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Input-to-state stability of impulsive and switching hybrid systems with time-delay☆

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ABSTRACT

This paper investigates input-to-state stability (ISS) and integral input-to-state stability (iISS) of impulsive and switching hybrid systems with time-delay, using the method of multiple Lyapunov–Krasovskii functionals. It is shown that, even if all the subsystems governing the continuous dynamics, in the absence of impulses, are not ISS/iISS, impulses can successfully stabilize the system in the ISS/iISS sense, provided that there are no overly long intervals between impulses, i.e., the impulsive and switching signal satisfies a dwell-time upper bound condition. Moreover, these impulsive ISS/iISS stabilization results can be applied to systems with arbitrarily large time-delays. Conversely, in the case when all the subsystems governing the continuous dynamics are ISS/iISS in the absence of impulses, the ISS/iISS properties can be retained if the impulses and switching do not occur too frequently, i.e., the impulsive and switching signal satisfies a dwell-time lower bound condition. Several illustrative examples are presented, with their numerical simulations, to demonstrate the main results.

1. Introduction

Due to their numerous applications in various fields of sciences and engineering, hybrid systems have become a large and growing interdisciplinary area of research. A hybrid system is a dynamical system that exhibits both continuous and discrete dynamic behavior. The interaction of continuous- and discrete-time dynamics in a hybrid system can lead to rich dynamical behavior and phenomena that are not encountered in purely continuous- or discrete-time systems and hence bring difficulties and challenges to the studies of hybrid systems, such as their stability analysis and control design (see, e.g., Goebel, Sanfelice, and Teel (2009), Liberzon (2003), Shorten, Wirth, Mason, Wulff, and King (2007), and van der Schaft and Schumacher (2000) and references therein).

Impulsive differential equations or impulsive dynamical systems model real world processes that undergo abrupt changes (impulses) in the state at discrete times (Lakshmikantham, Baĭnov, & Simeonov, 1989). In the past decades, impulsive control and stabilization has been shown to be a powerful tool in the theory and applications of nonlinear dynamical systems. While difficulties and challenges remain in the area of impulsive differential equations (Lakshmikantham et al., 1989), especially those involving time-delays (Ballinger & Liu, 1999), impulsive control and stabilization has gained increasing popularity and found successful applications in a wide variety of areas, such as control systems (Ballinger & Liu, 2001; Liu, 2004), control and synchronization of chaotic systems (Li, Liao, Yang, & Huang, 2005; Liu, 2001; Yang & Chua, 1997), complex dynamical networks (Zhang, Liu, & Ma, 2007), secure communication (Yang & Chua, 1997), spacecraft maneuvers (Carter, 1991), population growth and biological systems (Liu, 1995; Liu & Rohlf, 1998), neural networks (Liu & Wang, 2008), ecosystems management (Neuman & Costanza, 1990), and pharmacokinetics (Bellman, 1971). Impulsive dynamical systems can be naturally viewed as a class of hybrid systems that consist of three elements: a continuous differential equation, which governs the continuous evolution of the system between impulses; a difference equation, which governs the way the system states are changed at impulse times; and an impulsive law for determining when the impulses occur (see, e.g., Haddad, Chellaboina, and Nersesov (2006) and Lakshmikantham et al. (1989)).

Another important type of hybrid systems are switched systems, which can be described by a differential equation whose right-hand side is chosen from a family of functions according to a switching signal. For each switching signal, the switched system is a time-varying differential equation. We usually study the properties of a switched system not under a particular switching signal, but rather under various classes of switching signals (see, e.g., Hespanha (2004) and Liberzon (2003)).

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Impulsive systems and switched systems can be naturally combined to form a more comprehensive model, i.e., impulsive switched systems, in which the switching signal and the impulsive law can be integrated as an impulsive and switching law (to be described in this paper). Despite the apparent abundance of applications, impulsive switched systems received only moderate attention in the past decade (see, e.g., Haddad et al. (2006), Li, Soh, and Wen (2005), Liu, Liu, and Xie (2009), Xie and Wang (2004) and Xu, Liu, and Teo (2008)), although, earlier in 1984, switching and impulses had already been combined to provide control for a reflected diffusion (Liao, 1984).

The notions of input-to-state stability (ISS) and integral input-to-state stability (iISS), originally introduced in Sontag (1989, 1998), have proved very useful in characterizing the effects of external inputs to a control system. The ISS/iISS notions are subsequently extended to discrete-time systems in Jiang and Wang (2001) and to switched systems in Mancilla-Aguilar and Garcia (2001) and Vu, Chatterjee, and Liberzon (2007). ISS notions for hybrid systems are investigated in Cai and Teel (2005) and Cai and Teel (2009), where the hybrid systems are defined on hybrid time domains. More recently, the work of Hespanha, Liberzon, and Teel (2008) studies Lyapunov conditions for input-to-state stability of impulsive nonlinear systems.

The notions of ISS/iISS have been generalized to nonlinear time-delay systems by various authors. In the seminal paper Teel (1998), the notion of ISS is extended to time-delay systems and sufficient conditions for ISS are investigated using Lyapunov–Razumikhin theorems. In Pepe and Jiang (2006), the method of Lyapunov–Krasovskii functionals is proposed for studying ISS/iISS of time-delay systems and several sufficient conditions for ISS/iISS of time-delay systems are presented. In Yeganefar, Pepe, and Dambrine (2008), a link is established between exponential stability of an unforced system and the ISS of time-delay systems using the method of Lyapunov–Krasovskii functionals. It is also pointed out in Yeganefar et al. (2008) that the characterization of ISS/iISS for nonlinear time-delay systems remains a difficult task despite the recent progress. More recently, the work of Chen and Zheng (2009) investigates both ISS and iISS for nonlinear impulsive systems with time delays. Sufficient conditions for ISS/iISS are established using the Lyapunov–Razumikhin method.

In this paper, we study input-to-state properties of impulsive and switching hybrid delay systems using the method of multiple Lyapunov–Krasovskii functionals. In contrast with the Lyapunov–Razumikhin method presented in Chen and Zheng (2009), it is well-known that the method of Lyapunov–Krasovskii functionals are sometimes more general than the Lyapunov–Razumikhin method in the sense that the latter can be considered as a particular case of the method of Lyapunov–Krasovskii functionals (Kolmanovskii & Myshkis, 1999, Section 4.8, p. 254) (see also Pepe and Jiang (2006)). For a discussion on the advantages and disadvantages of both methods in different situations, see Hale and Lunel (1993, Section 5.5, pp. 164–165). Therefore, it is worthwhile to study ISS properties of time-delay systems using the method of Lyapunov–Krasovskii functionals, as shown in Pepe and Jiang (2006) and Yeganefar et al. (2008). However, as far as impulsive stabilization of time-delay systems are concerned, the Lyapunov–Krasovskii functional method is usually more difficult than the Lyapunov–Razumikhin method. The reason is that, in general, we cannot expect an impulse that occurs at a discrete time to bring the value of a functional down instantaneously, whereas, in the Lyapunov–Razumikhin method, the value of a function can subside simultaneously as the impulse occurs. For impulsive stabilization of time-delay systems using the Lyapunov–Krasovskii functional methods, see Liu and Wang (2007) and Shen, Luo, and Liu (1999). Moreover, to the best knowledge of the authors, there have been no studies on the input-to-state properties of hybrid time-delay systems with both switching and impulse effects. The main objective of this paper is to establish some results in this direction. It is also shown that the results in the current paper can be applied to systems with arbitrarily large delays and, therefore, improve those results in Chen and Zheng (2009), Liu and Wang (2007) and Shen et al. (1999). Moreover, due to existence of different switching modes in the hybrid systems being investigated, we employ multiple Lyapunov–Krasovskii functionals, in the spirit of work in Branicky (1998) on multiple Lyapunov functions for switched systems.

The rest of this paper is organized as follows. In Section 2, we give some necessary notations and then formulate an impulsive and switching hybrid time-delay system with external input. The concepts of input-to-state stability and integral input-to-state stability are presented. The main results of this paper, presented in Section 3, give sufficient conditions for input-to-state stability and integral input-to-state stability of impulsive and switching hybrid time-delay systems in terms of Lyapunov–Krasovskii functionals. Several examples are presented in Section 4 to illustrate the main results. The paper is concluded by Section 5, where the main contributions of this paper are highlighted.

2. Preliminaries

Let $\mathbb{Z}^+$ denote the set of nonnegative integers, $\mathbb{R}^+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ the n-dimensional real Euclidean space. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. For $-\infty < a < b < \infty$, we use the notation $\mathcal{PC}([a,b];\mathbb{R}^n)$ to denote the class of functions from $[a,b]$ to $\mathbb{R}^n$ satisfying the following: (i) it has at most a finite number of jump discontinuities on $(a,b]$, i.e., points at which the function has finite-valued but different left-hand and right-hand limits; (ii) it is continuous from the right at all points in $[a,b)$. We say that a function $\psi : [a,\infty) \mapsto \mathbb{R}^n$ belongs to the class $\mathcal{PC}([a,\infty);\mathbb{R}^n)$, if $\psi|_{[a,b]}$ (ψ restricted on $[a,b]$) is in $\mathcal{PC}([a,b];\mathbb{R}^n)$ for all $b > a$. Given $r > 0$, a norm on $\mathcal{PC}([-r,0];\mathbb{R}^n)$ is defined as $\|\phi\| := \sup_{-r\le s\le 0}|\phi(s)|$ for $\phi \in \mathcal{PC}([-r,0];\mathbb{R}^n)$. For simplicity, $\mathcal{PC}$ is used for $\mathcal{PC}([-r,0];\mathbb{R}^n)$ for the rest of this paper. Given $x \in \mathcal{PC}([-r,\infty);\mathbb{R}^n)$ and for each $t \in \mathbb{R}^+$, let x_t be an element of $\mathcal{PC}$ defined by $x_t(s) := x(t+s)$, $-r \le s \le 0$.

Let $\mathcal{N}_c$ and $\mathcal{N}_d$ be two arbitrary index sets. Consider the following impulsive switched delay system:

$$x'(t) = f_{i_k}(t, x_t, w(t)), \quad t \in (t_k, t_{k+1}),\ i_k \in \mathcal{N}_c,\ k \in \mathbb{Z}^+, \tag{2.1a}$$

$$\Delta x(t) = I_{j_k}(t, x_{t^-}, w(t^-)), \quad t = t_k,\ j_k \in \mathcal{N}_d,\ k \in \mathbb{Z}^+ \setminus \{0\}, \tag{2.1b}$$

$$x_{t_0} = \xi, \tag{2.1c}$$

where $\xi \in \mathcal{PC}$ is the initial data, $x(t) \in \mathbb{R}^n$ is the system state, $x'(t)$ the right-hand derivative of $x(t)$, $w(t) : \mathbb{R}^+ \mapsto \mathbb{R}^m$ the input function, $\{t_k : k \in \mathbb{Z}^+\} \subset \mathbb{R}^+$ a strictly increasing sequence such that $t_k \to \infty$ as $k \to \infty$, and x_{t^-} defined by $x_{t^-}(s) = x(t+s)$, for $s \in [-r, 0)$, and $x_{t^-}(0) = x(t^-)$, where $x(t^-) = \lim_{s\to t^-} x(s)$ (the limit is taken from below t; similarly, $w(t^-) = \lim_{s\to t^-} w(s)$). For each $i \in \mathcal{N}_c$ and $j \in \mathcal{N}_d$, $f_i : \mathbb{R}^+ \times \mathcal{PC} \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $I_j : \mathbb{R}^+ \times \mathcal{PC} \times \mathbb{R}^m \mapsto \mathbb{R}^n$. The input function w is assumed to be in $\mathcal{PC}(\mathbb{R}^+;\mathbb{R}^n)$. Given $w \in \mathcal{PC}(\mathbb{R}^+;\mathbb{R}^n)$ and $i \in \mathcal{N}_c$, define $g_i(t,\phi) = f_i(t,\phi,w(t))$ and suppose that $g_i : \mathbb{R}^+ \times \mathcal{PC} \mapsto \mathbb{R}^n$ is composite-PC (i.e., for any function $x \in \mathcal{PC}([-r,\infty);\mathbb{R}^n)$, the composite function $t \mapsto g_i(t, x_t)$ is in $\mathcal{PC}(\mathbb{R}^+;\mathbb{R}^n)$), quasi-bounded, and continuous in its second variable so that system (2.1) has at least one solution (see Ballinger and Liu (1999)[1]). Moreover,

[1] In Ballinger and Liu (1999), the existence and uniqueness results are established for impulsive delay systems without switching. The case for system (2.1) with switching is essentially the same, by an argument using the method of steps over all the switching/impulse intervals.

each solution x belongs to the space $\mathcal{PC}([t_0 - r, \infty); \mathbb{R}^n)$ and is continuous at each $t \neq t_k$ for $t \geq t_0$. We assume that, for each $i \in \mathcal{N}_c$ and $j \in \mathcal{N}_d$, $f_i(t, 0, 0) \equiv I_j(t, 0, 0) \equiv 0$ so that system (2.1), without input, admits a trivial solution.

In (2.1), the sequence of triples $\{(t_k, i_k, j_k)\}$ together imposes the following: (1) a sequence of indices i_k to *switch* the right-hand side of (2.1a) among the family $\{f_i : i \in \mathcal{N}_c\}$; (2) a sequence of indices j_k to select the *impulse* functions I_{j_k} from the family $\{I_j : j \in \mathcal{N}_d\}$ to reset the system state according to the difference equation (2.1b); and (3) a sequence of discrete times t_k, called the *impulse and switching times* (except the initial time t_0[2]), to determine when the switching and impulse occur. Besides the family of functions $\{f_i : i \in \mathcal{N}_c\}$ and $\{I_i : i \in \mathcal{N}_d\}$, which govern the continuous dynamics and the discrete dynamics of system (2.1), respectively, it is expected that properties of solutions to system (2.1) (e.g. input-to-state stability properties to be investigated in this paper) can also be highly affected by the sequence of triples $\{(t_k, i_k, j_k)\}$, which we call an *impulsive and switching signal*. Hence, it is of interest to characterize properties of solutions that are uniform over a certain class of impulsive and switching signals. We may use $\mathcal{S}$ to denote a certain class of such signals.

A function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to be of *class* $\mathcal{K}$ and we write $\alpha \in \mathcal{K}$, if α is continuous, strictly increasing, and $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ also satisfies $\alpha(t) \to \infty$ as $t \to \infty$, we say that α is of *class* $\mathcal{K}_\infty$ and write $\alpha \in \mathcal{K}_\infty$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to be of *class* $\mathcal{KL}$ and we write $\beta \in \mathcal{KL}$, if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \in \mathbb{R}^+$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \in \mathbb{R}^+$.

Definition 2.1. System (2.1) is said to be *uniformly input-to-state stable (ISS)* over a certain class of signals $\mathcal{S}$, if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, independent of the choice of impulsive and switching signals $\{(t_k, i_k, j_k)\}$ in $\mathcal{S}$, such that, for each initial data $\xi \in \mathcal{PC}$ and input function $w \in \mathcal{PC}(\mathbb{R}^+; \mathbb{R}^n)$, the solution x of (2.1) exists globally and satisfies

$$|x(t)| \leq \beta(\|\xi\|, t - t_0) + \gamma\left(\sup_{t_0 \leq s \leq t} |w(s)|\right).$$

Definition 2.2. System (2.1) is said to be *uniformly integral input-to-state stable (ISS)* over a certain class of signals $\mathcal{S}$, if there exist functions $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}_\infty$, independent of the choice of impulsive and switching signals $\{(t_k, i_k, j_k)\}$ in $\mathcal{S}$, such that, for each initial data $\xi \in \mathcal{PC}$ and input function $w \in \mathcal{PC}(\mathbb{R}^+; \mathbb{R}^n)$, the solution x of (2.1) exists globally and satisfies

$$\alpha(|x(t)|) \leq \beta(\|\xi\|, t - t_0) + \int_{t_0}^{t} \gamma(|w(s)|)\mathrm{d}s + \sum_{t_0 < t_k \leq t} \gamma(|w(t_k^-)|).$$

The above definitions are parallel to the ones given in Chen and Zheng (2009), Hespanha et al. (2008) and Pepe and Jiang (2006). A minor distinction from the definitions in Chen and Zheng (2009) is that we consider the inputs to the continuous dynamics and the discrete dynamics of system (2.1), namely $w(t)$ in both (2.1a) and (2.1b), to be the same. This simpler formulation is, nevertheless, without loss of generality. If given $w_c(t) \in \mathbb{R}^{m_1}$ as the input for the continuous dynamics and $w_d(t) \in \mathbb{R}^{m_2}$ as the input for the discrete dynamics (as in Chen and Zheng (2009)), we can let $w(t) = [w_c(t)\ w_d(t)] \in \mathbb{R}^{m_1+m_2}$ and redefine f_i and I_j accordingly to achieve the formulation in (2.1).

[2] While a switching mode is assigned by i_0 at $t = t_0$, we do not consider a solution to instantly undergo an impulse at the initial time t_0.

A function $v : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^+$ is said to belong to *class* ν_1 and we write $v \in \nu_1$, if, for each function $x \in \mathcal{PC}(\mathbb{R}^+; \mathbb{R}^n)$, the composite function $t \mapsto v(t, x(t))$ is also in $\mathcal{PC}(\mathbb{R}^+; \mathbb{R}^n)$ and can be discontinuous at some $t' \in \mathbb{R}^+$ only if t' is a discontinuity point of x. A functional $v : \mathbb{R}^+ \times \mathcal{PC} \mapsto \mathbb{R}^+$ is said to belong to *class* ν_2 and we write $v \in \nu_2$, if, for each function $x \in \mathcal{PC}([-r, \infty); \mathbb{R}^n)$, the composite function $t \mapsto v(t, x_t)$ is continuous in t for $t \geq 0$.

To investigate the ISS/iISS properties of system (2.1), which has different modes of the continuous dynamics given by $\{f_i : i \in \mathcal{N}_c\}$, we shall choose accordingly a family of multiple Lyapunov–Krasovskii functionals $\{V_i : i \in \mathcal{N}_c\}$, where each $V_i : \mathbb{R}^+ \times \mathcal{PC} \mapsto \mathbb{R}^+$ is given by $V_i(t, \phi) = V_1^i(t, \phi(0)) + V_2^i(t, \phi)$. We shall assume that the family $\{V_1^i : i \in \mathcal{N}_c\}$ are of class ν_1 and the family $\{V_2^i : i \in \mathcal{N}_c\}$ are of class ν_2. The intuitive idea is to break the Lyapunov–Krasovskii functionals V_i into a function part V_1^i, which can effectively reflect the impulse effects, and a functional part V_2^i, which is indifferent to impulses, so that the difficulties in analyzing the impulse effects using Lyapunov–Krasovskii functionals can be overcome.

To effectively analyze a family of multiple Lyapunov–Krasovskii functionals $\{V_i : i \in \mathcal{N}_c\}$ for system (2.1), we introduce the upper right-hand derivative of V_i with respect to ith mode of system (2.1), for each $i \in \mathcal{N}_c$, at $(t, \phi) \in \mathbb{R}^+ \times \mathcal{PC}$ is defined by

$$D_i^+ V_i(t, \phi) = \limsup_{h \to 0^+} \frac{1}{h}[V_i(t + h, x_{t+h}(t, \phi)) - V_i(t, \phi)], \tag{2.2}$$

where $x(t, \phi)$ is a solution to the ith mode of system (2.1) satisfying $x_t = \phi$, i.e., $x(t) := x(t, \phi)$ satisfies $x_{t_0} = \phi$ and $x'(t) = f_i(t, x_t, w(t))$ for $t \in (t_0, t_0 + h)$, where $h > 0$ is some positive number. Moreover, for a function $v : \mathbb{R} \mapsto \mathbb{R}$, $D^+v(t)$ is the upper right-hand derivative of $v(t)$ defined by $D^+v(t) = \limsup_{h \to 0^+} \frac{1}{h}[v(t+h) - v(t)]$. Let $\{(t_k, i_k, j_k)\}$ be an impulsive and switching signal and x be a solution to system (2.1) on $[t_k, t_{k+1})$. Define $v(t) = V_{i_k}(t, x_t) = V_1^i(t, x(t)) + V_2^i(t, x_t)$, for $t \in [t_k, t_{k+1})$. The above definitions for the upper right-hand derivative of a function $v(t)$ and a functional V_i, with respect to ith mode of system (2.1), are connected by $D^+v(t) = D_{i_k}^+ V_{i_k}(t, x_t)$ for $t \in (t_k, t_{k+1})$.

3. Main results

We first introduce two classes of impulsive and switching signals and then establish uniform input-to-state stability of system (2.1) over these particular classes of signals. The two classes of signals to be introduced here generalize the well-known dwell-time conditions to *dwell-time conditions with respect to specific switching modes*. We say that an impulsive and switching signal $\{(t_k, i_k, j_k)\}$ belongs to $\mathcal{S}_{\inf}^i(\delta)$, for some $\delta > 0$ and $i \in \mathcal{N}_c$, if it satisfies $\inf\{t_{k+1} - t_k : k \in \mathbb{Z}^+,\ i_k = i\} \geq \delta$, where t_0 is the initial time. If $\{(t_k, i_k, j_k)\}$ satisfies $\sup\{t_{k+1} - t_k : k \in \mathbb{Z}^+,\ i_k = i\} \leq \delta$, it is said to belong to $\mathcal{S}_{\sup}^i(\delta)$. For fixed $\delta > 0$, the well-known dwell-time signals are recovered by $\mathcal{S}_{\inf}(\delta) = \bigcap_{i \in \mathcal{N}_c} \mathcal{S}_{\inf}^i(\delta)$ and $\mathcal{S}_{\sup}(\delta) = \bigcap_{i \in \mathcal{N}_c} \mathcal{S}_{\sup}^i(\delta)$. In other words, an impulsive and switching signal $\{(t_k, i_k, j_k)\}$ belongs to $\mathcal{S}_{\inf}^i(\delta)$ or $\mathcal{S}_{\sup}^i(\delta)$, if it assumes a dwell-time lower bound or upper bound δ, respectively, with respect to the ith mode.

Our first result is concerned with ISS properties of system (2.1), in the case when all the subsystems governing the continuous dynamics of (2.1) are stable and the impulses, on the other hand, are destabilizing. Intuitively, the conditions in the following theorem consist of four aspects (corresponding to each of the conditions): (i) the Lyapunov–Krasovskii functionals satisfy certain positive definite and decrescent conditions; (ii) the jumps induced by the destabilizing impulses satisfy certain growth conditions; (iii) there exist some negative estimates of the upper right-hand derivatives of the functionals with respect to each stable mode of (2.1); (iv) the estimates on the derivatives and the growth by jumps satisfy certain balancing conditions in terms of the dwell-time lower bounds.

Theorem 3.1. *Suppose that there exist a family of functions* $\{V_1^i : i \in \mathcal{N}_c\}$ *of class* ν_1 *and a family of functionals* $\{V_2^i : i \in \mathcal{N}_c\}$ *of class* ν_2*, functions* α_1, α_2, α_3*, and* χ *of class* $\mathcal{K}_\infty$*, positive constants* $\lambda > 0$, $\rho_i \geq 1$, δ_i*, and* $\mu_i > \lambda$ $(i \in \mathcal{N}_c)$ *such that, for all* $i, \tilde{i} \in \mathcal{N}_c$, $j \in \mathcal{N}_d$, $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$*, and* $\phi \in \mathcal{PC}$,

(i) $\alpha_1(|x|) \leq V_1^i(t,x) \leq \alpha_2(|x|)$ *and* $0 \leq V_2^i(t,\phi) \leq \alpha_3(\|\phi\|)$;
(ii) $V_1^{\tilde{i}}(t, \phi(0) + I_j(t,\phi,y)) \leq \rho_i V_1^i(t^-, \phi(0)) + \chi(|y|)$ *and* $V_2^{\tilde{i}}(t,\phi) \leq \rho_i V_2^i(t,\phi)$;
(iii) $D^+V_i(t,\phi) \leq -\mu_i V_i(t,\phi) + \chi(|w(t)|)$*, where* $V_i(t,\phi) = V_1^i(t,\phi(0)) + V_2^i(t,\phi)$;
(iv) $\ln \rho_i < (\mu_i - \lambda)\delta_i$.

Moreover, suppose that $\sup_{i\in\mathcal{N}_c} \rho_i < \infty$, $\inf_{i\in\mathcal{N}_c} \delta_i > 0$*, and* $\inf_{i\in\mathcal{N}_c} \mu_i > \lambda$*. Then system (2.1) is uniformly ISS over* $\bigcap_{i\in\mathcal{N}_c} \mathcal{S}^i_{\inf}(\delta_i)$*. In particular, if* $\delta = \sup_{i\in\mathcal{N}_c} \delta_i < \infty$*, system (2.1) is uniformly ISS over* $\mathcal{S}_{\inf}(\delta)$.

Proof. In view of condition (iv) and that $\sup_{i\in\mathcal{N}_c} \rho_i < \infty$, $\inf_{i\in\mathcal{N}_c} \delta_i > 0$, and $\inf_{i\in\mathcal{N}_c} \mu_i > \lambda$, we can choose a positive constant c such that $\rho_i e^{-(\mu_i-\lambda)\delta_i} < 1$, $-\mu_i + 1/c + \lambda < 0$, and $c\rho_i e^{-\mu_i\delta_i} + \rho_i/\mu_i < c$, for all $i \in \mathcal{N}_c$. Let x be a solution to (2.1), (t_k, i_k, j_k) be the corresponding impulsive and switching signal, and $w(t)$ a given input function. Set $v_1(t) = V_1^{i_k}(t, x(t))$, $v_2(t) = V_2^{i_k}(t, x_t)$, and $v(t) = v_1(t) + v_2(t)$, for $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}^+$. It is clear that $v(t)$ defines a right-continuous function on $[t_0, \infty)$. We shall show that

$$v(t)e^{\lambda(t-t_0)} \leq \alpha(\|\xi\|) + ce^{\lambda(t-t_0)}\chi\left(\sup_{t_0\leq s\leq t}|w(s)|\right) + \sum_{t_0<t_k\leq t} e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|), \quad t \geq t_0, \tag{3.1}$$

where $\alpha = \alpha_2 + \alpha_3$. For convenience, write the RHS of (3.1) as $u(t)$ and $\bar{\chi}(t) = \chi(\sup_{t_0\leq s\leq t}|w(s)|)$. It is clear that (3.1) holds for $t = t_0$. Define $t^* := \inf\{t \in [t_0, t_1) : v(t)e^{\lambda(t-t_0)} > u(t) + \varepsilon\}$, where $\varepsilon > 0$ is an arbitrarily fixed number. It is clear that $t^*(\varepsilon) = t_1$ implies $v(t)e^{\lambda(t-t_0)} \leq u(t) + \varepsilon$ for all $t \in [t_0, t_1)$. Therefore, if $t^*(\varepsilon) = t_1$ for all $\varepsilon > 0$, we must have that (3.1) holds on $[t_0, t_1)$. Suppose this is not the case, i.e. $t^*(\varepsilon^*) < t_1$ for some $\varepsilon^* > 0$. It follows that $v(t^*)e^{\lambda(t^*-t_0)} = u(t^*) + \varepsilon^* > 0$, which, by (3.1), implies that $v(t^*) > c\bar{\chi}(t^*)$. Hence condition (iii) of the theorem shows that

$$D^+[v(t^*)e^{\lambda(t^*-t_0)}] < \left(-\mu_{i_0} + \frac{1}{c} + \lambda\right)v(t^*)e^{\lambda(t^*-t_0)} < 0,$$

which clearly contradicts how t^* is chosen. Therefore, (3.1) holds on $[t_0, t_1)$. Now suppose that (3.1) is true on $[t_0, t_m)$, where $m \geq 1$. We will show that (3.1) holds on $[t_m, t_{m+1})$ as well. First, based on the inductive assumption, we estimate $v(t_m^-)e^{\lambda(t_m-t_0)}$. Since $D^+v(t) \leq -\mu_{i_{m-1}}v(t) + \chi(|w(t)|)$ on $[t_{m-1}, t_m)$, we can obtain, by integration, that

$$e^{\mu_{i_{m-1}}t_m}v(t_m^-) - e^{\mu_{i_{m-1}}t_{m-1}}v(t_{m-1}) \leq \frac{1}{\mu_{i_{m-1}}}[e^{\mu_{i_{m-1}}t_m} - e^{\mu_{i_{m-1}}t_{m-1}}]\bar{\chi}(t_m^-),$$

which implies

$$v(t_m^-) \leq v(t_{m-1})e^{-\mu_{i_{m-1}}\delta_{i_{m-1}}} + \frac{1}{\mu_{i_{m-1}}}\bar{\chi}(t_m^-). \tag{3.2}$$

On the other hand, from (3.1) on $[t_0, t_m)$, we have

$$v(t_{m-1})e^{\lambda(t_{m-1}-t_0)} \leq u(t_{m-1}). \tag{3.3}$$

Combining (3.2) and (3.3) gives

$$\begin{aligned} v(t_m^-)e^{\lambda(t_m-t_0)} &\leq \frac{1}{\rho_{i_{m-1}}}\Bigg[\rho_{i_{m-1}}e^{-(\mu_{i_{m-1}}-\lambda)\delta_{i_{m-1}}}\alpha(\|\xi\|) \\ &\quad + \left(c\rho_{i_{m-1}}e^{-\mu_{i_{m-1}}\delta_{i_{m-1}}} + \frac{\rho_{i_{m-1}}}{\mu_{i_{m-1}}}\right)e^{\lambda(t_m-t_0)}\bar{\chi}(t_m^-) \\ &\quad + \rho_{i_{m-1}}e^{-(\mu_{i_{m-1}}-\lambda)\delta_{i_{m-1}}} \\ &\quad \times \sum_{t_0<t_k\leq t_{m-1}} e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|)\Bigg] \\ &\leq \frac{1}{\rho_{i_{m-1}}}u(t_m^-). \end{aligned}$$

Therefore, by condition (ii),

$$\begin{aligned} v(t_m)e^{\lambda(t_m-t_0)} &\leq [\rho_{i_{m-1}}v(t_m^-) + \chi(|w(t_m^-)|)]e^{\lambda(t_m-t_0)} \\ &\leq u(t_m^-) + e^{\lambda(t_m-t_0)}\chi(|w(t_m^-)|) = u(t_m), \end{aligned}$$

i.e., (3.1) holds for $t = t_m$. Applying the argument used to show (3.1) on $[t_0, t_1)$, we can prove that (3.1) is true on $[t_m, t_{m+1})$. By induction, (3.1) is true for all $t \geq t_0$. Rewrite (3.1) as

$$v(t) \leq \alpha(\|\xi\|)e^{-\lambda(t-t_0)} + c\chi\left(\sup_{t_0\leq s\leq t}|w(s)|\right) + \sum_{t_0<t_k\leq t} e^{-\lambda(t-t_k)}\chi(|w(t_k^-)|).$$

Note that

$$\begin{aligned} \sum_{t_0<t_k\leq t} e^{-\lambda(t-t_k)}\chi(|w(t_k^-)|) &\leq \chi\left(\sup_{t_0\leq s\leq t}|w(s)|\right)\sum_{t_0<t_k\leq t} e^{-\lambda(t-t_k)} \\ &\leq \frac{1}{1-e^{-\lambda\delta}}\chi\left(\sup_{t_0\leq s\leq t}|w(s)|\right). \end{aligned}$$

We have

$$\begin{aligned} \alpha_1(x(t)) &\leq v(t) \leq \alpha(\|\xi\|)e^{-\lambda(t-t_0)} \\ &\quad + \left[c + \frac{1}{1-e^{-\lambda\delta}}\right]\chi\left(\sup_{t_0\leq s\leq t}|w(s)|\right). \end{aligned}$$

Since this estimate is independent of a particular impulsive and switching signal, it follows from a standard argument that there exist functions β and γ, still independent of a particular impulsive and switching signal, such that the estimate in Definition 2.1 for $x(t)$ holds, which implies uniform ISS of (2.1). The above estimate also establishes boundedness of the state, which further implies global existence of solutions (see Ballinger and Liu (1999)). The proof is complete. □

Remark 3.1. Theorem 3.1 is sufficiently general to cover the situation that the index sets $\mathcal{N}_c$ and $\mathcal{N}_d$ are infinite sets. If $\mathcal{N}_c$ is a finite index set, then $\sup_{i\in\mathcal{N}_c} \rho_i < \infty$, $\inf_{i\in\mathcal{N}_c} \delta_i > 0$, and $\inf_{i\in\mathcal{N}_c} \mu_i > \lambda$ are trivially satisfied in view of the theorem conditions. Moreover, condition (iv) can be replaced by a simpler one $\ln \rho_i < \mu_i\delta_i$, since $\lambda > 0$ can be chosen to be sufficiently small such that condition (iv) holds.

Remark 3.2. Condition (iii) in Theorem 3.1 implies that each of the continuous dynamics is ISS (see Pepe and Jiang (2006)). Nevertheless, condition (ii) does not rule out that each of the discrete dynamics can be destabilizing ($\rho_i \geq 1$). Theorem 3.1 shows that system (2.1) is ISS, if it satisfies a dwell-time lower bound condition given by condition (iv). In other words, if the impulses and switching occur not too frequently, the ISS properties of a hybrid time-delay system with stable continuous dynamics can be retained despite the destabilizing impulses.

Remark 3.3. Condition (ii) not only imposes a condition on the impulse functions I_j, it also characterizes possible jumps in the values of the multiple Lyapunov–Krasovskii functionals V_i at the impulsive and switching times. Actually, even if there are no impulses, a comparison factor among the multiple Lyapunov functions is necessary for the average dwell-time approach to the stability analysis of switched systems (see, e.g., Liberzon (2003) and Vu et al. (2007) and the factor $\mu \geq 1$ in their results).

Remark 3.4. The positive constant λ plays a role in characterizing the ISS/iISS properties with a "stability margin" (similar to stability margin of linear systems). This is similar to the generalized concept of $e^{\lambda t}$-weighted ISS/iISS properties, introduced in Vu et al. (2007).

The second result is concerned with ISS properties of system (2.1), in the case when all the subsystems governing the continuous dynamics of (2.1) can be unstable and the impulses, on the other hand, are stabilizing. Intuitively, the conditions in the following theorem consist of four aspects (corresponding to each of the conditions): (i) the Lyapunov–Krasovskii functionals satisfy certain positive definite and decrescent conditions; (ii) the jumps induced by the stabilizing impulses satisfy certain diminishing conditions; (iii) there exist some positive estimates of the upper right-hand derivatives of the functionals with respect to each unstable mode of (2.1); (iv) the estimates on the derivatives and the growth by jumps satisfy certain balancing conditions in terms of the dwell-time upper bounds.

Theorem 3.2. *Suppose that there exist a family of functions $\{V_1^i : i \in \mathcal{N}_c\}$ of class ν_1 and a family of functionals $\{V_2^i : i \in \mathcal{N}_c\}$ of class ν_2, functions $\alpha_1, \alpha_2, \alpha_3, \chi$ of class $\mathcal{K}_\infty$, positive constants $\rho_i < 1$, λ, δ_i, κ_i, and μ_i ($i \in \mathcal{N}_c$) such that, for all $i, \tilde{i} \in \mathcal{N}_c, j \in \mathcal{N}_d, t \in \mathbb{R}^+, x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $\phi \in \mathcal{PC}$,*

(i) $\alpha_1(|x|) \leq V_1^i(t,x) \leq \alpha_2(|x|)$ *and* $0 \leq V_2^i(t,\phi) \leq \alpha_3(\|\phi\|)$;
(ii) $V_1^i(t,\phi(0) + I_j(t,\phi,y)) \leq \rho_i V_1^{\tilde{i}}(t^-,\phi(0)) + \chi(|y|)$ *and* $V_2^i(t,\phi) \leq \kappa_i \sup_{-r\leq s\leq 0} V_1^{\tilde{i}}(t+s,\phi(s))$;
(iii) $D^+V_i(t,\phi) \leq \mu_i V_i(t,\phi) + \chi(|w(t)|)$, *where* $V_i(t,\phi) = V_1^i(t,\phi(0)) + V_2^i(t,\phi)$;
(iv) $\ln \bar{\rho}_i < -(\mu_i + \lambda)\delta_i$, *where* $\bar{\rho}_i = \rho_i + \kappa_i e^{\lambda r}$.

Moreover, suppose that $\sup_{i\in\mathcal{N}_c} \delta_i < \infty$, $\delta = \inf_{i\in\mathcal{N}_c} \delta_i > 0$, *and* $\sup_{i\in\mathcal{N}_c} \mu_i < \infty$. *Then system (2.1) is uniformly ISS over* $\bigcap_{i\in\mathcal{N}_c} \mathcal{S}^i_{\sup}(\delta_i)$. *In particular, system (2.1) is uniformly ISS over* $\mathcal{S}_{\sup}(\delta)$.

Proof. Let $v(t)$ and $\bar{\chi}(t)$ be the same as in the proof for Theorem 3.1. In view of condition (iv) and that $\sup_{i\in\mathcal{N}_c} \rho_i < \infty$ and $\sup_{i\in\mathcal{N}_c} \mu_i < \infty$, we can choose positive constants M, c_1, and c_2 such that $M \geq e^{\mu_i\delta_i}$, $\bar{\rho}_i e^{\mu_i\delta_i} c_1 + \mu_i e^{\mu_i\delta_i} < c_1$, $c_2 \geq e^{(\mu_i+\lambda)\delta_i}$, and $\bar{\rho}_i e^{(\mu_i+\lambda)\delta_i} < 1$. We shall show that

$$v(t)e^{\lambda(t-t_0)} \leq M\alpha(\|\xi\|) + c_1 e^{\lambda(t-t_0)}\bar{\chi}(t) + c_2 \sum_{t_0<t_k\leq t} e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|), \quad t \geq t_0, \tag{3.4}$$

where $\alpha = \alpha_2 + \alpha_3$. For convenience, let $u(t)$ denote the RHS of (3.4). For $k \in \mathbb{Z}^+$, condition (iii) on $[t_k, t_{k+1})$ implies that

$$e^{-\mu_{i_k}t}v(t) - e^{-\mu_{i_k}t_k}v(t_k) \leq \int_{t_k}^t e^{-\mu_{i_k}s}\chi(|w(s)|)ds, \quad t \in [t_k, t_{k+1}),$$

which gives

$$v(t) \leq v(t_k)e^{\mu_{i_k}(t-t_k)} + \mu_{i_k}e^{\mu_{i_k}\delta_{i_k}}\bar{\chi}(t), \tag{3.5}$$

for $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}^+$. Note that (3.5) implies (3.4) on $[t_0, t_1)$. Now suppose that (3.4) is true on $[t_0, t_m)$. We will show that (3.4) holds on $[t_m, t_{m+1})$ as well. By condition (ii), the inductive assumption, and the continuity of v_2 at $t = t_m$,

$$\begin{aligned} v(t_m) &\leq \rho_{i_m} v_1(t_m^-) + \chi(|w(t_m^-)|) + v_2(t_m^-) \\ &\leq \rho_{i_m} u(t_m^-)e^{-\lambda(t_m-t_0)} + \kappa_{i_m} \sup_{-r\leq s<0} |v_1(t_m+s)| \\ &\quad + \chi(|w(t_m^-)|) \\ &\leq \bar{\rho}_{i_m} u(t_m^-)e^{-\lambda(t_m-t_0)} + \chi(|w(t_m^-)|). \end{aligned} \tag{3.6}$$

Applying (3.5) on $[t_m, t_{m+1})$ and by (3.6), we have, for $t \in [t_m, t_{m+1})$,

$$\begin{aligned} v(t)e^{\lambda(t-t_0)} &\leq \bar{\rho}_{i_m} e^{(\mu_{i_m}+\lambda)\delta_{i_m}} M\alpha(\|\phi\|) \\ &\quad + [\bar{\rho}_{i_m} e^{\mu_{i_m}\delta_{i_m}} c_1 + \mu_{i_m} e^{\mu_{i_m}\delta_{i_m}}] e^{\lambda(t-t_0)}\bar{\chi}(t) \\ &\quad + \bar{\rho}_{i_m} e^{(\mu_{i_m}+\lambda)\delta_{i_m}} c_2 \sum_{t_0<t_k\leq t_{m-1}} e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|) \\ &\quad + e^{(\mu_{i_m}+\lambda)\delta_{i_m}} e^{\lambda(t_m-t_0)}\chi(|w(t_m^-)|) \\ &\leq u(t), \end{aligned}$$

i.e., (3.4) holds on $[t_m, t_{m+1})$. Hence, by induction, (3.4) is true for all $t \geq t_0$. The rest of the proof is the same as that of Theorem 3.1. The proof is complete. □

The following corollary is an immediate consequence of Theorem 3.2 and gives alternative sufficient conditions for ISS of system (2.1).

Corollary 3.1. *Suppose that there exist positive constants $\alpha_1, \alpha_2, \alpha_3$, and p such that conditions (i), (ii), and (iv) in Theorem 3.2 are replaced by the following*

(i′) $\alpha_1|x|^p \leq V_1^i(t,x) \leq \alpha_2|x|^p$ *and* $0 \leq V_2^i(t,\phi) \leq \alpha_3\|\phi\|^p$;
(ii′) $V_1^i(t,\phi(0) + I_j(t,\phi,y)) \leq \rho_i V_1^{\tilde{i}}(t^-,\phi(0)) + \chi(|y|)$;
(iv′) $\ln \bar{\rho}_i < -(\mu_i + \lambda)\delta_i$, *where* $\bar{\rho}_i = \rho_i + \frac{\alpha_3}{\alpha_1}e^{\lambda r}$,

and condition (iii) and all other assumptions remain the same. Then the same conclusions hold as in Theorem 3.2.

Proof. It suffices to verify that condition (ii) in Theorem 3.2 is satisfied with $\kappa_i = \frac{\alpha_3}{\alpha_1}$. □

Remark 3.5. If $\mathcal{N}_c$ is a finite index set, then $\sup_{i\in\mathcal{N}_c} \delta_i < \infty$, $\inf_{i\in\mathcal{N}_c} \delta_i > 0$, and $\sup_{i\in\mathcal{N}_c} \mu_i < \infty$ are trivially satisfied in view of the theorem conditions, and, moreover, condition (iv) can be replaced by a simpler one $\ln(\rho_i + \kappa_i) < -\mu_i\delta_i$, since $\lambda > 0$ can be chosen to be sufficiently small such that condition (iv) holds.

Remark 3.6. Condition (iii) in Theorem 3.2 implies that each of the continuous dynamics can be unstable ($\mu_i > 0$). Nevertheless, condition (ii) implies that each of the discrete dynamics is stabilizing ($\rho_i < 1$). Theorem 3.2 shows that system (2.1) is ISS, if it satisfies a dwell-time upper bound condition given by condition (iv). In other words, if the impulses are applied sufficiently frequently, a hybrid time-delay system with unstable continuous dynamics can be impulsively stabilized in the ISS/iISS sense.

Remark 3.7. Condition (ii) characterizes the key distinction of the idea of impulsive stabilization of time-delay systems using the method of Lyapunov–Krasovskii functionals. According to condition (ii), it is only required that the function part of V_i (i.e., V_1^i) is stabilized by the impulses, which is reasonable, since we cannot expect an impulse that occurs at a discrete time to bring the value of a purely functional part of V_i (e.g. an integral of ϕ) down. The factor κ_i plays a role in estimating the functional part of V_i in terms of $V_{\tilde{i}}$ with memory (Corollary 3.1 gives such an example), which eventually leads to condition (iv), where the dwell-time conditions are given depending on the delay size r and the factor κ_i. It appears that the delay size r and the factor κ_i have to be sufficiently small such that $\rho_i + \kappa_i e^{\lambda r} < 1$ (implied by condition (iv)). However, as we can see from the examples in the next section, we can always add a tuning parameter as the coefficient of V_2^i and hence make κ_i sufficiently small (of course, by doing so, more burden is going to be placed on V_1^i and the estimates μ_i in condition (iii) can become larger, which eventually leads to more restrictions on δ_i). The restriction on the delay size r can also be resolved using this technique. It is therefore remarked that Theorem 3.2 can be applied to system (2.1) with arbitrarily large delays, whereas the results in Chen and Zheng (2009) and Liu and Wang (2007) both have restrictions on the delay size.

Remark 3.8. Theorems 3.1 and 3.2 each provide a set of sufficient Lyapunov conditions to check for ISS properties of an impulsive and switching hybrid time-delay system. Therefore, the construction of Lyapunov–Krasovskii functionals are crucial in applying these results. Although no universal rules exist on how to construct these functionals, there are some guidelines on how to choose them. As for Lyapunov–Krasovskii functionals for linear time-delay systems, there are some commonly used candidates to perform stability analysis of such systems, as summarized in the survey paper Richard (2003). It can be seen that these functionals usually consist of two parts, a function part, which can be a quadratic function, and a functional part, which is usually an integral whose form varies depending on the right-hand sides of the systems being studied. These guidelines are well demonstrated by the examples to be presented in Section 4. Under these guidelines, and with the help of linear or non-linear feedback controllers, hopefully one can find suitable Lyapunov functionals that can satisfy all the conditions for ISS/iISS in Theorems 3.1 and 3.2. Moreover, conditions in Theorems 3.1 and 3.2 are only sufficient conditions for ISS/iISS of the impulsive and switching hybrid time-delay systems being investigated. Whether these conditions are necessary or what are the necessary conditions for ISS/iISS of such systems remains an interesting problem to be investigated.

As shown in Sontag (1998), iISS is a weaker notion than that of ISS and can be characterized by a weaker Lyapunov condition. In a similar spirit, the following two theorems on iISS of system (2.1) are formulated with a weaker condition on continuous dynamics.

Theorem 3.3. *If all the conditions in Theorem 3.1 hold, except that condition (iii) is replaced by*

(iii)′ $D^+V_i(t,\phi) \le (\chi(|w(t)|) - \mu_i)V_i(t,\phi) + \chi(|w(t)|)$,

then system (2.1) is uniformly iISS over $\bigcap_{i\in\mathcal{N}_c}\mathcal{S}^i_{\inf}(\delta_i)$. In particular, (2.1) is uniformly iISS over $\mathcal{S}_{\inf}(\delta)$.

Proof. Choose a positive constant c such that $\rho e^{(-\mu_i+\lambda)\delta_i} < 1$ and $1 < c\rho e^{-\mu_i\delta_i} + \rho_i < c$ for all $i \in \mathcal{N}_c$. Instead of showing (3.1), we prove that

$$v(t)e^{\lambda(t-t_0)} \le \mathcal{E}(t,t_0)\left[\alpha(\|\xi\|) + ce^{\lambda(t-t_0)}\int_{t_0}^t \chi(|w(s)|)ds\right] + \sum_{t_0<t_k\le t} e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|), \quad t \ge t_0, \tag{3.7}$$

where $\alpha = \alpha_2 + \alpha_3$ and $\mathcal{E}(t,s)$, for $t \ge s \ge t_0$, is defined by $\mathcal{E}(t,s) = \exp(\int_s^t \chi(|w(s)|)ds)$. As usual, we let $u(t)$ denote the RHS of (3.7). For $k \ge 1$, condition (iii′) on $[t_{k-1},t_k)$ implies that

$$e^{-\int_{t_{k-1}}^t \chi(|w(s)|)ds + \mu_{i_{k-1}}t}v(t) - e^{\mu_{i_{k-1}}t_{k-1}}v(t_{k-1}) \le \int_{t_{k-1}}^t e^{-\int_{t_{k-1}}^s \chi(|w(\tau)|)d\tau + \mu_{i_{k-1}}s}\chi(|w(s)|)ds, \quad t \in [t_{k-1},t_k),$$

which gives

$$v(t) \le \mathcal{E}(t,t_{k-1})\left[v(t_{k-1})e^{-\mu_{i_{k-1}}(t-t_{k-1})} + \int_{t_{k-1}}^t \chi(|w(s)|)ds\right], \tag{3.8}$$

for $t \in [t_{k-1},t_k)$ and $k \ge 1$. Note that (3.8) implies (3.7) on $[t_0,t_1)$. Now suppose that (3.7) is true on $[t_0,t_m)$, where $m \ge 1$. We will show that (3.7) holds on $[t_m,t_{m+1})$ as well. First, based on the inductive assumption, we estimate $v(t_m^-)e^{\lambda(t_m-t_0)}$. Applying (3.8) on $[t_{m-1},t_m)$ gives

$$v(t_m^-) \le \mathcal{E}(t_m,t_{m-1})\left[v(t_{m-1})e^{-\mu_{i_{m-1}}(t_m-t_{m-1})} + \int_{t_{m-1}}^{t_m} \chi(|w(s)|)ds\right].$$

Moreover, by (3.7) on $[t_0,t_m)$, $v(t_{m-1})e^{\lambda(t_{m-1}-t_0)} \le u(t_{m-1})$. Combining this and the last display equation gives

$$\begin{aligned} v(t_m^-)e^{\lambda(t_m-t_0)} &\le \frac{1}{\rho_{i_{m-1}}}\Bigg\{\rho_{i_{m-1}}e^{-(\mu_{i_{m-1}}-\lambda)\delta_{i_{m-1}}}\mathcal{E}(t_m,t_0)\alpha(\|\xi\|) \\ &\quad + (c\rho_{i_{m-1}}e^{-\mu_{i_{m-1}}\delta_{i_{m-1}}} + \rho_{i_{m-1}})\mathcal{E}(t_m,t_0)e^{\lambda(t_m-t_0)} \\ &\quad \times \int_{t_0}^{t_m}\chi(|w(s)|)ds + \rho_{i_{m-1}}e^{-(\mu_{i_{m-1}}-\lambda)\delta_{i_{m-1}}} \\ &\quad \times \sum_{t_0<t_k\le t_{m-1}} e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|)\Bigg\} \\ &\le \frac{1}{\rho_{i_{m-1}}}u(t_m^-), \end{aligned}$$

where the relation $\mathcal{E}(t,s) = \mathcal{E}(t,\tau)\mathcal{E}(\tau,s)$, for $t \ge \tau \ge s$, are used. Therefore, by condition (ii),

$$\begin{aligned} v(t_m)e^{\lambda(t_m-t_0)} &\le [\rho_{i_{m-1}}v(t_m^-) + \chi(|w(t_m^-)|)]e^{\lambda(t_m-t_0)} \\ &\le u(t_m^-) + e^{\lambda(t_m-t_0)}\chi(|w(t_m^-)|) = u(t_m). \end{aligned} \tag{3.9}$$

Applying (3.8) on $[t_m,t_{m+1})$ and using (3.9), we obtain

$$\begin{aligned} v(t)e^{\lambda(t-t_0)} &\le e^{-(\mu_{i_m}-\lambda)(t-t_m)}\mathcal{E}(t,t_0)\alpha(\|\xi\|) \\ &\quad + \mathcal{E}(t,t_0)\left(ce^{-\mu_{i_m}(t-t_m)}e^{\lambda(t_m-t_0)}\int_{t_0}^{t_m}\chi(|w(s)|)ds + e^{\lambda(t-t_0)}\int_{t_m}^t \chi(|w(s)|)ds\right) \\ &\quad + e^{-(\mu_{i_m}-\lambda)(t-t_m)}\sum_{t_0<t_k\le t} e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|) \\ &\le u(t), \quad t \in [t_m,t_{m+1}). \end{aligned}$$

That is, (3.1) is true on $[t_m,t_{m+1})$. By induction, (3.1) is true for all $t \ge t_0$. To show uniform iISS of (2.1), note that

$$\begin{aligned} \mathcal{E}(t,t_0)\alpha(\|\xi\|) &= \alpha(\|\xi\|) + (\mathcal{E}(t,t_0)-1)\alpha(\|\xi\|) \\ &\le \alpha(\|\xi\|) + \frac{1}{2}\alpha^2(\|\xi\|) + \frac{1}{2}[\mathcal{E}(t,t_0)-1]^2. \end{aligned} \tag{3.10}$$

Define $\vartheta_1(r) = r + \frac{r^2}{2}$ and $\vartheta_2(r) = \frac{1}{2}(e^r-1)^2 + re^r$, $r \ge 0$. It is clear that ϑ_1 and ϑ_2 are both of class $\mathcal{K}_\infty$. Applying (3.10) to (3.7), we obtain

$$v(t) \le \vartheta_1(\alpha(\|\xi\|)) + \vartheta_2\left(\int_{t_0}^t \chi(|w(s)|)ds\right) + \sum_{t_0<t_k\le t} e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|).$$

Uniform iISS of (2.1) follows from the previous inequality by a standard argument, and global existence of solutions follows from the boundedness of the state (see Ballinger and Liu (1999)). The proof is complete. □

Since condition (iii′) in Theorem 3.3 is weaker than condition (iii) in Theorem 3.1, the following is an immediate consequence of Theorem 3.3.

Corollary 3.2. *Under the same conditions as in Theorem 3.1, system (2.1) is uniformly iISS over* $\bigcap_{i\in\mathcal{N}_c}\mathcal{S}^i_{\inf}(\delta_i)$ *and* $\mathcal{S}_{\inf}(\delta)$.

Theorem 3.4. *If all the conditions in Theorem 3.2 hold, except that condition (iii) is replaced by*

(iii″) $D^+V_i(t,\phi)\le(\chi(|w(t)|)+\mu)V_i(t,\phi)+\chi(|w(t)|)$,

then system (2.1) is uniformly iISS over $\bigcap_{i\in\mathcal{N}_c}\mathcal{S}^i_{\sup}(\delta_i)$. *In particular, system (2.1) is uniformly iISS over* $\mathcal{S}_{\sup}(\delta)$.

Proof. Instead of showing (3.4), we can show that

$$\begin{aligned}v(t)e^{\lambda(t-t_0)} &\le \mathcal{E}(t,t_0)\left[M\alpha\|\xi\|^p+c_1e^{\lambda(t-t_0)}\int_{t_0}^t\chi(|w(s)|)ds\right]\\&\quad+c_2\sum_{t_0<t_k\le t}e^{\lambda(t_k-t_0)}\chi(|w(t_k^-)|),\quad t\ge t_0,\end{aligned}$$

where $\mathcal{E}(t,t_0)$ is defined as in the proof of Theorem 3.3 and the constants are chosen as in the proof of Theorem 3.2. The proof can be completed by induction and the argument is similar to that in the proof of Theorem 3.2. The details are omitted. □

Since condition (iii″) in Theorem 3.4 is weaker than condition (iii) in Theorem 3.2, the following is an immediate consequence of Theorem 3.4.

Corollary 3.3. *Under the same conditions as in Theorem 3.2, system (2.1) is uniformly iISS over* $\bigcap_{i\in\mathcal{N}_c}\mathcal{S}^i_{\sup}(\delta_i)$ *and* $\mathcal{S}_{\sup}(\delta)$.

4. Examples

In this section, we present several examples to illustrate our main results.

Example 4.1. Consider the following nonlinear impulsive switched delay system

$$\begin{aligned}x'(t) &= -\text{sat}(x(t))+a_{i_k}\text{sat}(x(t-\tau))\\&\quad+b_{i_k}\text{sat}(w(t)),\quad t\in(t_k,t_{k+1}),\ k\in\mathbb{Z}^+,\end{aligned}\tag{4.1a}$$

$$\Delta x(t)=c_{j_k}\text{sat}(w(t^-)),\quad t=t_k,\ k\in\mathbb{Z}^+\setminus\{0\},\tag{4.1b}$$

where $a_{i_k}, b_{i_k}\in\{-0.2,-0.1,0.1,0.2\}$, $c_{j_k}\in[-\frac12,\frac12]$, and $\text{sat}(x)$ is a saturation function defined by $\text{sat}(x)=\frac12(|x+1|-|x-1|)$.

To investigate the ISS properties of (4.1), let $\mathcal{N}_c\triangleq\{1,2,3,4\}$ and $\mathcal{N}_d\triangleq[-\frac12,\frac12]$. For $i\in\mathcal{N}_c$ and $j\in\mathcal{N}_d$, let $[a_1\ a_2\ a_3\ a_4]=[b_1\ b_2\ b_3\ b_4]=[-0.2\ -0.1\ 0.1\ 0.2]$ and $c_j=j$. Choose Lyapunov–Krasovskii functionals $V_i(t,\phi)=V_1^i(t,\phi(0))+V_2^i(t,\phi)$, with

$$V_1^i(t,x)=\begin{cases}x^2, & |x|\le 1,\\ e^{2(|x|-1)}, & |x|>1,\end{cases}$$

$$V_2^i(t,\phi)=|a_i|\int_{-r}^0\text{sat}^2(\phi(s))\left[\kappa+1+\frac{\kappa s}{r}\right]ds,$$

where $\kappa>0$. Condition (i) of Theorem 3.1 is clearly satisfied. Next we verify condition (ii) of Theorem 3.1. For $|\phi(0)+I_j(t,\phi,y)|\le 1$,

$$\begin{aligned}V_1^{\tilde i}(t,\phi(0)+I_j(t,\phi,y)) &= [\phi(0)+j\,\text{sat}(y)]^2\\&\le 2\phi^2(0)+\frac12y^2\le 2V_1^i(t,\phi(0))+\frac12y^2.\end{aligned}$$

For $|\phi(0)+I_j(t,\phi,y)|>1$, it is implied that $|\phi(0)|>\frac12$. We have

$$\begin{aligned}V_1^{\tilde i}(t,\phi(0)+I_j(t,\phi,y)) &= e^{2|\phi(0)+j\,\text{sat}(y)|-2}\le e^{2|\phi(0)|-1}\\&\le\begin{cases}ee^{2(|\phi(0)|-1)}=eV_1^i(t,\phi(0)), & |\phi(0)|>1,\\ 2e|\phi(0)|^2=2eV_1^i(t,\phi(0)), & |\phi(0)|\in\left(\frac12,1\right],\end{cases}\end{aligned}$$

where the fact that $2x^2>e^{2(x-1)}$, for $x\in(\frac12,1]$, has been used. Moreover, $V_2^{\tilde i}(t,\phi)\le 2V_2^i(t,\phi)$. In either case, condition (ii) is verified with $\rho_i\equiv 2e$. Now we check condition (iv). If $|\phi(0)|\le 1$, we have

$$\begin{aligned}D^+V_i(t,\phi) &= 2\phi(0)[-\text{sat}(\phi(0))+a_i\,\text{sat}(\phi(-r))+b_i\,\text{sat}(w(t))]\\&\quad+|a_i|(\kappa+1)\,\text{sat}^2(\phi(0))-|a_i|\,\text{sat}^2(\phi(-r))\\&\quad-\frac{\kappa}{r}|a_i|\int_{-r}^0\text{sat}^2(\phi(s))ds\\&\le-(2-|a_i|(\kappa+2)-2|b_i|)V_1^i(t,\phi(0))\\&\quad-\frac{\kappa}{(\kappa+1)r}V_2^i(t,\phi)+\frac{|b_i|}{2}y^2\\&\le-\min\left\{(2-|a_i|(\kappa+2)-2|b_i|),\frac{\kappa}{(\kappa+1)r}\right\}\\&\quad\times V_i(t,\phi)+\frac{|b_i|}{2}y^2.\end{aligned}$$

If $|\phi(0)|>1$, we have

$$\begin{aligned}D^+V_i(t,\phi) &= 2e^{2(|\phi(0)|-1)}\text{sgn}(\phi(0))\\&\quad\times[-\text{sat}(\phi(0))+a_i\,\text{sat}(\phi(-r))+b_i\,\text{sat}(w(t))]\\&\quad+|a_i|(\kappa+1)\,\text{sat}^2(\phi(0))-|a_i|\,\text{sat}^2(\phi(-r))\\&\quad-\frac{\kappa}{r}|a_i|\int_{-r}^0\text{sat}^2(\phi(s))ds\\&\le-2e^{2(|\phi(0)|-1)}(1-|b_i|)V_i(t,\phi)\\&\quad+|a_i|[2e^{2(|\phi(0)|-1)}|\text{sat}(\phi(-r))|\\&\quad-\text{sat}^2(\phi(-r))+(\kappa+1)]\\&\le-(2-|a_i|(\kappa+2)-2|b_i|)V_1^i(t,\phi(0))\\&\quad-\frac{\kappa}{(\kappa+1)r}V_2^i(t,\phi)\\&\le-\min\left\{(2-|a_i|(\kappa+2)-2|b_i|),\frac{\kappa}{(\kappa+1)r}\right\}V_i(t,\phi),\end{aligned}$$

where, to derive the second inequality above, we have used the fact that $-x^2+2bx+(\kappa+1)\le 2b+\kappa\le b(\kappa+2)$, for $x\in[0,1]$, $b\ge 1$, and $\kappa>0$. Therefore, taking

$$\mu_i=-\min\left\{(2-|a_i|(\kappa+2)-2|b_i|),\frac{\kappa}{(\kappa+1)r}\right\},\tag{4.2}$$

condition (iii) of Theorem 3.1 is satisfied, provided that $2-|a_i|(\kappa+2)-2|b_i|>0$ for all $i\in\mathcal{N}_c$. According to Theorem 3.1 (and Remark 3.1), if $\delta_i>\frac{1+\ln 2}{\mu_i}$ for all $i\in\mathcal{N}_c$, then system (4.1) is uniformly ISS over $\bigcap_{i\in\mathcal{N}_c}\mathcal{S}^i_{\inf}(\delta_i)$. For illustration, choosing $\kappa=0.2$ and $r=0.1$, we can compute from (4.2) that $\mu_1=\mu_4=1.16$ and $\mu_2=\mu_3=1.58$, which gives dwell-time conditions $\delta_1=\delta_4>1.4596$ and $\delta_2=\delta_3>1.0716$. Simulation results for system (4.1) with these parameters are shown in Fig. 1.

Next, we apply Theorems 3.3 and 3.4 to study integral input-to-state stability of a bilinear impulsive switched delay system.

Example 4.2. Consider the following bilinear impulsive switched delay system

$$\begin{aligned}x'(t) &= A_{i_k}x(t)+\sum_{p=1}^m w_p(t)(A_p^{i_k}x(t)+B_p^{i_k}x(t-r))\\&\quad+C_{i_k}w(t),\quad t\in(t_k,t_{k+1}),\ k\in\mathbb{Z}^+,\end{aligned}\tag{4.3a}$$

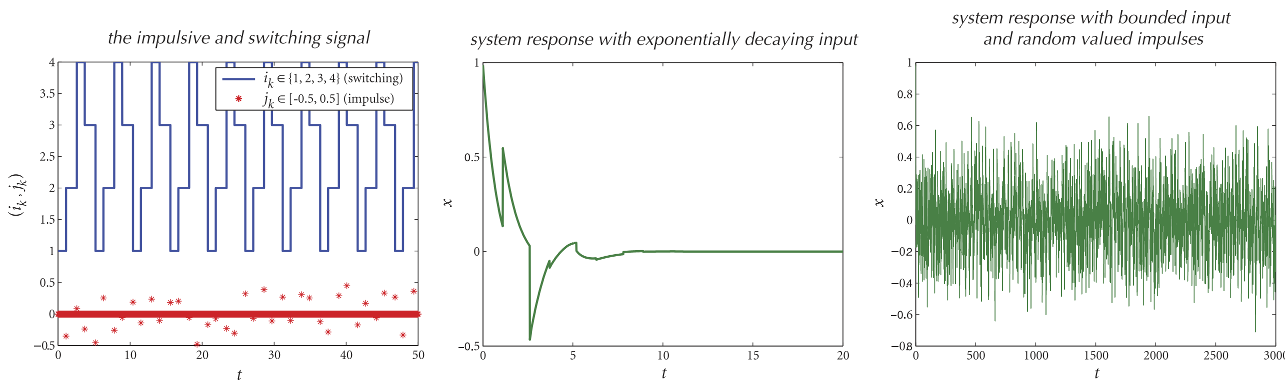

Fig. 1. Simulation results for Example 4.1.

$$\Delta x(t) = D_{j_k}x(t^-) + E_{j_k}w(t^-), \quad t = t_k,\ k \in \mathbb{Z}^+ \setminus \{0\}, \tag{4.3b}$$

where $i_k \in \mathcal{N}_c, j_k \in \mathcal{N}_d$, and both $\mathcal{N}_c$ and $\mathcal{N}_d$ are finite index sets. For each $i \in \mathcal{N}_c$ and $j \in \mathcal{N}_d$, $p = 1, 2, \ldots, m$, A_i, A_p^i, B_p^i, and D_j are in $\mathbb{R}^{n\times n}$, and C_i and E_j are in $\mathbb{R}^{n\times m}$. The input function w is in $\mathbb{R}^m$ and its components are w_p, $p = 1, 2, \ldots, m$.

The ISS properties of (4.3) obtained by Theorems 3.3 and 3.4 are summarized in the following proposition.

Proposition 4.1. *If*

(i) *all A_i are Hurwitz and P_i are positive definite matrices such that $A_i^TP_i + P_iA_i = -I$ and there exist triples of positive numbers $(\mu_i, \rho_i, \delta_i)$, $i \in \mathcal{N}_c$, such that $\delta_i > \frac{\ln \rho_i}{\mu_i} \geq 0$ and*

$$0 < \mu_i < \min\left\{\frac{1}{\lambda_{\max}(P_i)}, \frac{1}{2r}\right\},$$

$$\rho_i > \max_{j\in\mathcal{N}_d,\, \bar{i}\in\mathcal{N}_c} \lambda_{\max}(P_i^{-1}(\mathcal{I}_n + D_j)^TP_{\bar{i}}(\mathcal{I}_n + D_j)), \tag{4.4}$$

for all $i \in \mathcal{N}_c$, where $\mathcal{I}_n$ denotes the $n \times n$ identity matrix, then system (4.3) is uniformly iISS over $\bigcap_{i\in\mathcal{N}_c} \mathcal{S}^i_{\inf}(\delta_i)$. In particular, for $\delta = \max_{i\in\mathcal{N}_c} \delta_i$, system (4.3) is uniformly iISS over $\mathcal{S}_{\inf}(\delta)$.

(ii) *there exist triples of positive numbers $(\mu_i, \rho_i, \delta_i)$, $i \in \mathcal{N}_c$, such that all positive and*

$$\delta_i > -\frac{\ln(\rho_i)}{\mu_i}, \quad \mu_i > \lambda_{\max}(A_i^T + A_i),$$

$$\rho_i > \max_{j\in\mathcal{N}_d} \lambda_{\max}(\mathcal{I}_n + D_j)^T(\mathcal{I}_n + D_j), \tag{4.5}$$

for all $i \in \mathcal{N}_c$, then system (4.3) is uniformly iISS over $\bigcap_{i\in\mathcal{N}_c} \mathcal{S}^i_{\sup}(\delta_i)$. In particular, for $\delta = \min_{i\in\mathcal{N}_c} \delta_i$, system (4.3) is uniformly iISS over $\mathcal{S}_{\sup}(\delta)$.

Proof. We outline the proof as follows. For part (i), choose $V_i(t, \phi) = V_1^i(t, \phi(0)) + V_2^i(t, \phi)$, with $V_1^i(t, x) = x^2$ and $V_2^i(t, \phi) = \varepsilon\int_{-r}^0 (2 + \frac{s}{r})|\phi(0)|^2 ds$, where $\varepsilon \in (0, \frac{1}{3})$. For part (ii), choose $V_i(t, \phi) = V_1^i(t, \phi(0)) + V_2^i(t, \phi)$, with $V_1^i(t, x) = x^2$ and $V_2^i(t, \phi) = \int_{-r}^0 |\phi(0)|^2 ds$, where $\varepsilon > 0$. The conclusion follows from that of Theorem 3.4. □

Remark 4.1. Proposition 4.1 partially extends Theorem 3.10 in Pepe and Jiang (2006), where necessary and sufficient conditions for the iISS of a bilinear delay system are studied, to impulsive hybrid systems with time-delay. Proposition 4.1 also shows that, if each of the bilinear delay systems is unstable (i.e., A_i is not Hurwitz), we can still apply impulse control to achieve iISS for the hybrid time-delay system.

The following example presents a network based impulse control strategy to achieve ISS/iISS properties of estimation errors over a hybrid delayed system.

Example 4.3. Consider the following hybrid delayed networked control system

$$\begin{aligned} x'(t) &= A_{i_k}x(t) + f_{i_k}(x(t-r)) \\ &\quad + B_{i_k}w(t), t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}^+, \end{aligned} \tag{4.6a}$$

$$y(t) = x(t) + v(t), \quad t \geq t_0, \tag{4.6b}$$

$$\hat{x}'(t) = A_{i_k}\hat{x}(t) + \hat{f}_{i_k}(\hat{x}(t-r)), \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}^+, \tag{4.6c}$$

$$\hat{x}_l(t) = \begin{cases} y_{j_k}(t^-), & l = j_k, \\ \hat{x}_l(t^-), & l \neq j_k, \end{cases} \quad l \in \{1, 2, \ldots, n\}, \tag{4.6d}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^m$ is the disturbance input, $y(t) \in \mathbb{R}^n$ is the state measurement, $v(t) \in \mathbb{R}^n$ is the measurement noise, $\hat{x}(t) \in \mathbb{R}^n$ is the remote estimate of $x(t)$, $f_i(x(t-r))$ and $\hat{f}_i(\hat{x}(t-r))$ are the nonlinear delayed perturbations of the state and their estimations, respectively, with $r > 0$ as a constant time-delay. Moreover, $i_k \in \mathcal{N}_c$ and $\mathcal{N}_c$ is a finite index set; $\{t_k\}$ is a monotonically increasing transmission time sequence satisfying $t_k \to \infty$ as $k \to \infty$; at each transmission time $t = t_k$, a try-once-discard (TOD)-like protocol (Walsh, Beldiman, & Bushnell, 2001) to determine the index $j_k \in \{1, 2, \ldots, n\}$, i.e., j_k is the index j corresponding to the largest $|\hat{x}_j(t_k^-) - y_j(t_k^-)| = |e_j(t_k^-) - v_j(t_k^-)|$, where $j \in \{1, 2, \ldots, n\}$. When $t \in (t_k, t_{k+1})$, we can estimate $x(t)$ by letting $\hat{x}(t)$ evolve according to (4.6c); at $t = t_k$, a measurement y_{j_k} (based on a TOD-like protocol) is sent to the remote estimator and provides feedback impulse control to the estimate $\hat{x}_{j_k}$. The objective is to achieve ISS/iISS properties of the estimation error $E(t)$, which is defined by $E(t) = \hat{x}(t) - x(t)$ and can be shown to satisfy the following impulsive and switching hybrid delayed systems

$$\begin{aligned} E'(t) &= A_{i_k}E(t) + \hat{f}_{i_k}(\hat{x}(t-r)) - f_{i_k}(x(t-r)) \\ &\quad - B_{i_k}w(t), \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}^+, \end{aligned} \tag{4.7a}$$

$$\hat{E}_l(t) = \begin{cases} v_{j_k}(t^-), & l = j_k, \\ \hat{E}_l(t^-), & l \neq j_k, \end{cases} \quad l \in \{1, 2, \ldots, n\}. \tag{4.7b}$$

It is assumed that there exist positive constants L_i ($i \in \mathcal{N}_c$) such that

$$|f_i(x) - \hat{f}_i(\hat{x})| \leq L_i|x - \hat{x}|, \quad \forall x,\ \hat{x} \in \mathbb{R}^n,\ i \in \mathcal{N}_c. \tag{4.8}$$

To investigate the ISS properties of (4.7), choose Lyapunov–Krasovskii functionals $V_i(t, \phi) = V_1^i(t, \phi(0)) + V_2^i(t, \phi)$, with $V_1^i(t, E) = E^2$ and $V_2^i(t, \phi) = \kappa_i\int_{-r}^0 |\phi(s)|^2 ds$, where $\kappa_i > 0$. According to Hespanha et al. (2008), for each $\rho \in ((n-1)/n, 1)$, one can find a function $\chi \in \mathcal{K}_\infty$ such that, for all $k \in \mathbb{Z}^+$ and $i \in \mathcal{N}_c$,

$$V_1^i(t_k, E(t_k)) \leq \rho V_1^i(t_k^-, E(t_k^-)) + \chi(v(t_k^-)). \tag{4.9}$$

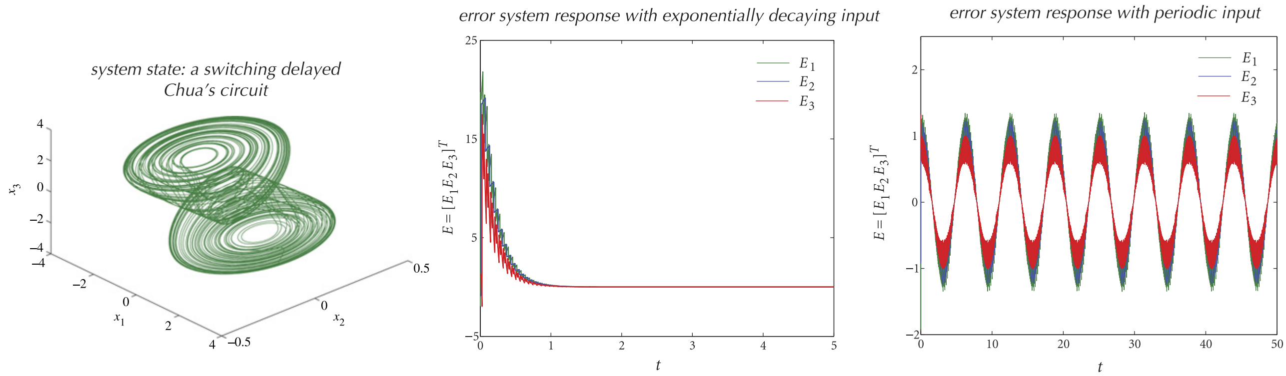

Fig. 2. Simulation results for Example 4.3.

Moreover, one can easily verify

$$V_2^i(t,\phi) \le \kappa_i \int_{-r}^{0} |\phi(s)|^2 \mathrm{d}s \le \kappa_i r \sup_{-r\le s\le 0} V_1^{\tilde{i}}(t+s,\phi(s)), \tag{4.10}$$

where $t \ge t_0$, i, $\tilde{i} \in \mathcal{N}_c$, and $\phi \in \mathcal{PC}$. Therefore, condition (ii) of Theorem 3.2 is satisfied with the constants ρ and $\kappa_i r L_i^2$ in place of ρ_i and κ_i, respectively. Computing the upper right-hand derivative of $V_i(t,\phi)$ along the ith mode of (4.7) gives

$$\begin{aligned} D_i^+ V_i(t,\phi) &= 2\phi^T(0)(A_i\phi(0) + \hat{f}_i(\hat{x}(t-r)) - f_i(x(t-r)) \\ &\quad - B_i w(t)) + \kappa_i(|\phi(0)|^2 - |\phi(-r)|^2) \\ &\le \left(\lambda_{\max}(A_i^T + A_i) + \frac{L_i^2}{\kappa_i} + \kappa_i + \varepsilon\right) V_1^i(t,\phi(0)) \\ &\quad + \chi_\varepsilon(|w(t)|), \end{aligned} \tag{4.11}$$

where $\varepsilon > 0$ is an arbitrary positive constant and χ_ε is a function in $\mathcal{K}_\infty$ which depends on ε. We claim that, if

$$\ln\left(\frac{n-1}{n} + \kappa_i r\right) < -\left(\lambda_{\max}(A_i^T + A_i) + \frac{L_i^2}{\kappa_i} + \kappa_i\right)\delta_i < 0, \tag{4.12}$$

holds for some $\delta_i > 0$ ($i \in \mathcal{N}_c$), then the error system (4.7) is uniformly ISS over $\bigcap_{i\in\mathcal{N}_c} \mathcal{S}^i_{\sup}(\delta_i)$. In particular, for $\delta = \min_{i\in\mathcal{N}_c} \delta_i$, system (4.3) is uniformly ISS over $\mathcal{S}_{\sup}(\delta)$. Actually, if (4.12) holds, one can choose $\rho \in ((n-1)/n, 1)$, $\mu_i > \lambda_{\max}(A_i^T + A_i) + L_i^2/\kappa_i + \kappa_i > 0$, and $\lambda > 0$ sufficiently small such that $\ln(\rho + \kappa_i r e^{\lambda r}) < -(\mu_i + \lambda)\delta_i$, i.e., condition (iv) of Theorem 3.2 is satisfied. In view of (4.11), we can choose $\varepsilon > 0$ sufficiently small such that $D_i^+ V_i(t,\phi) \le \mu_i V_i(t,\phi)$, i.e., condition (iii) of Theorem 3.2 is satisfied. Our claim follows from the conclusion of Theorem 3.2. As a numerical example, we take $B_i = [1 \quad 0.1 \quad 0]$, $f_i(x) = \hat{f}_i(x) = [\alpha_i(a-b) \quad 0 \quad 0]^T \mathrm{sat}(x_1)$, and

$$A_i = \begin{bmatrix} -\alpha_i(1-b) & \alpha_i & 0 \\ 1 & -1 & 1 \\ 0 & -\beta_i & 0 \end{bmatrix},$$

where $i \in \mathcal{N}_c = \{1,2\}$, and $\alpha_1 = 9$, $\beta_1 = 100/7$, $\alpha_2 = 10$, $\beta_2 = 16$, $a = 8/7$, $b = 5/7$. Therefore, the state system can be regarded as a hybrid system switching between two delayed Chua's circuits with slightly different parameters, which both exhibit chaotic behaviors under the given parameters. It is easy to verify that (4.8) is satisfied with $L_1 = 27/7$ and $L_2 = 30/7$. Moreover, $\lambda_{\max}(A_1^T + A_1) = 14.8685$ and $\lambda_{\max}(A_2^T + A_2) = 16.7839$. Now (4.12) specifies a condition on the dwell-time upper bound δ_i for the ith mode with $\kappa_i > 0$ as a tuning parameter. With $r = 0.02$ and $n = 3$ and choosing $\kappa_1 = 2.2$ and $\kappa_2 = 2.4$, (4.12) boils down to $\delta_1 < 0.0143$ and $\delta_2 < 0.0125$. Therefore, with $\delta_1 = 0.014$ and $\delta = 0.012$, Theorem 3.2 guarantees that the error system (4.7) is uniformly ISS over $\bigcap_{i=1,2} \mathcal{S}^i_{\sup}(\delta_i)$. In particular, for $\delta = 0.012$, system (4.3) is uniformly ISS over $\mathcal{S}_{\sup}(\delta)$. Simulation results for both the state system (4.6) and the error system (4.7), under the above parameters, are shown in Fig. 2.

Remark 4.2. As discussed in Remark 3.7 after Theorem 3.2, the parameters κ_i here play important roles in allowing our results to be applied to systems with arbitrarily large delays. Especially, in this example, the impulse amplitude, characterized by the factor ρ, is not arbitrarily chosen, since we have $\rho \in ((n-1)/n, 1)$. The result in Chen and Zheng (2009) (if adapted to hybrid systems with switching modes and applied to deal with this example) would have a restriction on the delay size as (according to their analysis, but using our notation)

$$r < -\frac{\ln(\rho)}{\lambda_{\max}(A_i^T + A_i) + 2L_i} < \frac{\ln\left(\frac{n}{n-1}\right)}{\lambda_{\max}(A_i^T + A_i) + 2L_i},$$

which, in our numerical example, gives $r < 0.0160$. Therefore, the result in Chen and Zheng (2009) cannot be applied to deal with $r = 0.02$ in the above example. Using our results, the parameters κ_i can be chosen sufficiently small so that, even if the delay size r is arbitrarily large, one can still verify inequality (4.12) for $\delta_i > 0$ sufficiently small, i.e., the transmission sequence that gives the feedback impulse control is sufficiently frequent. Therefore, we remark that one of the contributions of our results is making the impulse control strategy applicable to systems with arbitrarily large delays.

5. Conclusions

A method of multiple Lyapunov–Krasovskii functionals for investigating input-to-state stability properties for impulsive and switching hybrid time-delay systems has been presented. We have established sufficient conditions for input-to-state stability and integral input-to-state stability of hybrid time-delay systems with both switching and impulse effects. The formulation of hybrid systems are quite general in that it allows both the continuous dynamics and the discrete dynamics to be chosen from a certain family, according to a general impulsive and switching signal. The idea of impulsive stabilization for time-delay systems using the method of Lyapunov–Krasovskii functionals is exploited, whereas, even for the classical notion of stability (Lyapunov stability), there are very few results concerning the impulsive stabilization of time-delay systems using the Lyapunov–Krasovskii functional method.

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References

Ballinger, G., & Liu, X. (1999). Existence and uniqueness results for impulsive delay differential equations. *Dynamics of Continuous, Discrete & Impulsive Systems*, *5*, 579–591.

Ballinger, G., & Liu, X. (2001). Practical stability of impulsive delay differential equations and applications to control problems. In *Optimization methods and applications*. Dordrecht: Kluwer Acad. Publ..

Bellman, R. (1971). Topics in pharmacokinetics, III: repeated dosage and impulsive control. *Mathematical Biosciences*, *12*, 1–5.

Branicky, M. S. (1998). Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, *43*, 475–482.

Cai, C., & Teel, A.R. (2005). Results on input-to-state stability for hybrid systems. In *Proc. of the 44th conf. on decision and control* (pp. 5403–5408).

Cai, C., & Teel, A. R. (2009). Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, *58*, 47–53.

Carter, T. (1991). Optimal impulsive space trajectories based on linear equations. *Journal of Optimization Theory and Applications*, *70*, 277–297.

Chen, W.-H., & Zheng, W. X. (2009). Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. *Automatica*, *45*, 1481–1488.

Goebel, R., Sanfelice, R. G., & Teel, A. R. (2009). Hybrid dynamical systems: robust stability and control for systems that combine continuous-time and discrete-time dynamics. *IEEE Control Systems Magazine*, *29*, 28–93.

Haddad, W. M., Chellaboina, V., & Nersesov, S. G. (2006). *Impulsive and hybrid dynamical systems*. New Jersey: Princeton Univ. Press.

Hale, J. K., & Lunel, S. M. V. (1993). *Introduction to functional differential equations*. New York: Springer-Verlag.

Hespanha, J. P. (2004). Uniform stability of switched linear systems: extension of LaSalle's invariance principle. *IEEE Transactions on Automatic Control*, *49*, 470–482.

Hespanha, J. P., Liberzon, D., & Teel, A. R. (2008). Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, *44*, 2735–2744.

Jiang, Z. P., & Wang, Y. (2001). Input-to-state stability for discrete-time nonlinear systems. *Automatica*, *37*, 857–869.

Kolmanovskii, V., & Myshkis, A. (1999). *Introduction to the theory and applications of functional differential equations*. Dordrecht: Kluwer Avademic Publishers.

Lakshmikantham, V., Baĭnov, D., & Simeonov, P. (1989). *Theory of impulsive differential equations*. Teaneck, NJ: World Scientific Publishing.

Liao, Y. C. (1984). Switching and impulsive control of a reflected diffusion. *Applied Mathematics and Optimization*, *11*, 153–159.

Liberzon, D. (2003). *Switching in systems and control*. Boston: Birkhäuser.

Li, C., Liao, X., Yang, X., & Huang, T. (2005). Impulsive stabilization and synchronization of a class of chaotic delay systems. *Chaos*, *15*, 043103. 9 pp.

Li, Z., Soh, Y., & Wen, C. (2005). *Lecture notes in control and information sciences*: *Vol. 313*. *Switched and impulsive systems: analysis, design, and applications*. Berlin: Springer-Verlag.

Liu, X. (2004). Stability of impulsive control systems with time delay. *Mathematical and Computer Modelling*, *39*, 511–519.

Liu, X. (2001). Impulsive stabilization and control of chaotic system. *Nonlinear Analysis*, *47*, 1081–1092.

Liu, X. (1995). Impulsive stabilization and applications to population growth models. *The Rocky Mountain Journal of Mathematics*, *25*, 381–395.

Liu, J., Liu, X., & Xie, W.-C. (2009). Invariance principles for impulsive switched systems. *Dynamics of Continuous, Discrete & Impulsive Systems. Series B. Applications & Algorithms*, *16*, 631–654.

Liu, X., & Rohlf, K. (1998). Impulsive control of a Lotka–Volterra system. *IMA Journal of Mathematical Control and Information*, *15*, 269–284.

Liu, X., & Wang, Q. (2008). Impulsive stabilization of high-order Hopfield-type neural networks with time-varying delays. *IEEE Transactions on Neural Networks*, *19*, 71–79.

Liu, X., & Wang, Q. (2007). The method of Lyapunov functionals and exponential stability of impulsive systems with time delay. *Nonlinear Analysis*, *66*, 1465–1484.

Mancilla-Aguilar, J. L., & Garcia, R. A. (2001). On converse Lyapunov theorems for ISS and iISS switched nonlinear systems. *Systems & Control Letters*, *42*, 47–53.

Neuman, C., & Costanza, V. (1990). Deterministic impulse control in native forest ecosystems management. *Journal of Optimization Theory and Applications*, *66*, 173–196.

Pepe, P., & Jiang, Z. P. (2006). A Lyapunov–Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters*, *55*, 1006–1014.

Richard, J.-P. (2003). Time-delay systems: an overview of some recent advances and open problems. *Automatica*, *39*, 1667–1694.

Shen, J., Luo, Z., & Liu, X. (1999). Impulsive stabilization of functional-differential equations via Lyapunov functionals. *Journal of Mathematical Analysis and Applications*, *240*, 1–15.

Shorten, R., Wirth, F., Mason, O., Wulff, K., & King, C. (2007). Stability criteria for switched and hybrid systems. *SIAM Review*, *49*, 543–732.

Sontag, E. D. (1989). Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, *34*, 435–443.

Sontag, E. D. (1998). Comments on integral variants of ISS. *Systems & Control Letters*, *34*, 93–100.

Teel, A. R. (1998). Connection between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Transactions on Automatic Control*, *43*, 960–964.

van der Schaft, A., & Schumacher, H. (2000). *Lecture notes in control and information sciences*: *Vol. 251*. *An introduction to hybrid dynamical systems*. London: Springer-Verlag.

Vu, L., Chatterjee, D., & Liberzon, D. (2007). Input-to-state stability of switched systems and switching adaptive control. *Automatica*, *42*, 639–646.

Walsh, G. C., Beldiman, O., & Bushnell, L. G. (2001). Asymptotic behavior of nonlinear networked control systems. *IEEE Transactions on Automatic Control*, *46*, 1093–1097.

Xie, G., & Wang, L. (2004). Necessary and sufficient conditions for controllability and observability of switched impulsive control systems. *IEEE Transactions on Automatic Control*, *49*, 960–966.

Xu, H., Liu, X., & Teo, K. L. (2008). A LMI approach to stability analysis and synthesis of impulsive switched systems with time delays. *Nonlinear Analysis Hybrid Systems*, *2*, 38–50.

Yang, T., & Chua, L. (1997). Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, *44*, 976–988.

Yeganefar, N., Pepe, P., & Dambrine, M. (2008). Input-to-state stability of time-delay systems: a link with exponential stability. *IEEE Transactions on Automatic Control*, *53*, 1526–1531.

Zhang, G., Liu, Z., & Ma, Z. (2007). Synchronization of complex dynamical networks via impulsive control. *Chaos*, *17*, 043126. 9 pp.

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