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Impulsive stabilization of stochastic functional differential equations☆

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ABSTRACT

This paper investigates impulsive stabilization of stochastic delay differential equations. Both moment and almost sure exponential stability criteria are established using the Lyapunov–Razumikhin method. It is shown that an unstable stochastic delay system can be successfully stabilized by impulses. The results can be easily applied to stochastic systems with arbitrarily large delays. An example with its numerical simulation is presented to illustrate the main results.

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1. Introduction

In the past decades, impulsive control and stabilization has been shown to be a powerful tool in the theory and applications of nonlinear dynamical systems. As the area of impulsive delay differential equations is still an interesting and challenging theoretical subject, yet to be fully developed [1,2], impulsive control and stabilization has gained increasing popularity [3–8] and found successful applications in a wide variety of areas, such as control systems [9,10], control and synchronization of chaotic systems [11–15], complex dynamical networks [16], large-scale dynamical systems [17], secure communication [11], spacecraft maneuvers [18], population growth and biological systems [19,20], neural networks [13,21], ecosystem management [22], and pharmacokinetics [23].

Recently, the impulsive control theory has been generalized from deterministic systems to stochastic systems and has been shown to have applications in chaos control, chaos synchronization, and neural networks [13]. However, to the best of the authors' knowledge, there are almost no studies on impulsive stabilization of stochastic delay systems. To fill this gap, in this letter, stability criteria for impulsive stochastic functional differential equations are established by the Razumikhin technique and Lyapunov functions. It is shown that an unstable stochastic delay system can be successfully stabilized by impulses and the results can be easily applied to stochastic systems with arbitrarily large delays.

2. Preliminaries

Let $\mathbb{Z}^+$ denote the set of all positive integers, $\mathbb{R}^n$ the n-dimensional real Euclidean space, and $\mathbb{R}^{n\times m}$ the space of $n \times m$ real matrices. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. For $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, define $|A| := \sqrt{\mathrm{trace}(A^T A)} =$

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$\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^2}$ and $\|A\| := \sqrt{\lambda_{\max}(A^T A)}$, i.e. $|A|$ and $\|A\|$ denote the Frobenius norm and the spectral norm of the matrix A, respectively.

For $-\infty < a < b < \infty$, we say that a function from $[a, b]$ to $\mathbb{R}^n$ is *piecewise continuous*, if the function has at most a finite number of jump discontinuities on $(a, b]$ and are continuous from the right for all points in $[a, b)$. Given $r > 0$, $\mathcal{PC}([-r, 0]; \mathbb{R}^n)$ denotes the family of piecewise continuous functions from $[-r, 0]$ to $\mathbb{R}^n$. A norm on $\mathcal{PC}([-r, 0]; \mathbb{R}^n)$ is defined as $\|\phi\| := \sup_{-r\le s\le 0}|\phi(s)|$ for $\phi \in \mathcal{PC}([-r, 0]; \mathbb{R}^n)$. For simplicity, $\mathcal{PC}$ is used for $\mathcal{PC}([-r, 0]; \mathbb{R}^n)$ for the rest of this paper.

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space with $\{\mathcal{F}_t\}_{t\ge 0}$ as a filtration satisfying the usual conditions and $W(t)$ be an m-dimensional standard Wiener process defined on $(\Omega, \mathcal{F}, P)$ and adapted to $\{\mathcal{F}_t\}_{t\ge 0}$. For $p > 0$ and $t \ge 0$, let $\mathcal{L}^p_{\mathcal{F}_t}$ denote the family of all $\mathcal{F}_t$-measurable $\mathcal{PC}$-valued random variables ϕ such that $\mathbb{E}(\|\phi\|^p) < \infty$. Let $\mathcal{L}^{\mathrm{b}}_{\mathcal{F}_t}$ be the family of $\mathcal{PC}$-valued random variables that are bounded and $\mathcal{F}_t$-measurable.

Consider the following impulsive stochastic functional differential system:

$$dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), \quad t \ne t_k,\ t \ge t_0, \tag{2.1a}$$

$$\Delta x(t) = I(t, x_{t^-}), \quad t = t_k, \tag{2.1b}$$

$$x_{t_0} = \xi, \tag{2.1c}$$

where $\{t_k : k \in \mathbb{Z}^+\}$ is a strictly increasing sequence such that $t_k \to \infty$ as $k \to \infty$, x_t is defined by $x_t(s) = x(t + s)$, for $-r \le s \le 0$, and can be treated as a $\mathcal{PC}$-valued stochastic process. Similarly, the $\mathcal{PC}$-valued stochastic process x_{t^-} is defined by $x_{t^-}(s) = x(t+s)$, for $s \in [-r, 0)$, and $x_{t^-}(0) = x(t^-)$, where $x(t^-) = \lim_{s\to t^-} x(s)$. Both $f : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{R}^n$, and $g : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{R}^{n\times m}$, are Borel measurable. Moreover, f, g, and I are assumed to satisfy necessary assumptions so that, for any initial data $\xi \in \mathcal{L}^{\mathrm{b}}_{\mathcal{F}_{t_0}}$, system (2.1) has a unique global solution, denoted by $x(t; \xi)$, and, moreover, $x(t; \xi) \in \mathcal{L}^p_{\mathcal{F}_t}$ for all $t \ge t_0$ and $p > 0$ (e.g., see [24] for existence and uniqueness results for general impulsive hybrid stochastic delay systems including (2.1) and see [25,26] for stochastic functional (delay) differential equations without impulses). In addition, it is assumed that $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, and $I(t, 0) \equiv 0$, for all $t \in [0, \infty)$, so that system (2.1) admits a trivial solution.

The main objective of this paper is to find conditions such that, even if the continuous dynamics given by (2.1a) are highly unstable, the impulsive control, introduced by (2.1b), can successfully stabilize system (2.1). Both moment stability and almost sure stability will be considered, which are formulated in the following definition.

Definition 2.1. For $p > 0$, the trivial solution of system (2.1) is said to be *pth moment globally uniformly exponentially stable (g.u.e.s.)*, if, for any initial data $\xi \in \mathcal{L}^{\mathrm{b}}_{\mathcal{F}_{t_0}}$, the solution $x(t; \xi)$ satisfies

$$\mathbb{E}\left(|x(t;\xi)|^p\right) \le C\mathbb{E}\left(\|\xi\|^p\right) e^{-\varepsilon(t-t_0)}, \quad t \ge t_0, \tag{2.2}$$

where ε and C are positive constants independent of t_0. It follows from (2.2) that

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}\left(|x(t;\xi)|^p\right) \le -\varepsilon. \tag{2.3}$$

The left-hand side of (2.3) is called the *pth moment Lyapunov exponent* for the solution. Moreover, define

$$\limsup_{t\to\infty} \frac{1}{t}\log |x(t;\xi)| \tag{2.4}$$

to be the *Lyapunov exponent* of the solution. The trivial solution of system (2.1) is said to be *almost surely exponentially stable* if the Lyapunov exponent is almost surely negative for any $\xi \in \mathcal{L}^{\mathrm{b}}_{\mathcal{F}_{t_0}}$.

Definition 2.2. Let $\mathcal{C}^{1,2}$ denote the set of all functions from $[t_0 - r, \infty) \times \mathbb{R}^n$ to $\mathbb{R}^+$ that are continuously differentiable in t and twice continuously differentiable in x. For each $V \in \mathcal{C}^{1,2}$, define an operator from $\mathbb{R}^+ \times \mathcal{PC}$ to $\mathbb{R}$ by

$$\mathcal{L}V(t,\phi) := V_t(t,\phi(0)) + V_x(t,\phi(0))f(t,\phi) + \frac{1}{2}\mathrm{trace}\left[g^T(t,\phi)V_{xx}(t,\phi(0))g(t,\phi)\right].$$

3. Main results

In this section, we first investigate the pth moment exponential stability of system (2.1). It is shown that, even if the pth moment of a stochastic delay system can be highly unstable, the impulsive control can effectively stabilize the system and achieve pth moment global exponential stability.

Theorem 3.1. *Let Λ, p, c_1, c_2, $\rho < 1$, δ, and $\bar{\mu}$ be positive constants. Suppose that*

(i) *there exists a function $V \in \mathcal{C}^{1,2}$ such that $c_1|x|^p \le V(t, x) \le c_2|x|^p$, for $(t, x) \in [t_0 - r, \infty) \times \mathbb{R}^n$,*

(ii) *there exists a nonnegative and piecewise continuous function $\mu : [t_0, \infty) \to \mathbb{R}^+$, satisfying $\int_t^{t+\delta}\mu(s)ds \le \bar{\mu}\delta$ for all $t \ge t_0$, such that*

$$\mathbb{E}\left(\mathcal{L}V(t,\phi)\right) \le \mu(t)\mathbb{E}\left(V(t,\phi(0))\right), \tag{3.1}$$

whenever $t \geq t_0$ and $\phi \in \mathcal{L}^p_{\mathcal{F}_t}$ are such that $\mathbb{E}(V(t+s, \phi(s))) \leq q\mathbb{E}(V(t, \phi(0)))$, for all $s \in [-r, 0]$, where q is a constant such that $q \geq \exp(\Lambda r + \Lambda\delta + \bar{\mu}\delta)$,

(iii) *there exist positive constants d_k, with $\prod_{k=1}^{\infty} d_k < \infty$, such that*

$$\mathbb{E}(V(t_k, \phi(0) + I(t_k, \phi))) \leq \rho d_k \mathbb{E}\left(V(t_k^-, \phi(0))\right), \tag{3.2}$$

for $k \in \mathbb{Z}^+$ and $\phi \in \mathcal{L}^p_{\mathcal{F}_t}$, and

(iv) $\sup_{k\in\mathbb{Z}^+}\{t_k - t_{k-1}\} = \delta < -\frac{\ln(\rho)}{\Lambda+\bar{\mu}}$.

Then the trivial solution of system (2.1) is pth moment globally uniformly exponentially stable and its pth moment Lyapunov exponent is not greater than $-\Lambda$.

Proof. Given any initial data $\xi \in \mathcal{L}^b_{\mathcal{F}_{t_0}}$, the global solution $x(t;\xi)$ of (2.1) is written as $x(t)$ in this proof. Without loss of generality, assume that the initial data ξ is nontrivial so that $x(t)$ is not a trivial solution. Let $v(t) = \mathbb{E}(V(t, x(t)))$, for $t \geq t_0 - r$, and $\tilde{\Lambda} = \Lambda - \eta$, where $\eta > 0$ being an arbitrary number such that $\tilde{\Lambda} > 0$. It is clear that $v(t)$ is right-continuous for $t \geq t_0$. Choose $M \in (e^{(\Lambda+\bar{\mu})\delta}, qe^{\Lambda\delta})$ so that

$$\|v_{t_0}\| < M\|v_{t_0}\|e^{-(\Lambda+\bar{\mu})\delta} < M\|v_{t_0}\|e^{-\Lambda\delta} < q\|v_{t_0}\|, \tag{3.3}$$

where $\|v_{t_0}\| = \max_{-r\leq s\leq 0} v(t_0+s)$. We will show that

$$v(t) \leq M\|v_{t_0}\|e^{-\tilde{\Lambda}(t_1-t_0)}, \quad \forall t \in [t_0, t_1), \tag{3.4}$$

by proving a stronger claim:

$$v(t) \leq M\|v_{t_0}\|e^{-\Lambda\delta}, \quad \forall t \in [t_0, t_1). \tag{3.5}$$

Suppose (3.5) is not true and observe that

$$v(t) \leq \|v_{t_0}\| < M\|v_{t_0}\|e^{-\Lambda\delta}, \tag{3.6}$$

holds on $[t_0 - r, t_0]$. Define $t^* = \inf\{t \in [t_0, t_1) : v(t) > M\|v_{t_0}\|e^{-\Lambda\delta}\}$. Then $t^* \in (t_0, t_1)$ and, by continuity of $v(t)$,

$$v(t) \leq v(t^*) = M\|v_{t_0}\|e^{-\Lambda\delta}, \quad \forall t \in [t_0, t^*]. \tag{3.7}$$

In view of (3.6), define $t_* = \sup\{t \in [t_0, t^*) : v(t) \leq \|v_{t_0}\|\}$. Then $t_* \in [t_0, t^*)$ and, by continuity of $v(t)$,

$$v(t) \geq v(t_*) = \|v_{t_0}\|, \quad \forall t \in [t_*, t^*]. \tag{3.8}$$

Now in view of (3.3), (3.7) and (3.8), one has, for $t \in [t_*, t^*]$ and $s \in [-r, 0]$,

$$v(t+s) \leq v(t^*) = M\|v_{t_0}\|e^{-\Lambda\delta} < q\|v_{t_0}\| \leq qv(t).$$

By the Razumikhin-type condition (ii), one has

$$\mathbb{E}(\mathcal{L}V(t, x_t)) \leq \mu(t)\mathbb{E}(V(t, x(t))), \quad \forall t \in [t_*, t^*]. \tag{3.9}$$

Applying Itô's formula on $[t_*, t^*]$ and by (3.9), one obtains that

$$e^{\int_{t_0}^{t^*}\mu(\tau)d\tau}v(t^*) - e^{\int_{t_0}^{t_*}\mu(\tau)d\tau}v(t_*) = \int_{t_*}^{t^*} e^{\int_{t_0}^{s}\mu(\tau)d\tau}\left[\mathbb{E}(\mathcal{L}V(s, x_s)) - \mu(s)\mathbb{E}(V(s, x(s)))\right]ds \leq 0,$$

which implies

$$v(t^*) \leq v(t_*)e^{\int_{t_*}^{t^*}\mu(s)ds} \leq v(t_*)e^{\bar{\mu}\delta}. \tag{3.10}$$

Since (3.10) contradicts what is implied by (3.3), (3.7) and (3.8), claim (3.5) must be true and so is (3.4).

Now, assume that

$$v(t) \leq M_k\|v_{t_0}\|e^{-\tilde{\Lambda}(t_k-t_0)}, \quad \forall t \in [t_{k-1}, t_k), \tag{3.11}$$

for all $k \leq m$, where $k, m \in \mathbb{Z}^+$ and M_k is defined as $M_1 = M$ and $M_k = M\prod_{1\leq l\leq k-1} d_l$, for $k \geq 2$. We proceed to show that

$$v(t) \leq M_{m+1}\|v_{t_0}\|e^{-\tilde{\Lambda}(t_{m+1}-t_0)}, \quad \forall t \in [t_m, t_{m+1}), \tag{3.12}$$

by proving a stronger claim:

$$v(t) \leq M_{m+1}\|v_{t_0}\|e^{-\tilde{\Lambda}[(t_m-t_0)+\delta]}, \quad \forall t \in [t_m, t_{m+1}). \tag{3.13}$$

From (3.11) and (3.2), one has

$$v(t_m) \le \rho d_m v(t_m^-) \le \rho d_m M_m \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}(t_m - t_0)} = \rho M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}(t_m - t_0)},$$

which implies

$$v(t_m) < \mathrm{e}^{-\bar{\mu}\delta} M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}[(t_m - t_0)+\delta]}. \tag{3.14}$$

Choose $\varepsilon > 0$ sufficiently small such that

$$v(t_m) < \mathrm{e}^{-\bar{\mu}\delta-\varepsilon} M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}[(t_m - t_0)+\delta]} < \mathrm{e}^{-\bar{\mu}\delta} M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}[(t_m - t_0)+\delta]}. \tag{3.15}$$

Suppose claim (3.13) is not true. Define

$$\bar{t} = \inf\left\{ t \in [t_m, t_{m+1}) : v(t) > M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}[(t_m - t_0)+\delta]} \right\}.$$

In view of (3.14), one has $\bar{t} \in (t_m, t_{m+1})$ and, by continuity of $v(t)$,

$$v(t) \le v(\bar{t}) = M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}[(t_m - t_0)+\delta]}, \quad \forall t \in [t_m, \bar{t}]. \tag{3.16}$$

In view of (3.15), define

$$\underline{t} = \sup\left\{ t \in [t_m, \bar{t}) : v(t) \le \mathrm{e}^{-\bar{\mu}\delta-\varepsilon} M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}[(t_m - t_0)+\delta]} \right\}.$$

Then $\underline{t} \in (t_m, \bar{t})$ and, by continuity of $v(t)$,

$$v(t) \ge v(\underline{t}) = \mathrm{e}^{-\bar{\mu}\delta-\varepsilon} M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}[(t_m - t_0)+\delta]} = \mathrm{e}^{-\bar{\mu}\delta-\varepsilon} v(\bar{t}), \tag{3.17}$$

for all $t \in [\underline{t}, \bar{t}]$. Now for $t \in [\underline{t}, \bar{t}]$ and $s \in [-r, 0]$, from (3.11) and (3.17), and the fact that $q \ge \exp(\Lambda r + \Lambda\delta + \bar{\mu}\delta)$ and $t + s \in [t_{m-1}, \bar{t}]$, one has

$$v(t+s) \le M_{m+1} \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}(t+s-t_0)} \le \mathrm{e}^{\tilde{\Lambda}r + \tilde{\Lambda}\delta + \bar{\mu}\delta + \varepsilon} v(\underline{t}) \le q v(t),$$

provided that ε defined in (3.15) is chosen sufficiently small. Similar to the argument on $[t_*, t^*]$, an application of Itô's formula on $[\underline{t}, \bar{t}]$ will lead to $v(\bar{t}) \le v(\underline{t})\mathrm{e}^{\bar{\mu}\delta}$, which would contradict (3.17). Therefore, claim (3.13) must be true and so is (3.12). By induction on m and from the definition of M_m, one can conclude that

$$v(t) \le M \prod_{\{k: t_0 < t_k \le t\}} d_k \|v_{t_0}\| \mathrm{e}^{-\tilde{\Lambda}(t-t_0)}, \quad \forall t \ge t_0.$$

By condition (i) and the facts that $\hat{d} = \prod_{k=1}^{\infty} d_k < \infty$ and $\eta > 0$ is arbitrary and independent of t, we actually have shown

$$\mathbb{E}\left(|x(t)|^p\right) \le M\hat{d}\frac{c_2}{c_1}\mathbb{E}\left(\|\xi\|^p\right)\mathrm{e}^{-\Lambda(t-t_0)}, \quad \forall t \ge t_0,$$

which shows that the trivial solution of (2.1) is pth moment g.u.e.s., with its pth moment Lyapunov exponent not greater than $-\Lambda$. □

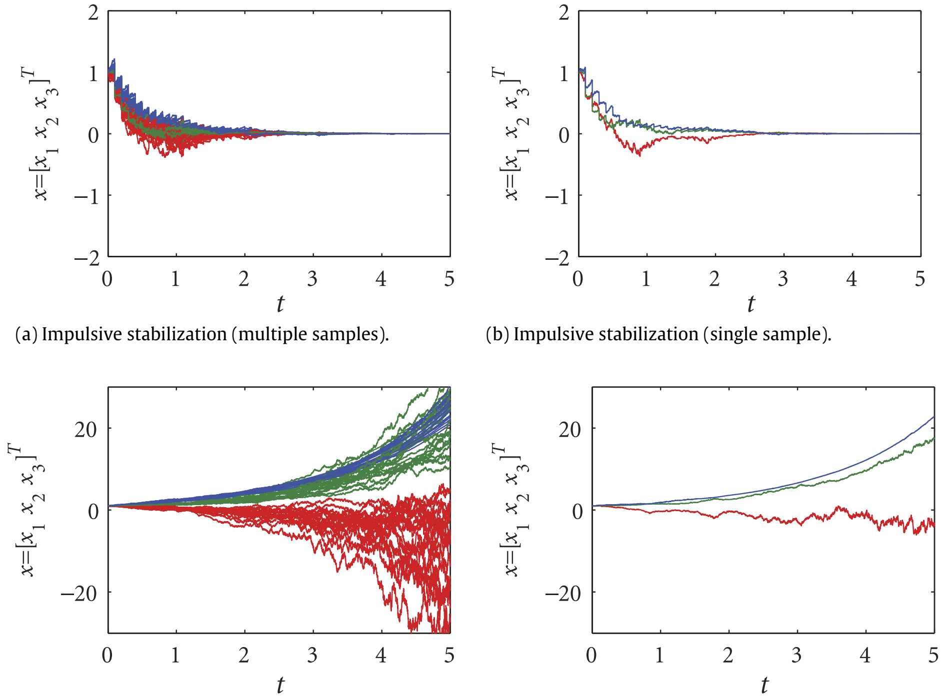
Remark 3.1. It is clear that Theorem 3.2 allows the continuous dynamics of system (2.1) to be unstable, since the function $\mu(t)$ in (3.1), which characterizes the changing rate of $V(t, x(t))$ at t, is assumed to be nonnegative. Theorem 3.1 shows that an unstable stochastic delay system can be successfully stabilized by impulses.

Remark 3.2. The only condition in Theorem 3.1 that involves the delay size r is condition (ii), where q is required to satisfy

$$q \ge \exp(\Lambda r + \Lambda\delta + \bar{\mu}\delta), \tag{3.18}$$

with $\bar{\mu}$ usually being a quantity depending on q due to the Razumikhin-type condition. To apply the theorem, it is required that (3.18) as an inequality of q has at least one solution. When r becomes larger (while Λ can still be arbitrarily chosen but fixed), δ has to become sufficiently small to guarantee that (3.18) has a solution, i.e. the impulse frequency has to reach a certain level to cope with the increase in time-delay, which is reasonable. Based on this observation, Theorem 3.1 can be easily applied to a system with arbitrarily large delays.

Remark 3.3. In condition (3.2), the constants d_k make it possible to tolerate certain perturbations in the overall impulsive stabilization process, i.e. it is not strictly required by Theorem 3.1 that each impulse contributes to stabilize the system, as long as the overall contribution of the impulses are stabilizing. Without these d_k (i.e. $d_k \equiv 1$), it is required that each impulse is a stabilizing factor ($\rho < 1$), which is more restrictive.



(a) Impulsive stabilization (multiple samples). (b) Impulsive stabilization (single sample).

(c) Impulse free (multiple samples). (d) Impulse free (single sample).

Fig. 1. Simulation results for Example 4.1: (a) impulsive stabilization of system (4.1) (multiple samples); (b) impulsive stabilization of system (4.1) (single sample); (c) system response without impulses (multiple samples); (d) system response without impulses (single sample).

The following theorem shows that the trivial solution of system (2.1) is also almost surely exponentially stable, under some additional conditions.

Theorem 3.2. *Suppose that $p \geq 1$ and the same conditions as in Theorem 3.1 hold. Moreover, there exists a constant $K > 0$ such that*

$$\mathbb{E}\left(|f(t,\phi)|^p \vee |g(t,\phi)|^p \vee |I(t,\phi)|^p\right) \leq K \sup_{-r\leq s\leq 0} \mathbb{E}\left(|\phi(s)|^p\right), \tag{3.19}$$

and there exists a positive integer N such that there are at most N impulse times within any interval of length r. Then the trivial solution of system (2.1) is also almost surely exponentially stable, with its Lyapunov exponent not greater than $-\Lambda/p$.

Proof. The detailed proof is omitted due to limited space. The readers can refer to [25, p.175–178] for the proof of a similar result on stochastic functional differential equations without impulses. To deal with the impulse effects, note that, by the theorem condition, there are at most N impulse times within an interval of length r, which, by Hölder's inequality, can give a useful estimate of the impulse effects. □

Now consider the linear impulsive stochastic delay system

$$\begin{cases} \mathrm{d}x(t) = \left[Ax(t) + Bx(t-r(t))\right]\mathrm{d}t + \left[Cx(t) + Dx(t-r(t))\right]\mathrm{d}W(t), & t \neq t_k,\ t > t_0, \\ \Delta x(t) = E_k x(t^-), & t = t_k, \end{cases} \tag{3.20}$$

where A, B, C, D, and E_k are all $n \times n$ matrices and W is a one-dimensional standard Wiener process. A single time-varying delay is given by $r(t)$, which is continuous on $[t_0, \infty)$ and satisfies $0 \leq r(t) \leq r$, for some constant $r > 0$. The initial data is omitted, but it is assumed to be in $\mathcal{L}^{\mathrm{b}}_{\mathcal{F}_{t_0}}$.

Theorem 3.3. *If there exist constants $p \geq 2$, $\Lambda > 0$, and $q > 1$ such that $q \geq \exp(\Lambda r + \Lambda\delta + \mu\delta)$, where $\mu = \kappa + \tilde{\kappa} q > 0$ with*

$$\kappa = \frac{p}{2}\lambda_{\max}(A^T + A) + \frac{p}{2}\|B\| + \frac{p(p-1)}{2}\|C\|^2 + (p-1)^2\|C\|\,\|D\| + \frac{(p-1)(p-2)}{2}\|D\|^2,$$
$$\tilde{\kappa} = \|B\| + (p-1)\|C\|\,\|D\| + (p-1)\|D\|^2,$$

and

$$\log(\|I + E_k\|) < -\frac{1}{2}(\Lambda + \mu)\delta + \ln d_k, \quad \forall k \in \mathbb{Z}^+,$$

where $\delta = \sup_{k\in Z^+}\{t_k - t_{k-1}\}$, $d_k > 0$, and $\prod_{k=1}^{\infty} d_k < \infty$, then the trivial solution of (3.20) is pth moment g.u.e.s., with its pth moment Lyapunov exponent not greater than $-\Lambda$. If, in addition, there are at most N impulse times within any interval of length r, for some fixed integer N, then the trivial solution of system (3.20) is also almost surely exponentially stable, with its Lyapunov exponent not greater than $-\Lambda/p$.

Proof. The conclusions follow from Theorems 3.1 and 3.2 by considering $V(x) = |x|^p$. □

4. An example

Example 4.1. Consider the linear impulsive stochastic delay system

$$\begin{cases} \mathrm{d}x(t) = \big[Ax(t) + Bx(t-1)\big]\mathrm{d}t + \big[Cx(t) + Dx(t-1)\big]\mathrm{d}W(t), & t \neq t_k,\ t > 0, \\ \Delta x(t) = E_k x(t^-), & t = t_k, \end{cases} \tag{4.1}$$

with $t_k = k/10$, $k \in \mathbb{Z}^+$,

$$A = \begin{pmatrix} -0.01 & -0.26 & -0.13 \\ -0.16 & 0.14 & 0.17 \\ 0.05 & 0.32 & 0.25 \end{pmatrix}, \qquad B = \begin{pmatrix} -0.31 & -0.08 & 0.14 \\ -0.28 & 0.13 & 0.25 \\ 0.11 & 0.16 & 0.13 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.23 & -0.03 & -0.21 \\ -0.24 & -0.32 & 0.28 \\ 0 & 0.05 & -0.16 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.1 & -0.43 & 0.12 \\ 0.04 & -0.01 & 0.1 \\ -0.18 & -0.20 & 0.33 \end{pmatrix}, \qquad E_k = \begin{pmatrix} -0.3 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & -0.3 \end{pmatrix}.$$

Choosing $p = 2$, then κ and $\tilde{\kappa}$ in Theorem 3.3 can be computed to be $\kappa = 2.1255$ and $\tilde{\kappa} = 1.1196$. If we further choose $\Lambda = 0.5$ and $\delta = 0.1$, then there exists $q = 2.9992$ such that $q = \exp(\Lambda r + \Lambda\delta + \mu\delta)$, where $\mu = \kappa + \tilde{\kappa}q = 5.4834$. It is easy to verify that $\log(\|I + E_k\|) = -0.3567 < -0.2992 = -(\Lambda + \mu)\delta/2$. Theorem 3.3 guarantees that the trivial solution of system (4.1) is second moment g.u.e.s., with its second moment Lyapunov exponent not greater than $-\Lambda = -0.5$, and also almost surely exponentially stable, with its Lyapunov exponent not greater than $-\Lambda/2 = -0.25$. Numerical simulations for this example are shown in Fig. 1. It is clearly demonstrated that impulses can successfully stabilize an otherwise unstable stochastic delay system.

References

[1] G. Ballinger, X. Liu, Existence and uniqueness results for impulsive delay differential equations, Dynam. Contin. Discrete Impuls. Systems 5 (1999) 579–591.
[2] V. Lakshmikantham, D. Baĭnov, P. Simeonov, Theory of Impulsive Differential Equations, World Scientific Publishing, 1989.
[3] L. Berezansky, E. Braverman, Impulsive stabilization of linear delay differential equations, Dynam. Systems Appl. 5 (1996) 263–276.
[4] X. Liu, Impulsive stabilization of nonlinear systems, IMA J. Math. Control Inform. 10 (1993) 11–19.
[5] Z. Luo, J. Shen, Impulsive stabilization of functional differential equations with infinite delays, Appl. Math. Lett. 16 (2003) 695–701.
[6] J. Shen, Z. Luo, X. Liu, Impulsive stabilization of functional-differential equations via Liapunov functionals, J. Math. Anal. Appl. 240 (1999) 1–15.
[7] Q. Wang, X. Liu, Impulsive stabilization of delay differential systems via the Lyapunov–Razumikhin method, Appl. Math. Lett. 20 (2007) 839–845.
[8] J. Yan, J. Shen, Impulsive stabilization of functional-differential equations by Lyapunov–Razumikhin functions, Nonlinear Anal. Ser. A: TMA 37 (1999) 245–255.
[9] G. Ballinger, X. Liu, Practical stability of impulsive delay differential equations and applications to control problems, in: Optimization Methods and Applications, Kluwer Acad. Publ., Dordrecht, 2001.
[10] X. Liu, Stability of impulsive control systems with time delay, Math. Comput. Modelling 39 (2004) 511–519.
[11] T. Yang, L. Chua, Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication, IEEE Trans. Circuits Syst. I 44 (1997) 976–988.
[12] C. Li, X. Liao, X. Yang, T. Huang, Impulsive stabilization and synchronization of a class of chaotic delay systems, Chaos 15 (2005) 043103. 9 pp.
[13] C. Li, L. Chen, K. Aihara, Impulsive control of stochastic systems with applications in chaos control, chaos synchronization, and neural networks, Chaos 18 (2008) 023132. 11 pp.
[14] Y. Li, X. Liu, H. Zhang, Dynamical analysis and impulsive control of a new hyperchaotic system, Math. Comput. Modelling 42 (2005) 1359–1374.
[15] X. Liu, Impulsive stabilization and control of chaotic system, Nonlinear Anal. 47 (2001) 1081–1092.
[16] G. Zhang, Z. Liu, Z. Ma, Synchronization of complex dynamical networks via impulsive control, Chaos 17 (2007) 043126. 9 pp.
[17] Z. Yang, D. Xu, Impulsive stabilization of large-scale dynamical systems, Dynam. Systems Appl. 15 (2006) 317–332.
[18] T. Carter, Optimal impulsive space trajectories based on linear equations, J. Optim. Theory Appl. 70 (1991) 277–297.
[19] X. Liu, Impulsive stabilization and applications to population growth models, Rocky Mountain J. Math. 25 (1995) 381–395.
[20] X. Liu, K. Rohlf, Impulsive control of a Lotka–Volterra system, IMA J. Math. Control Inform. 15 (1998) 269–284.
[21] X. Liu, Q. Wang, Impulsive stabilization of high-order hopfield-type neural networks with time-varying delays, IEEE Trans. Neural Netw. 19 (2008) 71–79.
[22] C. Neuman, V. Costanza, Deterministic impulse control in native forest ecosystems management, J. Optim. Theory Appl. 66 (1990) 173–196.
[23] R. Bellman, Topics in pharmacokinetics, III: repeated dosage and impulsive control, Math. Biosci. 12 (1971) 1–5.
[24] J. Liu, X. Liu, W.-C. Xie, Existence and uniqueness results for impulsive hybrid stochastic delay systems, Commun. Appl. Nonlinear Anal. 17 (3) (2010) 37–54.
[25] X. Mao, Stochastic Differential Equations and Applications, second ed., Horwood Publishing Limited, Chichester, 2007.
[26] S.-E.A. Mohamedd, Stochastic Functional Differential Equations, Longman Scientific and Technical, New York, 1986.