Stochastic consensus seeking with communication delays

Jun Liu, Zhenyu Cao, Wei-Chau Xie, Hongtao Zhang

1. Introduction

While consensus problems have a long history in both computer science (Lynch, 1997) and statistics (DeGroot, 1974), there has been a recent surge of interests among various disciplines of engineering and science in problems related to networked systems of multi-agents that emphasize consensus or agreement (see, e.g., Bauso, Di Marco, & Masci, 2009; Bliman & Ferrari-Trecate, 2008; Cao, Mancini, & Anderson, 2006a,b; Carli, Fagnani, Speranzon, & Zampieri, 2006; Fax & Murray, 2004; Hatano & Mesbahi, 2005; Huang & Manton, 2009, 2010a,b; Jadbabaie, Lin, & Morse, 2003; Li & Zhang, 2009; Moreau, 2005; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Ren, Beard, & Kingston, 2005; Xiao, Boyd, & Kim, 2007; also see Olfati-Saber, Fax, & Murray, 2007 for a recent survey and extensive references therein). Consensus problems naturally arise when a group of agents, often distributed over a network, are seeking agreement upon a certain quantity of interest, which might be attitude, position, velocity, voltage, direction, temperature, and so on, depending on different applications.

Networked systems are often subject to environmental uncertainties and communication delays, which make it difficult or impossible for a networked agent to obtain timely and accurate information of its neighbors. Moreover, link gains/failures and formation reconfiguration make it necessary to address consensus problems for networks with switching network topology. The recent work of Huang and Manton (2009, 2010a,b) studies stochastic consensus problems of networked agents, with or without switching topology, in the discrete-time setting using algorithms from stochastic approximation. In Li and Zhang (2009), the work of Huang and Manton (2009) is extended to the continuous-time setting, and both necessary and sufficient conditions for stochastic consensus have been obtained for networks that are both balanced and containing a spanning tree (equivalent to strongly connected and balanced; Li & Zhang, 2009). Other work on consensus problems that explicitly takes into account measurement and environmental noises in different contexts includes (Bauso et al., 2009; Borkar & Varaiya, 1982; Carli et al., 2006; de Castro & Paganini, 2004; Hatano & Mesbahi, 2005; Ren et al., 2005; Tsitsiklis & Athans, 1984; Tsitsiklis, Bertsekas, & Athans, 1986; Xiao et al., 2007), in some of which the noises are modeled as deterministic but unknown disturbances (e.g., Bauso et al., 2009; de Castro & Paganini, 2004). On the other hand, consensus problems with communication delays have also been studied extensively in recent years (see, e.g., Bliman & Ferrari-Trecate, 2008; Lin & Jia, 2009; Lu, Ho, &
Kurths, 2009; Munz, Papachristodoulou, & Allgower, 2010; Olfati-Saber & Murray, 2004; Tian & Liu, 2008; Wang & Slotine, 2006; Xiao & Wang, 2008; Yang & Fang, 2010; Zhu & Cheng, 2010). None of the above mentioned work, however, has investigated stochastic consensus problems of networks with communication delays, either in a discrete- or continuous-time setting, while delays are ubiquitous in communication networks.

The purpose and main contribution of this paper is to investigate stochastic consensus problems with communication time-delays. Following the time-varying consensus protocol introduced by Huang and Manton (2009) for discrete-time systems and by Li and Zhang (2009) for continuous-time systems, both without communication delays, we propose a time-varying consensus protocol that takes into account both the measurement noises and general time-varying communication time-delays. We take a continuous-time approach using differential equations and stochastic calculus, and aim to provide conditions under which the proposed consensus protocol actually leads to consensus for networks with strongly connected and balanced topology. Moreover, the consensus results are extended to networks with arbitrary deterministic switching topology and with Markovian random switching topology. Explicit delay upper bounds for guaranteeing consensus are obtained in each case.

The rest of this paper is organized as follows. In Section 2, we formulate the consensus problem, propose the consensus protocol, and introduce two notions of stochastic consensus. The main consensus results are presented in Section 3, followed by demonstrations through numerical simulations in Section 4. The detailed proofs for the main results are included in Section 5, where a Gronwall–Bellman–Halany type inequality is established, which plays an essential role in proving the consensus results. The paper is concluded by Section 6, which also points out some possible future research along the line of this paper.

2. Problem formulation and consensus protocols

2.1. Network topology

The interaction topology of a network of $n$-agents is modeled by a weighted digraph (or directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order $n$ with set of nodes $\mathcal{V} = \{v_1, \ldots, v_n\}$, set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a weighted adjacent matrix $\mathcal{A} = [a_{ij}]_{n \times n}$ with nonnegative elements $a_{ij}$. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (v_i, v_j)$. An edge $e_{ij}$ exists if and only if $a_{ij} > 0$. It is assumed that $a_{ii} = 0$ for $i = 1, \ldots, n$. The set of neighbors of a node $v_i$ is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. Let $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \text{diag}(d_1, \ldots, d_n)$ is the degree matrix of $\mathcal{G}$ with elements $d_i = \sum_{j \in \mathcal{N}_i} a_{ji}$ and $\mathcal{A}$ is the weighted adjacent matrix. A digraph (and the corresponding network) is strongly connected if there is a directed path connecting any two arbitrary nodes in the graph. A digraph (and the corresponding network) is said to be balanced if $\sum_{j \in \mathcal{N}_i} a_{ij} = \sum_{j \in \mathcal{N}_i} a_{ji}$ for all $i \in \mathcal{I}$.

2.2. Consensus protocols

Consider each node of the graph to be a dynamic agent with dynamics

$$\dot{x}_i = u_i, \quad i \in \mathcal{I},$$

where the state feedback $u_i = u_i(x_1, \ldots, x_n)$ is called a protocol with topology $\mathcal{G}$ if the set of nodes $\{x_1, \ldots, x_n\}$ is all taken from the set $\{v_1\} \cup \mathcal{N}_i$, i.e. only the information of $v_i$ itself and its neighbors are available in forming the state feedback for the node $v_i$.

We consider the following consensus protocol by Olfati-Saber and Murray (2004)

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i), \quad i \in \mathcal{I}.$$  

(2.3)

The above protocol requires that agent $i$ can obtain information from its neighbors in $\mathcal{N}_i$ timely and accurately, i.e. it assumes zero communication time-delay and accurate information exchange among agents. Let $y_{ij}$ be a measurement of $x_j$ by $x_i$ given by

$$y_{ij} = x_j + \sigma_{ij} w_{ij}(t), \quad i \in \mathcal{I},$$

(2.4)

where $\{w_{ij}(t) : i,j = 1, \ldots, n\}$ are independent standard white noises and $\sigma_{ij} \geq 0$ represent the noise intensity. Replacing $x_j$ in (2.3) with the noisy measurement $y_{ij}$ gives the following stochastic consensus protocol

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij}(y_{ji} - x_i), \quad i \in \mathcal{I}.$$  

(2.5)

If, in addition, time-varying communication delays are considered, we propose the following delayed stochastic consensus protocol

$$u_i(t) = c(t) \sum_{j \in \mathcal{N}_i} a_{ij}[y_{ji}(t) - x_i(t - \tau_{ij}(t))], \quad i \in \mathcal{I}$$

(2.6)

where

$$y_{ij} = x_j(t - \tau_{ij}(t)) + \sigma_{ij} w_{ij}(t), \quad i \in \mathcal{I},$$

(2.7)

and the time-varying delays $\tau_{ij}(t)$ lie in $[0, \tau]$ for some $\tau > 0$ and are assumed to be continuous in $t$. The function $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ in (2.6) is a piecewise continuous function satisfying

$$\int_0^\infty c(s)ds = \infty \quad \text{and} \quad \int_0^\infty c^2(s)ds < \infty.$$  

(2.8)

The role of the function $c(t)$ is to attenuate the noise effects as $t \rightarrow \infty$. Condition (2.7), on the other hand, implies that $c(t)$ is vanishing as $t \rightarrow \infty$, but, on the other hand, not too fast due to $\int_0^\infty c(s)ds = \infty$. Without loss of generality, we can assume that $\sup_{t \geq 0} c(t) \leq 1$. If $\mathcal{N}_i$ is fixed, (2.6) gives a fixed topology protocol. If $\mathcal{N}_i$ is time-varying, we have a switching topology protocol. The communication delays and noisy measurements in the protocol (2.6) are illustrated by Fig. 1.

In this paper, we shall focus on the case when the time-varying delays are uniform, i.e. $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$.

2.3. Network dynamics

If the time-delays are uniform, i.e. $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$, the collective dynamics of system (2.2) under the consensus protocol (2.6) can be written in a compact form of a stochastic delay differential equation (SDDE) as

$$dx_i(t) = c(t)(-\Delta x_i(t - \tau(t))dt + \Theta dW(t)),$$

(2.8)

where $W(t)$ is an $n^2$-dimensional standard Wiener process; $\mathcal{L}$ is the graph Laplacian of the network; and $\Theta \in \mathbb{R}^{n \times n^2}$ is a constant matrix defined by $\Theta = \text{diag}(\Theta_1, \ldots, \Theta_n)$, where $\Theta_i$ is an $n$-dimensional row vector given by $\Theta_i = [a_{i1} \sigma_{i1} a_{i2} \sigma_{i2} \cdots a_{in} \sigma_{in}]$. 

![Fig. 1. Delayed measurements with additive noises.](image-url)
2.4. Consensus notion

We introduce the following notion of mean square average-consensus for the multi-agent systems (2.2) under the consensus protocol (2.6) in an uncertain environment.

**Definition 2.1 (Li and Zhang (2009)).** The agents in (2.2) are said to reach mean square average-consensus if $E|x(t)|^2 < \infty$ for all $t \geq 0$ and $i \in I$ and there exists a random variable $x^*$ such that $E|x(t)| = \text{avg}(x(0)) = \sum_{i=1}^{n} x_i(0)/n$ and $\lim_{t \to \infty} E|x(t) - x^*|^2 = 0$ for all $i \in I$.

Huang and Manton (2009) defined and investigated both mean square consensus and almost sure consensus (called strong consensus in the discrete-time setting, without emphasizing average-consensus and considering communication delays. Continuous-time mean square average-consensus has been defined and studied by Li and Zhang (2009), without considering communication delays.

3. Consensus results

In this section, we analyze the consensus properties of the dynamics of system (2.2).

3.1. Networks with fixed topology

We start by analyzing networks with fixed topology, i.e. the weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is time-invariant.

**Theorem 3.1.** Suppose that $\mathcal{G}$ is a strongly connected and balanced digraph with $\mathcal{L}$ as its Laplacian. Let $\lambda_2(\mathcal{L})$ denote the second smallest eigenvalue of $\mathcal{L} = (\mathcal{L} + \mathcal{L}^T)/2$. If

$$\tau < \frac{\lambda_2(\mathcal{L})}{\|\mathcal{L}\|},$$

(3.1)

where $\tau > 0$ is an upper bound for the uniform time-varying delay $\tau(t)$ and $\|\cdot\|$ denotes the spectral norm, then the consensus protocol (2.6) leads to mean square average-consensus for the agents in (2.2).

To prove this theorem, we introduce a so-called displacement vector as in Li and Zhang (2009) and Olfati-Saber and Murray (2004)

$$\delta(t) = x(t) - \mathbf{1}x(t) = (I - J)x(t),$$

(3.2)

where $\mathbf{1}$ stands for the column n-vector with all ones, $\alpha(t) = \text{avg}(x(t)) = \frac{1}{n} \mathbf{1}^T x(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t)$, $I$ is the $n \times n$ identity matrix, and $J = \frac{1}{n} \mathbf{1} \mathbf{1}^T$. It is easy to see that

$$\mathbf{1}^T \delta(t) = \sum_{i=1}^{n} x_i(t) - n \alpha(t) = 0, \quad t \geq 0.$$  

(3.3)

The dynamics of $\delta(t)$ are given by

$$d\delta(t) = c(t) [-\mathcal{L} \delta(t - \tau(t)) dt + (I - J) \Theta dW(t)],$$

(3.4)

where we have used the fact that both $\mathbf{1}^T \mathcal{L} \mathbf{1}$ and $\mathbf{1} \mathbf{1}^T$ are zero vectors. The consensus analysis relies on the Lyapunov function candidate

$$V(t) = \mathbf{1}^T \delta(t) \delta(t) = \|\delta(t)\|^2, \quad t \geq 0.$$  

The quantity $\lambda_2(\mathcal{L})$, called the algebraic connectivity of the graph $\mathcal{G}$, was originally introduced by Fiedler (1973) for undirected graphs and later extended to digraphs by Olfati-Saber and Murray (2004). The following property of the graph Laplacian $\mathcal{L}$ for strongly connected and balanced digraphs (see Theorem 7 of Olfati-Saber & Murray, 2004),

$$\mathbf{1}^T \delta \geq \lambda_2(\mathcal{L}) |\delta|^2, \quad \forall \mathbf{1}^T \delta = 0,$$  

(3.5)

plays an important role in ensuring that the protocol (2.6) leads to consensus of the agents in (2.2): The detailed proof for Theorem 3.1 is included in Section 5.

3.2. Networks with arbitrarily switching topology

In general, the network topology specified by the weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ can be time-varying due to node and link failures/creations, packet-loss, asynchronous consensus, reconfiguration, evolution, and flocking as pointed out by Olfati-Saber et al. (2007). To effectively model the dynamic changing of the network structures, we consider a collection of digraphs and introduce a general time-dependent switching signal, either deterministic or stochastic, to switch the network structures among the collection of digraphs.

We consider deterministic time-dependent switching in this subsection. Let $\mathcal{P}$ denote a finite index set and $\{\mathcal{G}_p : p \in \mathcal{P}\}$ a family of digraphs. Let $\lambda_2(\mathcal{L}_p)$ denote the second smallest eigenvalue of $\mathcal{L}_p = (\mathcal{L}_p + \mathcal{L}_p^T)/2$. If

$$\tau < \max_{p \in \mathcal{P}} \|\mathcal{L}_p\|^2,$$  

(3.7)

then the consensus protocol (2.6) leads to mean square average-consensus for the agents in (2.2) under any arbitrary deterministic switching signals.

3.3. Networks with Markovian switching topology

Hybrid systems driven by continuous-time Markov chains have long been used to model many practical systems where abrupt changes in their structures and parameters caused by phenomena such as component failures and repairs as pointed out in Mao and Yuan (2006). In this subsection, we consider the case where the switching signal is modeled by a continuous-time Markov chain. More specifically, let $\sigma : \mathbb{R}^+ \to \mathcal{P}$ be a right-continuous Markov chain with generator $\Gamma = (\gamma_{ij})_{n \times n}$ given by

$$P(\sigma(t + \Delta) = j | \sigma(t) = i) = \begin{cases} \gamma_{ij} + o(\Delta), & i \neq j, \\ 1 + \gamma_{ii} + o(\Delta), & i = j. \end{cases}$$

where $\Delta > 0$, $N$ is the cardinality of $\mathcal{P}$, $\gamma_{ij} \geq 0$ for $i \neq j$, and $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. Such switching signals are called Markovian switching signals.

**Theorem 3.3.** If all the conditions in Theorem 3.2 are satisfied, then the consensus protocol (2.6) leads to mean square average-consensus for the agents in (2.2) under any Markovian switching signals.

Note that both Theorems 3.2 and 3.3 require that each of the digraphs is strongly connected and balanced digraph, while the switching can be arbitrary. This is in accordance with the stability theory for switched systems, where stability under arbitrary switching must imply each of the subsystems itself is stable. Relaxed conditions such as joint connectivity together with some constraints on the switching signals can also lead to consensus under switching topology (see, e.g., Hong, Gao, Cheng, & Hu, 2007; Jadbabaie et al., 2003; Li & Zhang, 2010).
4. Simulation results

Consider dynamical networks of three agents. Fig. 2 shows three different topologies denoted by the family \( \{g_1, g_2, g_3\} \). While all digraphs in the figure have 0–1 weights, they are also all strongly connected and balanced. The intensity of the measurement noises satisfies \( a_0 = 1 \) for all \( a_j = 1 \) \((i, j \in 1) \). It can be calculated that \( \lambda_2(\hat{L}_1) = 1.5, \lambda_2(\hat{L}_2) = \lambda_2(\hat{L}_3) = 1, \) and \( \|L_1\| = 3, \|L_2\| = \|L_3\| = 9 \). We simulate two different situations. First, we consider a fixed network topology given by \( g_1 \). According to Theorem 3.1, if the communication delays are less than \( \lambda_c(\hat{L}_1)/(\|L_1\|) < 0.5 \), then the average-consensus protocol (2.6) will lead to mean square average-consensus for the network \( g_1 \). The initial states are chosen so that \( \text{avg}(x(0)) = 0 \). Average-consensus is confirmed by simulation as shown in Fig. 4, where we choose \( c(t) = 1/(t + 1) \). It is also shown in this figure that if we choose \( c(t) \equiv 1 \), the noises cannot be attenuated, and the states tend to diverge from each other, and average-consensus is not reached. Second, we consider the situation where the network topologies are randomly switching among the three different configurations in \( \{g_1, g_2, g_3\} \) according to a continuous-time Markov chain. It follows from Theorem 3.3 that if the delays are less than \( \min_{1 \leq i \leq 3} \lambda_2(\hat{L}_i)/(\max_{1 \leq i \leq 3} \|L_i\|)^2 < 1/9 = 0.1111 \), then mean square average-consensus is reached. This is confirmed by simulation as shown in Fig. 4, where we choose \( c(t) = 1/(t + 1) \). It is also shown in this figure that if we choose \( c(t) = 1/(t + 1)^2 \), while the noises seem to be over attenuated, the states are settled at different values, and again consensus is not reached. Therefore, condition (2.7) on the function \( c(t) \) plays a critical role in both attenuating the noise and achieving consensus.

5. Proofs

In this section, we present the proofs for the main theorems. Most of the proofs rely on a generalized Gronwall–Bellman–Halanay type inequality, which we will present in the next subsection.

5.1. A Gronwall–Bellman–Halanay type inequality

We establish a general Gronwall–Bellman–Halanay type inequality for estimating a function based on a delay differential inequality, which is essential to prove the main theorem and might be of independent interest as well, since it generalizes the classical Halanay inequality in the sense that it can be applied to non-autonomous systems.

Lemma 5.1. Let \( t_0 \) and \( r \) be nonnegative constants. Let \( m : [t_0 - r, \infty) \mapsto \mathbb{R}^+ \) be continuous and satisfy

\[
D^+ m(t) \leq \gamma(t) - \mu c(t)m(t) + \lambda c(t) \sup_{-r \leq s \leq 0} m(t + s),
\]

on \([t_0, \infty)\), where \( \gamma \) and \( c \) are piecewise continuous functions with \( \gamma(t) \geq 0 \) and \( c(t) \in (0, 1) \) for all \( t \geq 0 \), and \( \mu \) and \( \lambda \) are constants satisfying \( \mu > \lambda > 0 \). Then

\[
m(t) \leq m_0 \exp \left\{ -\rho \int_{t_0}^{t} c(s) \, ds \right\} + \int_{t_0}^{t} \exp \left\{ -\rho \int_{s}^{t} c(r) \, dr \right\} \gamma(s) \, ds,
\]

holds on \([t_0, \infty)\), where \( \rho > 0 \) is the root of \( -\rho = -\mu + \lambda e^{\rho r} \) and \( m_0 = \sup_{-r \leq s \leq 0} m(t_0 + s) \).

Fig. 3. Simulation results for fixed topology \( g_1 \): the left figure shows results for \( c(t) = 1/(t + 1) \) and \( r = 0.499 \); the right figure shows results for \( c(t) = 1 \) and \( r = 0.499 \).

Fig. 4. Simulation results for a Markovian switching topology among \( \{g_1, g_2, g_3\} \) driven by the generator \( \gamma' = \{-0.5 0.2 0.3; 0.4 -0.7 0.3; 0.2 0.55 -0.75\} \); the left figure shows results for \( c(t) = 1/(t + 1) \) and \( r = 0.111 \); the right figure shows results for \( c(t) = 1/(t + 1)^2 \) and \( r = 0.111 \).
Define
\[ u(t) = \begin{cases} \text{RHS of (5.2)}, & t \in [t_0, \infty), \\ m_0, & t \in [t_0 - r, t_0), \end{cases} \]
and
\[ t^* = t^*(\varepsilon) = \inf\{t \in [t_0 - r, \infty) : m(t) > u(t) + \varepsilon\}, \]
where \( \varepsilon > 0 \) is an arbitrary positive constant. Note that, for \( t \in [t_0 - r, t_0) \), we have
\[ m(t) \leq \sup_{-r \leq s \leq 0} m(t_0 + s) = m_0 = u(t). \]
Hence \( t^* \geq t_0 \). If \( t^* = \infty \) for all \( \varepsilon \), then the lemma is proved. Suppose \( t^* \in [t_0, \infty) \) for some \( \varepsilon > 0 \). By the definition of \( t^* \) and the continuity of \( m(t) \), we have \( m(t^*) = u(t^*) + \varepsilon \) and
\[ \sup_{-r \leq s \leq 0} m(t^* + s) \leq \sup_{-r \leq s \leq 0} u(t^* + s) + \varepsilon = e^{\varepsilon t} u(t^*) + \varepsilon < e^{\varepsilon t} m(t^*). \] (5.3)

Therefore, by (5.1) and (5.3),
\[ D^+[m - u](t^*) < (\alpha + \lambda e^{\varepsilon t} + \rho)c(t^*) m(t^*) - \rho c(t^*) \varepsilon < 0, \] (5.4)
which contradicts how \( t^* \) is defined. Therefore, we must have \( t^* = \infty \) for all \( \varepsilon > 0 \) and the lemma is proved. \( \square \)

5.2. Proof of Theorem 3.1

Applying Itô's formula (see, e.g., Revuz & Yor, 1999, Theorem 3.3, Chapter IV) to \( V(t) \) in view of (3.2), we have
\[ dV(t) = -2c(t)\delta^2(t) L \delta(t) dt + 2c(t) \delta^2(t) L \delta(t) dt + 2c(t)\delta^2(t) (I - J) \Theta ds, \] (5.5)
where
\[ C_0 = \text{trace}[(I - J)^2 \Theta \Theta^T]. \] (5.6)
Eq. (5.5) implies that
\[ E(V(t)) - E(V(\tau)) = -2 \int_{\tau}^{t} c(s) E[\delta^2(s) L \delta(s)] ds + 2 \int_{\tau}^{t} c(s) E[\delta^2(s) L \delta(s) - \delta(s - t + \tau))] ds + C_0 \int_{\tau}^{t} \delta^2(s) ds. \] (5.7)
Let \( m(t) = E(V(t)) \) for \( t \geq 0 \). Writing the above integral equation in differential form gives
\[ D^+ m(t) = -2c(t) E[\delta^2(t) L \delta(t)] + C_0 \delta^2(t) + 2c(t) E[\delta^2(t) L \delta(t) - \delta(t - \tau(t))]. \] (5.8)

Note that
\[ E[\delta^2(t) L \delta(t)] \geq \lambda_2(\hat{L}) E(V(t)), \]
and
\[ 2E[\delta^2(t) L \delta(t) - \delta(t - \tau(t))] \leq \varepsilon E(V(t)) + \frac{1}{\varepsilon} E[L \delta(t) - \delta(t - \tau(t))]^2, \]
where \( \varepsilon > 0 \) is a constant to be determined later. On the other hand, Eq. (3.4) implies that
\[ E[\|\delta(t) - \delta(t - \tau(t))\|^2] \leq (1 + \beta) E\left[ \int_{t^-}^{t} c(s) L^2 \delta(s - \tau(s)) ds \right]^2 + \left(1 + \frac{1}{\beta}\right) E \int_{t^-}^{t} c(s) L(I - J) \Theta ds. \]

Therefore, \( \delta(t) \) is bounded. Then, putting the above three estimates together into (5.8) and setting \( \varepsilon = \tau \|L\|^2 \), we obtain
\[ D^+ m(t) \leq -2\lambda_2(\hat{L}) c(t) m(t) + 2(1 + \beta) \tau \|L\|^2 c(t) \sup_{-r \leq s \leq 0} m(t + s) + \left(1 + \frac{1}{\beta}\right) C_0 \int_{t^-}^{t} \delta^2(r) ds + C_0 \delta^2(t), \quad t \geq \tau. \]

Inequality (3.1) implies that we can choose \( \beta > 0 \) sufficiently small such that \( 2\lambda_2(\hat{L}) > (2 + \beta) \tau \|L\|^2 \). Therefore, there exists \( \rho > 0 \) such that \( -2\lambda_2(\hat{L}) + (2 + \beta) \tau \|L\|^2 e^{\rho \tau} = -\rho \). Lemma 5.1 shows that
\[ m(t) \leq m_0 \exp\left\{ -\rho \int_{\tau}^{t} c(s) ds \right\} + \int_{\tau}^{t} \exp\left\{ -\rho \int_{\tau}^{s} c(r) dr \right\} \gamma(s) ds, \] (5.9)
on \( [\tau, \infty) \) with
\[ m_0 = \sup_{-r \leq s \leq t} m(s), \]
\[ \gamma(t) = \left(1 + \frac{1}{\beta}\right) C_0 \int_{t^-}^{t} \delta^2(r) dr + C_0 \delta^2(t). \]
It follows from (2.7) that
\[ \exp\left\{ -\rho \int_{\tau}^{t} c(s) ds \right\} \to 0, \] (5.10)
as \( t \to \infty \). On the other hand, note that, for all \( t \geq T \geq \tau \), we have
\[ \int_{\tau}^{t} \exp\left\{ -\rho \int_{\tau}^{s} c(r) dr \right\} \int_{s}^{t} \delta^2(r) dr ds \]
\[ = \int_{\tau}^{T} \exp\left\{ -\rho \int_{\tau}^{s} c(r) dr \right\} \int_{s}^{t} \delta^2(r) dr ds \]
\[ \to 0, \] (5.11)
It follows from both $E(V(t)) \to 0$ and $E|\alpha(t) - x^*|^2 \to 0$ that $E|x_i(t) - x^*|^2 \to 0$, as $t \to \infty$. Moreover, it is easy to check that $E|x_i(t)|^2 \leq 2E\alpha^2(t) < \infty, \quad t \geq 0$.

Therefore, mean square consensus is reached and the proof is complete. \hfill \Box

5.3. Proof of Theorem 3.2

Let $\alpha(t), t \geq 0$, be a given switching signal. Then $V(t)$ can serve as a common Lyapunov function for the displacement dynamics

$$d\delta(t) = c(t)[-L_{\sigma(t)}(t - \tau(t))dt + (I - J)\theta dW(t)],$$

(5.17)

which follows from (3.6). Repeating the same argument as in the proof of Theorem 3.1, we can obtain

$$\begin{align*}
D^m_m(t) & \leq -2\min_{p \in P} \lambda_2(\hat{L}_p)c(t)m(t) + (2 + \beta)\tau \max_{p \in P} \|L_p\|^2 c(t) \sup_{-2r \leq \delta \leq 0} m(t + s) \\
& \quad + \left(1 + \frac{1}{\beta}\right) c_0^2 \int_0^t c^2(r) dr + c_0^2(t), \quad t \geq \tau.
\end{align*}$$

Since we can choose $\beta > 0$ such that $2\min_{p \in P} \lambda_2(\hat{L}_p) > (2 + \beta)\tau \max_{p \in P} \|L_p\|^2$, there exists $\rho > 0$ such that $-2\lambda_2(\hat{L}) + (2 + \beta)\tau \|L\|^2e^{\rho \tau} = -\rho$. Lemma 5.1 implies that the same estimate (5.9) holds and the rest of the proof is essentially the same as the proof of Theorem 3.1. \hfill \Box

5.4. Proof of Theorem 3.3

The proof is essentially the same as that of Theorem 3.2, except that we should apply the generalized Itô’s formula Mao and Yuan (2006, Theorem 1.45) due to the Markovian switching. Since a common Lyapunov function $V = |\delta|^2$ is used, we obtain the same integral equation as (5.7) for the expectation $E(V(t))$ with $L_{\sigma(t)}$ in place of $L$. The rest of the proof is the same. \hfill \Box

6. Conclusions

We have investigated the average-consensus problem of networked multi-agents systems subject to measurement noises. A time-varying consensus protocol that takes into the account both the measurement noises and general time-varying communication delays has been proposed. We have considered general networks with fixed topology, with arbitrary deterministic switching topology, and with Markovian switching topology. For each of these three cases, we have obtained sufficient conditions under which the proposed consensus protocol leads to mean square average-consensus. The sufficient conditions provide explicit delay upper bounds guaranteeing mean square average-consensus in terms of the graph Laplacians. We conclude our paper by pointing out some possible future research.

First, this paper focuses only on the mean square average-consensus. For future research, it would be interesting to propose consensus protocols to reach both general $p$th moment consensus and almost sure consensus.

Moreover, while characterizing exact delay bounds for stability of general linear time-delay systems remain challenging issues, obtaining exact delay upper bounds for consensus are of practical importance. In Olfati-Saber and Murray (2004), the exact delay bound for average-consensus is obtained for a fixed digraph.
network with a single constant delay, and in Bliman and Ferrari-Trecate (2008), exact delay bounds for average consensus are obtained for a fixed undirected network with a single time-varying delay or multiple constant delays. It would be interesting to know if these bounds are still optimal under the measurement noises considered in the current paper. If so, one might be able to obtain necessary and sufficient conditions for mean square consensus despite both measurement noises and communication delays following the treatment in Li and Zhang (2009) for networked systems without communication time-delays. It would also be of practical importance to deal with more general (nonuniform, multiple, time-varying or distributed) delays and communication noises occurring at the same time.

Finally, when considering switching topology network, it is possible that some modes of the network fail to have a strongly connected and/or balanced network. These modes may be characterized as unstable modes and it would be interesting to investigate if the ideas from stability analysis of switched systems with both stable and unstable subsystems (see, e.g. Liu, Liu, & Xie, 2009) can be borrowed here to investigate consensus problems under such situations.

Acknowledgments

The authors are grateful to the anonymous reviewers for their detailed comments and suggestions, which have helped to improve the quality of this paper.

References

Wei-Chau Xie received his B.Sc. degree in Precision Engineering from Shanghai Jiao-Tong University, Shanghai, China, in 1984 and his M.Sc. and Ph.D. degrees in Civil Engineering from the University of Waterloo, Waterloo, Ontario, Canada, in 1987 and 1990, respectively. He was a Stress Analyst and Design Engineer for the Atomic Energy of Canada Limited, Mississauga, Ontario, Canada, from September 1990 to December 1991. He joined the Department of Civil Engineering, University of Waterloo, Waterloo, Ontario, Canada, as an Assistant Professor in January 1992, where he became an Associate Professor and a Full Professor in 1997 and 2002, respectively. His principal areas of research include dynamic stability of structures, structural dynamics and random vibration, nonlinear dynamics, and stochastic mechanics, seismic analysis and design of engineering structures, reliability and safety analysis of engineering systems. He is the author of the books Dynamic Stability of Structures (Cambridge University Press, 2006), Differential Equations for Engineers (Cambridge University Press, 2010). He is currently an Associate Editor of the ASME Journal of Applied Mechanics and serves on the Editorial Board of several other professional journals. Dr. Xie is a licensed Professional Engineer (PEng) in Ontario, Canada, and a member of the American Society of Mechanical Engineers. He won the Distinguished Teacher Award from the University of Waterloo in 2007.

Hongtao Zhang received his B.Sc. degree in Power and Mechanical Engineering from Wuhan University, Wuhan, China, in 2001, his M.Sc. degree in Control Theory and Control Engineering from Huazhong University of Science and Technology, Wuhan, China, in 2004, and his Ph.D. degree in Electrical and Computer Engineering from the University of Waterloo, Waterloo, Ontario, Canada, in 2010. He is currently a Postdoctoral Fellow in Applied Mathematics at the University of Waterloo. His research interests include hybrid dynamical systems, chaos synchronization and network synchronization, and their applications to secure communication, biological systems, etc.