theta \in I_{\theta_0}$. From (6.7), either $\underbrace{D^+ V_{p_0}(\phi, \psi)}_{(6.5.p_0)} = 0$ or $\underbrace{D^- V_{p_0}(\phi, \psi)}_{(6.5.p_0)} = 0$ implies that $\phi(\theta) - \phi(0) = 0$ for $\theta \in I_{\theta_0}$. From Proposition 4.1, $E_{p_0} \subset \{(\phi, \psi) \colon \phi(\theta) = \phi(0), \forall \theta \in I_{\theta_0}\}$. Now choose any $(\phi, \psi) \in M_{p_0}(\tau)$. Let $(x, y)$ be a solution piece of $(6.5.p_0)$ on $[\alpha, \beta]$, with $0 \in [\alpha, \beta]$ and $\beta - \alpha \geqslant \tau$, such that $(x_0, y_0) = (\phi, \psi)$ and $(x_t, y_t) \in M_{p_0}(\tau)$ for all $t \in [\alpha, \beta]$. It follows that $x(t - \theta) = x(t)$ for all $t \in [\alpha, \beta]$ and $\theta \in I_{\theta_0}$. Therefore, $x(t) \equiv c$, a constant, on $[\alpha, \beta]$. From (6.5), this implies that $y(t) = 0$ on $[\alpha, \beta]$ and hence $H_{p_0} c = 0$. Either $H_{p_0} < 0$ or $H_{p_0} < 0$ in (i), (ii) or (iii). We must have $c = 0$. Therefore, $M_{p_0}(\tau) \subset \mathscr{C}_0$.

(1) If conditions in (i) are satisfied, Theorem 4.3 implies that the trivial solution $(0, 0)$ of (6.5) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau, p_0)$ with $\tau \geqslant r$. If, for each $p \in \mathscr{P}$, there exists a $\theta_0$ such that $\dot{F}_p(\theta_0) < 0$, then the trivial solution $(0, 0)$ of (6.5) is globally asymptotically stable w.r.t. $\mathscr{S}(\tau)$ with $\tau \geqslant r$.

(2) If conditions in (iii) are satisfied, then, by defining $G = \{V_{p_0}(\phi, \psi) < 0\}$, Theorem 4.4 implies that $(0, 0)$ is unstable.

(3) If conditions in (ii) are satisfied, then $\dot{F}_{p_0} \equiv 0$ and either $\underbrace{D^+ V_{p_0}(\phi, \psi)}_{(6.5.p_0)} = 0$ or $\underbrace{D^- V_{p_0}(\phi, \psi)}_{(6.5.p_0)} = 0$ would imply that $\phi(-r) = \phi(0)$. From Proposition 4.1, $E_{p_0} \subset \{(\phi, \psi) \colon \phi(-r) = \phi(0)\}$. Now choose any $(\phi, \psi) \in M_{p_0}(\tau)$ and let $(x, y)$ be a solution piece of $(6.5.p_0)$ on $[\alpha, \beta]$, with $0 \in [\alpha, \beta]$ and $\beta - \alpha \geqslant \tau$, such that $(x_0, y_0) = (\phi, \psi)$ and $(x_t, y_t) \in M_{p_0}(\tau)$ for all $t \in [\alpha, \beta]$. It follows that $x(t)$ satisfies $x(t - r) = x(t)$ for all $t \in [\alpha, \beta]$ and so does $y(t)$. The fact that $x(t - r) = x(t)$ for all $t \in [\alpha, \beta]$ implies that $\int_{t-r}^{t} x(s)\mathrm{d}s \equiv c$, a constant, on $[\alpha, \beta]$. Integrating the second equation of (6.5), with $\sigma = p_0$, from $t - r$ to $t$, where $t \in [\alpha, \beta]$, gives

$$\begin{aligned} 0 &= A_{p_0} y(t) - A_{p_0} y(t - r) \\ &= -H_{p_0} \int_{t-r}^{t} x(s)\mathrm{d}s + \int_{t-r}^{t} \int_{0}^{r} F_{p_0}[x(s - \theta) - x(s)]\mathrm{d}\theta\, \mathrm{d}s \\ &= -H_{p_0} c. \end{aligned}$$

Then $H_{p_0} > 0$ implies that $c = 0$. Therefore, with $\sigma = p_0$, (6.5) reduces to (6.6) and $(x, y)$ satisfies (6.6) on $[\alpha, \beta]$ and $(x, y)$ is periodic with period $r$.

The proof is complete. □

## 7. Conclusions

The classical LaSalle's principle is extended to switched delay systems with weak dwell-time switchings. This work generalizes the earlier work of Hale (1965) on functional differential equations to the hybrid switching setting, by using the notions of weak $\tau$-invariance, where $\tau > 0$ is a weak dwell-time for the class of switching signals considered in this paper. In proving the weak invariance principle, the notions of persistent mode and persistent limit play important roles. Stability and instability criteria are established, using MLFs, for switched delay systems, as important applications of the invariance principle obtained. Moreover, the idea of Haddock & Terjéki (1983) is also incorporated in this paper. Based on results obtained using MLFs, an invariance principle and several stability and instability criteria by MLRFs are obtained, which reveal a connection between the MLRF approach and the MLF approach.

The results developed in this paper can be applied to stability analysis of various practical models involving both switching and time delays. Applications of these results can provide interesting topics for future research. One particularly interesting example would be using the generalized invariance principles developed in this paper for switched delay systems to study the stability of both the disease-free solutions and the endemic solutions of delayed infectious disease models with switching parameters.

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