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Generalized Solutions to Hybrid Systems with Delays

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Abstract— **We propose a framework that allows to study hybrid systems with delays through generalized concepts of solutions. Such generalized solutions have proved to be suitable for the analysis and design of hybrid control systems. Extending these concepts to systems with delays is motivated by control applications where hybrid feedback control laws play an important role but delays are often inevitable. Our first step characterizes a suitable phase space that is equipped with a suitable notion of convergence. The space of hybrid memory arcs under graphical convergence is considered. Regularity conditions on the hybrid data are then formulated in terms of the graphical topology. The new formulation enables two basic results regarding existence and well-posedness of solutions being established.**

I. Introduction

Hybrid dynamical systems, which combine the features of classical continuous-time systems and discrete-time systems, have been attracting significant research interest from different research communities, such as computer science, control engineering, and dynamical systems, over the past two decades. Much consideration has been placed on the modeling and solution definitions for hybrid systems. Some notable references include [1]–[10]. See the recent tutorial article [11] and book [12] and extensive references therein for detailed expositions on hybrid systems, including references to different points of view about hybrid systems.

Motivated by robust stability issues in hybrid control systems, generalized solutions have been considered for hybrid systems [12]–[14]. These generalized solutions not only allow the solutions to be defined on a generalized domain, called hybrid time domain, but also allow the right-hand sides of the system to be set-valued. Using these concepts of generalized solutions, it has been shown that, to a very large extent, most of the stability analysis tools and results for classical nonlinear systems can be extended to hybrid systems [11]–[17].

As noted in [11], one of the main motivations to consider hybrid systems in the set-valued setting has been to provide tools to study robust stability in control engineering applications where small measurement noise is present. For the same consideration, we note that the many feedback control systems encountered in control engineering often process another feature, namely time delays, which can cause instability and/or loss of robustness [18]. In most control applications, delays are often inevitable, largely due to the fact that it takes time to sense and process information, to make control decisions, and to execute these decisions [19].

Solution definition and asymptotic stability have been addressed for hybrid systems with delays, which include, e.g., impulsive delay systems [20]–[22], switching delay systems [22]–[25], and systems in more abstract formulation [26], [27] following that of [10]. However, to the best of our knowledge, general results on robust asymptotic stability for hybrid systems with delays are still missing from the literature. This is reflected by the fact that almost all the current tools and results rely on standard concepts like uniform convergence, while this concept is not well-suited to handle discontinuities caused by jumps in hybrid systems, especially when structural properties of the solutions are concerned. It is from this perspective that we believe it is necessary to formulate hybrid systems with delays using generalized concepts of solutions.

In this paper, we propose a framework that extends the generalized concepts of hybrid solutions to systems with delays. The main contributions include a generalized formulation of hybrid systems with memory and an explicit consideration of the phase space equipped with the graphical convergence topology. This formulation would allow the use of graphical convergence concepts to solutions of such systems to study their structural properties. Regularity conditions on the hybrid data are then formulated in terms of the graphical topology. Based on such regularity assumptions, two main results, namely the basic existence and nominal well-posedness results, have been established.

II. Hybrid systems with memory

A. Solution description

For brevity, we write $\mathbb{R}_{\geq 0}$ for $[0, \infty)$, $\mathbb{R}_{\leq 0}$ for $(-\infty, 0]$, $\mathbb{Z}_{\geq 0}$ for $\{0,\, 1,\, 2,\, \cdots\}$, and $\mathbb{Z}_{\leq 0}$ for $\{0,\, -1,\, -2,\, \cdots\}$. We start with the definition of hybrid time domains and hybrid arcs [11], [12] for hybrid systems and generalize them in order to define hybrid systems with memory.

Definition 1: A subset $E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact hybrid (positive) time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$. The set E is called a *hybrid (positive) time domain* if, for all $(T, J) \in E$, $E \cap ([0, T] \times \{0,\, 1,\, \cdots,\, J\})$ is a compact hybrid time domain.

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Definition 2: A subset $E \subseteq \mathbb{R}_{\le 0} \times \mathbb{Z}_{\le 0}$ is called a *compact hybrid memory domain* if

$$E = \bigcup_{j=1}^{J} ([t_j, t_{j-1}], -j+1)$$

for some finite sequence of times $t_J \le \cdots \le t_{J-1} \le \cdots \le t_1 \le t_0 = 0$. The set E is called a *hybrid memory domain* if, for all $(T, J) \in \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0}$, $E \cap ([-T, 0] \times \{-J, -J+1, \cdots, 0\})$ is a compact hybrid memory domain. A subset $S \subseteq \mathbb{R} \times \mathbb{Z}$ is called a *hybrid time domain with memory* if it can be written as $S = E_1 \cup E_2$, where E_1 is a hybrid memory domain and E_2 is a hybrid (positive) time domain.

Note that there is a slight difference between the definitions of a hybrid time domain and a hybrid memory domain: E is a hybrid time domain if it is a union of a finite or infinite sequence of sets of the form $[t_j, t_{j+1}] \times \{j\} \subseteq \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0}$, with the last interval (if existent) possibly of the form $[t_j, T)$ with T finite or $T = \infty$, while E is a hybrid memory domain if it is a union of a finite or infinite sequence of sets of the form $[t_j, t_{j-1}] \times \{-j+1\} \subseteq \mathbb{R}_{\le 0} \times \mathbb{Z}_{\le 0}$, with the last interval (if existent) possibly of the form $(T, t_{j-1}]$ only if $T = -\infty$. This difference stems from the fact that for hybrid memory domains we consider any (T, J) in $\mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0}$.

Definition 3: A *hybrid arc (with memory)* is a pair consisting of a domain $dom\, x$, which is a hybrid time domain (with memory), and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(\cdot, j)$ is locally absolutely continuous on $I^j = \{t : (t, j) \in dom\, x\}$ for each $j \in \mathbb{Z}$ such that I^j has nonempty interior. We sometimes break the domain of x into $\operatorname{dom}_{\ge 0}(x) := \{(t, j) \in \operatorname{dom} x : t + j \ge 0\}$ and $\operatorname{dom}_{\le 0}(x) := \{(t, j) \in \operatorname{dom} x : t + j \le 0\}$ and call them *positive domain* and *memory domain* of x, respectively. A *hybrid memory arc* is a special case of a hybrid arc, which can be defined as a pair consisting of a hybrid memory domain $dom\, x$ and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(\cdot, j)$ is locally absolutely continuous on $I^j = \{t : (t, j) \in dom\, x\}$ for each $j \in \mathbb{Z}$ such that I^j has nonempty interior. We simply use the term *hybrid arc* if we do not have to distinguish between the above two hybrid arcs.

Since all hybrid time domains consider in this paper will include a memory domain, we shall simply refer to them as hybrid time domains. Given a hybrid time domain E, define

- $\sup_t E = \sup \{t \in \mathbb{R} : \exists j \in \mathbb{Z} \text{ s.t. } (t, j) \in E\}$;
- $\sup_j E = \sup \{j \in \mathbb{Z} : \exists t \in \mathbb{R} \text{ s.t. } (t, j) \in E\}$.

That is, the operators $\sup_t E$ and $\sup_j E$ on a hybrid time domain E return the supremum of the t and j coordinates of the points in its positive domain, respectively. Furthermore, $\sup E = (\sup_t E, \sup_j E)$ and $\text{length}_{\ge 0}(E) = \sup_t E + \sup_j E$. Given $j \in \mathbb{Z}$, we use I^j to denote the set $\{t : (t, j) \in E\}$, which is a set that can be empty, a single point, or an interval with nonempty interior.

For convenience, we denote the collection of all hybrid memory arcs by $\mathcal{M}$ and the collection of all hybrid memory domains by $\operatorname{dom}(\mathcal{M})$. In addition, we denote the collection of all hybrid arcs with memory by $\mathcal{X}$ and the collection of all hybrid time domains with memory by $\operatorname{dom}(\mathcal{X})$.

Definition 4: Given a hybrid arc $x \in \mathcal{X}$ and any $(t, j) \in \operatorname{dom} x$, we define an operator $\mathcal{A}_{[t,j]}$ that maps x to $\mathcal{A}_{[t,j]}x$, a hybrid memory arc in $\mathcal{M}$ given by

$$\mathcal{A}_{[t,j]}x(s, k) = x(t+s, j+k),$$

for all $(s, k) \in \operatorname{dom} \mathcal{A}_{[t,j]}x$, where $\operatorname{dom} \mathcal{A}_{[t,j]}x \in \operatorname{dom}(\mathcal{M})$ is defined by $(s, k) \in \operatorname{dom} \mathcal{A}_{[t,j]}x$ if and only if $(t+s, j+k) \in \operatorname{dom} x$. The hybrid memory arc $\mathcal{A}_{[t,j]}x$ can be seen as the memory of x at (t, j).

Definition 5: Data of a *hybrid system with memory* in $\mathbb{R}^n$ consists of four elements:

- a set $C \subseteq \mathbb{R}^n$, called the *flow set*;
- a set-valued functional $\mathcal{F} : \mathcal{M} \rightrightarrows \mathbb{R}^n$, called the *flow map*;
- a set $D \subseteq \mathbb{R}^n$, called the *jump set*;
- a set-valued functional $\mathcal{G} : \mathcal{M} \rightrightarrows \mathbb{R}^n$, called the *jump map*.

Here we use $\rightrightarrows$ to denote that $\mathcal{F}$ and $\mathcal{G}$ are set-valued mapping from $\mathcal{M}$ to $\mathbb{R}^n$. Formal definitions and properties of such mappings will be postponed to Section II-D. A hybrid system with memory defined by the data above will be represented by the notation $\mathcal{H}_{\mathcal{M}} = (C, \mathcal{F}, D, \mathcal{G})$. We use the subscript $\mathcal{M}$ to emphasize that it carries a memory in the space $\mathcal{M}$.

Definition 6: A hybrid arc $x \in \mathcal{X}$ is a *solution to the hybrid system* $\mathcal{H}_{\mathcal{M}}$ if $x(0, 0) \in C \cup D$ and:

(S1) for all $j \in \mathbb{Z}_{\ge 0}$ and almost all t such that $(t, j) \in \operatorname{dom} x$,

$$x(t, j) \in C, \quad \dot{x}(t, j) \in \mathcal{F}(\mathcal{A}_{[t,j]}x), \tag{1}$$

(S2) for all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j+1) \in \mathcal{G}(\mathcal{A}_{[t,j]}x). \tag{2}$$

The solution x is called *nontrivial* if its positive domain $\operatorname{dom}_{\ge 0}(x)$ has at least two points. It is called *complete* if $\operatorname{dom}_{\ge 0}(x)$ is unbounded. It is called *maximal* if there does not exist another solution y to $\mathcal{H}_{\mathcal{M}}$ such that $\operatorname{dom} x$ is a proper subset of $\operatorname{dom} y$ and $x(t, j) = y(t, j)$ for all $(t, j) \in \operatorname{dom} x$. The set of all maximal solutions to $\mathcal{H}_{\mathcal{M}}$ is denoted by $\mathcal{S}_{\mathcal{H}_{\mathcal{M}}}$.

B. Preliminaries on set-valued analysis

We need a few regularity conditions on the hybrid data to establish certain results on basic existence and well-posedness for $\mathcal{H}_{\mathcal{M}}$. To formulate these regularity conditions, we need to recall a few definitions from set-valued analysis. The set-valued analysis concepts recalled here for mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$ can be found in Chapter 5 of [28] (see also Chapter 5 of [12] for set-valued analysis in the hybrid systems setting).

Definition 7 (Set convergence): Consider a sequence of sets $\{H_i\}_{i=1}^{\infty}$ in $\mathbb{R}^n$. The *outer limit* of the sequence, denoted by $\limsup_{i\to\infty} H_i$ is the set of all $x \in \mathbb{R}^n$ for which there exists a subsequence $x_{i_k} \in S_{i_k}$, $k = 1, 2, \cdots$, such that $x_{i_k} \to x$. The *inner limit* of $\{H_i\}_{i=1}^{\infty}$, denoted by $\liminf_{i\to\infty} H_i$, is the set of all $x \in \mathbb{R}^n$ for which there exists a sequence $x_i \in S_i$ such that $x_i \to x$. The *limit* of $\{H_i\}_{i=1}^{\infty}$ exists if $\limsup_{i\to\infty} H_i = \liminf_{i\to\infty} H_i$ and it is then given by $\lim_{i\to\infty} H_i = \limsup_{i\to\infty} H_i = \liminf_{i\to\infty} H_i$.

Definition 8 (Set-valued mappings): Let $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$. Its domain, range, and graph are defined by $\operatorname{dom} S := \{x : S(x) \neq \emptyset\}$, $\operatorname{rge} S := \{y : \exists\, x \text{ s.t. } y \in S(x)\}$, and $\operatorname{gph} S := \{(x, y) : y \in S(x)\}$, respectively. The mapping S is called *outer semicontinuous* at $x \in \mathbb{R}^m$ if for every sequences of points $x_i \to x$ and $y_i \to y$ with $y_i \in S(x_i)$, we have $y \in S(x)$. It is *locally bounded* at $x \in \mathbb{R}^m$ if there exists a neighborhood U_x of x such that the set $S(U_x) := \bigcup_{x' \in U_x} S(x') \subseteq \mathbb{R}^n$ is bounded. It is said to be outer semicontinuous (respectively, locally bounded) relative to a set $H \subseteq \mathbb{R}^m$, if the mapping defined by $S(x)$ for $x \in H$ and by $\emptyset$ for $x \notin H$ is outer semicontinuous (respectively, locally bounded) at each $x \in H$.

By convention, a mapping S is said to be outer semicontinuous or locally bounded if it is so relative to its domain.

Definition 9 (Tangent cone): The tangent cone to a set $X \subseteq \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted by $T_X(x)$, is the set of all points $v \in \mathbb{R}^n$ for which there exist sequences $v_i \in X$ and $\tau_i \in \mathbb{R}_{\geq 0}$ such $v_i \to v$, $\tau_i \to 0$, and $\lim_{i\to\infty}(v_i - v)/\tau_i = x$.

Definition 10 (Graphical convergence): A sequence of mappings $S_i : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to *converge graphically* to some $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ if $\lim_{i\to\infty} \operatorname{gph} S_i = \operatorname{gph} S$. In particular, a sequence of hybrid arcs $\varphi_i : \operatorname{dom}\varphi_i \to \mathbb{R}^n$ converges graphically to some $\varphi : \mathbb{R}^2 \rightrightarrows \mathbb{R}^n$ if $\lim_{i\to\infty} \operatorname{gph}\varphi_i = \operatorname{gph}\varphi$. We use $\xrightarrow{\text{gph}}$ to denote graphical convergence.

In the above definition, since a set-valued mapping can be uniquely characterized by its graph, the limit $\operatorname{gph} S$ of the sequence of convergent sets $\operatorname{gph} S_i \subseteq \mathbb{R}^m \times \mathbb{R}^n$ always defines a set-valued map S, which is called the graphical limit of S_i. In general, even for a sequence of single-valued functions, its graphical limit can be set-valued. In particular, the graphical limit of a sequence of hybrid arcs can be set-valued and in general may not be an hybrid arc. We will be interested in graphical convergence of hybrid arcs that are bounded in following sense, a concept that was used in [11].

Definition 11 (Local eventually boundedness): A sequence of hybrid arcs $\varphi_i : \operatorname{dom}\varphi_i \to \mathbb{R}^n$ is *locally eventually bounded* if for any $m > 0$, there exists $i_0 > 0$ and a compact $K \subseteq \mathbb{R}^n$ such that for all $i > i_0$, all $(t, j) \in \operatorname{dom}\varphi_i$ with $|t + j| < m$, $\varphi_i(t, j) \in K$.

One basic property for a hybrid system to have is its nominal well-posedness [12]. The general idea behind a hybrid system $\mathcal{H}_{\mathcal{M}}$ being nominally well-posed is that the limit of a graphically convergent sequence of solutions to $\mathcal{H}_{\mathcal{M}}$ having a mild boundedness property should also be a solution to $\mathcal{H}_{\mathcal{M}}$, while a graphically convergent sequence of solutions to $\mathcal{H}_{\mathcal{M}}$ without that boundedness property should lead to a solution that blows up in finite time.

Definition 12 (Nominal well-posedness): A hybrid system $\mathcal{H}_{\mathcal{M}}$ is said to be *nominally well-posed* if the following properties hold: for every graphically convergent sequence $\{x_i\}_{i=1}^{\infty}$ of solutions to $\mathcal{H}_{\mathcal{M}}$ with $A_{[0,0]}x_i \xrightarrow{\text{gph}} \varphi$ for some $\varphi \in \mathcal{M}$,

(a) if the sequence $\{x_i\}_{i=1}^{\infty}$ is locally eventually bounded, then its graphical limit x is a solution to $\mathcal{H}_{\mathcal{M}}$ with $\mathcal{A}_{[0,0]}x = \varphi$ and $\operatorname{length}_{\geq 0}(\operatorname{dom} x) = \lim_{i\to\infty} \operatorname{length}_{\geq 0}(\operatorname{dom} x_i)$;

(b) if the sequence $\{x_i\}_{i=1}^{\infty}$ is not locally eventually bounded, then there exists a number $m \in (0, \infty)$ for which there exists $(t_i, j_i) \in \operatorname{dom}_{x_i}$, $i = 1, 2, \cdots$, such that $\limsup_{i\to\infty} |x_i(t_i, j_i)| = \infty$ and the graphical limit x restricted to $\{(t, j) \in \operatorname{dom} x : t + j < m\}$ is maximal solution to $\mathcal{H}_{\mathcal{M}}$ with $\operatorname{length}_{\geq 0}(\operatorname{dom} x) = m$ and $\limsup_{t\to\sup_t(\operatorname{dom} x)} |x(t, \sup_j(\operatorname{dom} x))| = \infty$.

One important consequences of a hybrid system $\mathcal{H}_{\mathcal{M}}$ being nominally well-posed is that its *graphical* limit set in $\mathcal{H}_{\mathcal{M}}$ possesses a certain invariance property, which will enable us to further develop invariance principles for such systems.

C. The space $(\mathcal{M}, \mathbf{d})$

The space of all hybrid memory arcs is not a vector space, since, by definition, different hybrid arcs can have different domains and are hence not amenable to additions. In this section, we recall from [28] a quantity that characterizes graphical convergence of outer semicontinuous mappings from $\mathbb{R}^2$ to $\mathbb{R}^n$ and use this distance to define a metric on $\mathcal{M}$. Let osc-maps$_{\not\equiv\emptyset}(\mathbb{R}^2; \mathbb{R}^n)$ denote the collection of all outer semicontinuous set-valued functions from $\mathbb{R}^2$ to $\mathbb{R}^n$ that are not constantly $\emptyset$, i.e., each of their domains include at least one point. Given $\rho > 0$, for each pair $S, T \in$ osc-maps$_{\not\equiv\emptyset}(\mathbb{R}^2; \mathbb{R}^n)$, define

$$\mathbf{d}_\rho(S, T) := \max_{|z| \leq \rho} \left| d(z, \operatorname{gph} S) - d(z, \operatorname{gph} T) \right|.$$

where $d(z, H)$ for $z \in \mathbb{R}^{n+2}$ and $H \subseteq \mathbb{R}^{n+2}$ is defined by $\inf_{w\in H} |w - z|$. Furthermore, define

$$\mathbf{d}(S, T) := \int_0^\infty \mathbf{d}_\rho(S, T) e^{-\rho} d\rho,$$

which is called the *graphical distance* for mappings in osc-maps$_{\not\equiv\emptyset}(\mathbb{R}^2; \mathbb{R}^n)$. This graphical distance indeed characterizes graphical convergence of mappings in osc-maps$_{\not\equiv\emptyset}(\mathbb{R}^2; \mathbb{R}^n)$ and the space osc-maps$_{\not\equiv\emptyset}(\mathbb{R}^2; \mathbb{R}^n)$, equipped with the graphical distance $\mathbf{d}$, becomes a metric space, which is formally stated as follows.

Theorem 1 (Rockafellar and Wets [28], Theorem 5.50): A sequence $S_i \in$ osc-maps$_{\not\equiv\emptyset}(\mathbb{R}^2; \mathbb{R}^n)$ converges graphically to S if and only $\mathbf{d}(S_i, S) \to 0$. Moreover, the space (osc-maps$_{\not\equiv\emptyset}(\mathbb{R}^2; \mathbb{R}^n), \mathbf{d}$) is a separable, locally compact, and complete metric space.

Though single-valued, a hybrid memory arc $\varphi : \operatorname{dom}\varphi \to \mathbb{R}^2$ can also be seen as a set-valued mapping from $\mathbb{R}^2$ to $\mathbb{R}^n$. It can be easily shown that any hybrid memory arc is outer semicontinuous. To see this, we note that the last interval (to the left) of a hybrid memory domain is of the form $[T, t_j]$ or $(-\infty, t_j]$. Therefore, the graph of a hybrid memory arc is always closed, which implies its outer semicontinuity [28, Theorem 5.7(a)] as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^n$. Therefore, the graphical distance, restricted to $\mathcal{M}\times\mathcal{M}$, defines a metric space $(\mathcal{M}, \mathbf{d})$.

Corollary 1: The space $(\mathcal{M}, \mathbf{d})$ is a separable metric space.

As noted earlier, $\mathcal{M}$ is not closed under graphical convergence in the sense that the limit of a sequence of graphically convergent hybrid memory arcs may not be a hybrid memory arc. We will be relying on certain boundedness properties, namely local eventual boundedness, of hybrid arcs to show that a graphical limit of a sequence of hybrid memory arcs is indeed an hybrid memory arc.

While the graphical distance $\mathbf{d}$ fully characterizes graphical convergence on $\mathcal{M}$, in some cases it is not convenient to directly use it. There are other quantities can be used in company with $\mathbf{d}$ to characterize closeness of two hybrid memory arcs as shown in the following.

Uniform distance: Given $\varphi, \psi \in \mathcal{M}$ such that $\operatorname{dom}\varphi = \operatorname{dom}\psi$, we can define

$$\|\varphi - \psi\| = \sup_{(t,j)\in \operatorname{dom}\varphi} |\varphi(t,j) - \psi(t,j)|,$$

which is called the *uniform distance* between φ and ψ.

(τ, ε)***-closeness:*** For hybrid arcs with possibly different domains, a notion called (τ, ε)-closeness [11] can be used to measure their closeness. Here we modify the notion slightly to use it on hybrid memory arcs. Given $\varphi, \psi \in \mathcal{M}$ and $\tau, \varepsilon > 0$, φ and ψ are said to be (τ, ε)-*closeness* if

(a) for all $(t, j) \in \operatorname{dom}\varphi$ with $|t + j| \le \tau$, there exists some s such that $(s, j) \in \operatorname{dom}\psi$, $|t - s| \le \varepsilon$, and $|\varphi(t, j) - \psi(s, j)| \le \varepsilon$;

(b) for all $(t, j) \in \operatorname{dom}\psi$ with $|t + j| \le \tau$, there exists some s such that $(s, j) \in \operatorname{dom}\varphi$, $|t - s| \le \varepsilon$, and $|\varphi(t, j) - \psi(s, j)| \le \varepsilon$.

(ρ, ε)***-closeness of graphs:*** More generally, we can also use the following to characterize the closeness of the graphs of two hybrid arcs $\varphi, \psi \in \mathcal{M}$: there exists $\rho > 0$ and $\varepsilon > 0$ such that $\operatorname{gph}\varphi \cap \rho\mathbb{B} \subseteq \operatorname{gph}\psi + \varepsilon\mathbb{B}$ and $\operatorname{gph}\psi \cap \rho\mathbb{B} \subseteq \operatorname{gph}\varphi + \varepsilon\mathbb{B}$. If the above holds, we say that the graphs of φ and ψ are (ρ, ε)-*close*.

The following proposition shows how the uniform distance, (τ, ε)-closeness, and (ρ, ε)-closeness can be used to provide an estimate of the graphical distance.

Proposition 1: Consider two hybrid memory arcs $\varphi, \psi \in \mathcal{M}$. The following two statements hold:

(a) If $\operatorname{dom}\varphi = \operatorname{dom}\psi$, then

$$\mathbf{d}(\varphi, \psi) \le \|\varphi - \psi\|. \tag{3}$$

(b) If the graphs of $\varphi, \psi \in \mathcal{M}$ are (ρ, ε)-close with $\varepsilon \ge 0$ and $\rho \ge \bar\rho + m$, where $m = \max(|\varphi(0,0)|, |\psi(0,0)|)$ and $\bar\rho \ge 0$, then

$$\mathbf{d}(\varphi, \psi) \le \varepsilon(1 - e^{-\bar\rho}) + (\bar\rho + m + 1)e^{-\bar\rho}. \tag{4}$$

In particular, given $\varepsilon \ge 0$, if the graphs of φ and ψ are (ρ, ε)-close for all $\rho \ge 0$, then

$$\mathbf{d}(\varphi, \psi) \le \varepsilon. \tag{5}$$

On the other hand, if $\mathbf{d}(\varphi, \psi) \le \delta$ for some $\delta \ge 0$, then the graphs of φ and ψ are $(\rho, \delta e^{\rho})$-close, for any $\rho \ge 0$.

(c) If $\varphi, \psi \in \mathcal{M}$ are (τ, ε)-close with $\varepsilon \ge 0$ and $\tau \ge \bar\rho + m$, where $m = \max(|\varphi(0,0)|, |\psi(0,0)|)$ and $\bar\rho \ge 0$, then

$$\mathbf{d}(\varphi, \psi) \le \sqrt{2}\varepsilon(1 - e^{-\bar\rho}) + (\tau + m + 1)e^{-\bar\rho}. \tag{6}$$

On the other hand, if $\mathbf{d}(\varphi, \psi) \le \delta$ for some $\delta \ge 0$, then φ and ψ are $(\tau, \delta e^{r})$-close, for any $\tau \ge 0$ and $r = \max(\sup_{|t+j|\le\tau} |\varphi(t, j)|, \sup_{|t+j|\le\tau} |\varphi(t, j)|)$.

In general, we cannot expect to bound the uniform distance of two hybrid memory arcs by their graphical distance, since their graphical distancing being finite may not even guarantee that their uniform distance is finite.

The following lemma shows that given a hybrid arc $x \in \mathcal{X}$, its memory $\mathcal{A}_{[t,j]}x$ at (t, j) can be regarded as a continuous function from I^j to $(\mathcal{M}, \mathbf{d})$, for each $j \in \mathbb{Z}$ such that I^j has nonempty interior.

Lemma 1: Let $x \in \mathcal{X}$ be a hybrid arc with memory. For each $j \in \mathbb{Z}$ such that I^j has nonempty interior, the function $a : I^j \to \mathcal{M}$ defined by $a(t) := \mathcal{A}_{[t,j]}x$ is uniformly continuous on each compact subinterval of I^j.

This lemma shows that by considering the graphical convergence topology on $\mathcal{M}$, we can establish the above continuity property of $\mathcal{A}_{[t,j]}x$, which is a fundamental property that is needed for studying functional differential equations [29]. If one considers the uniform topology, it is shown in the context of impulsive delay differential equations [20] that such continuity is entirely missing and one needs to impose a certain composite continuity property, which is a continuity property of the composite function $\mathcal{F}(\mathcal{A}_{[t,j]}x)$ in t. For consideration of robustness and well-posedness, in particular, nominal well-posedness to be considered in this paper, it is more convenient to formulate the regularity conditions on the hybrid data in terms of the graphical convergence topology.

D. Regularity assumptions on hybrid data of $\mathcal{H}_{\mathcal{M}}$

In this section, we introduce a few regularity conditions on the hybrid data, especially on $\mathcal{F}$ and $\mathcal{G}$, which are regarded as set-valued mappings from the space $(\mathcal{M}, \mathbf{d})$ to $\mathbb{R}^n$. These regularity conditions will allow us to establish certain basic existence and well-posedness results in the next section. Given a subset $\mathcal{M}' \subseteq \mathcal{M}$ and a functional $\mathcal{F} : \mathcal{M} \rightrightarrows \mathbb{R}^n$, we use the notation $\mathcal{F}|_{\mathcal{M}'}$ to denote the mapping defined by $\mathcal{F}(\varphi)$ for $\varphi \in \mathcal{M}'$ and by $\emptyset$ for $\varphi \notin \mathcal{M}$.

Definition 13 (Outer semicontinuous): A functional $\mathcal{F} : \mathcal{M} \rightrightarrows \mathbb{R}^n$ is said to be *outer semicontinuous* at $\varphi \in \mathcal{M}$, if for every sequences of hybrid memory arcs $\varphi_i \xrightarrow{\text{gph}} \varphi$ and $y_i \to y$ with $y_i \in \mathcal{F}(\varphi_i)$, we have $y \in \mathcal{F}(\varphi)$.

Definition 14 (Local boundedness): A functional $\mathcal{F} : \mathcal{M} \rightrightarrows \mathbb{R}^n$ is said to be *locally bounded* at $\varphi \in \mathcal{M}$ if there exists a neighborhood $\mathcal{U}_\varphi$ of φ such that the set $\mathcal{F}(\mathcal{U}_\varphi) := \bigcup_{\psi\in\mathcal{U}_\varphi} \mathcal{F}(\psi) \subseteq \mathbb{R}^n$ is bounded.

In the above definitions, $\mathcal{F}$ is said to be outer semicontinuous (respectively, locally bounded) *relative* to a set $\mathcal{M}' \subseteq \mathcal{M}$, if the mapping $\mathcal{F}|_{\mathcal{M}'}$ is outer semicontinuous (respectively, locally bounded) at each $\varphi \in \mathcal{M}'$. Finally, the mapping $\mathcal{F}$ is said to be outer semicontinuous (respectively, locally bounded) if it is so relative to its domain.

Given $X \subseteq \mathbb{R}^n$, we use $\mathcal{M}_X$ to denote the collection of hybrid memory arcs whose entire range is a subset of X, i.e.,

$$\mathcal{M}^X := \{\varphi \in \mathcal{M} : \varphi(s,k) \in X, \forall (s,k) \in \operatorname{dom}\varphi\}.$$

The regularity assumptions on $\mathcal{H}_\mathcal{M}$ that will enable us to prove basic existence and well-posedness results in the next section are:

(A1) C and D are closed subsets of $\mathbb{R}^n$;

(A2) $\mathcal{F}$ is outer semicontinuous and locally bounded relative to the set $\mathcal{M}_C := \{\varphi \in \mathcal{M}^{C\cup D} : \varphi(0,0) \in C\}$, and $\mathcal{F}(\varphi)$ is nonempty and convex for each $\varphi \in \mathcal{M}_C$;

(A3) $\mathcal{G}$ is outer semicontinuous and locally bounded relative to $\mathcal{M}_D := \{\varphi \in \mathcal{M}^{C\cup D} : \varphi(0,0) \in D\}$, and $\mathcal{G}(\varphi)$ is nonempty for each $\varphi \in \mathcal{M}_D$.

(A3') $\mathcal{G}$ is nonempty for each $\varphi \in \mathcal{M}_D$.

From the above assumptions, it is clear that we are interested in hybrid arcs that can jump when its current value is in D, can flow if its current value is in C, and completely lie in the set $C \cup D$, except possibly at $(T,J) = \sup\operatorname{dom} x$ when the latter is finite. A more general formulation would allow that the flow set C and jump set to be subsets of $\mathcal{M}$, in which case, whether or not a solution of $\mathcal{H}_\mathcal{M}$ can jump or flow not only depends on its current value, but also its entire history.

Example 1 (Hybrid systems with time-delay): Given $r > 0$, consider a hybrid system with a constant time-delay r. Let $x \in \mathcal{X}$ be a hybrid arc. Having a constant time-delay r means that the change (either by flow or jump) of x at $(t,j) \in \operatorname{dom} x$ depends on what happened r unit of time ago, i.e., the value of $x(t-r,k)$. Here k is such that $(t-r,k) \in \operatorname{dom} x$. Due to possibility of multiple instantaneous jumps at $t-r$, there can be more than one k satisfying $(t-r,k) \in \operatorname{dom} x$. A choice has to be made regarding which value of k to use. Given $\varphi \in \mathcal{M}$, we may define

$$\hat{\mathcal{F}}(\varphi) = \overline{\operatorname{conv}} \bigcup_{k \in J_{-r}} f(\varphi(-r,k)), \tag{7}$$

where $J_{-r} := \{k : (-r,k) \in \operatorname{dom}\varphi\}$ and $\overline{\operatorname{conv}}$ denotes the closure of the convex hull. It can be verified that $\hat{\mathcal{F}}$ is both outer semicontinuous and locally bounded at φ.

The idea behind defining $\hat{\mathcal{F}}$ by (7) is that in order to satisfy the regularity assumptions in terms of graphical distance, the definition of $\hat{\mathcal{F}}$ should have certain robustness to small variations in the size of delays.

E. Basic existence and nominal well-posedness

Under regularity assumptions (A1), (A2), and (A3'), we can prove the following basic existence result for hybrid systems with memory in the set-valued setting.

Theorem 2: Let $\mathcal{H}_\mathcal{M} = (C,\mathcal{F},D,\mathcal{G})$ satisfy (A1), (A2), and (A3'). Let $\varphi \in \mathcal{M}^{C\cup D}$. If either $\varphi(0,0) = \xi \in D$ or

(VC) there exists a neighborhood U of ξ such that for every $\zeta \in U \cap C$ and all $\varphi \in \mathcal{M}^{C\cup D}$ such that $\varphi(0,0) = \zeta$,

$$\mathcal{F}(\varphi) \cap T_C(\zeta) \neq \emptyset,$$

then there exists a nontrivial solution x to $\mathcal{H}_\mathcal{M}$ with $\mathcal{A}_{[0,0]}x = \varphi$. If (VC) holds for every $\xi \in C\backslash D$, then there exists a nontrivial solution to $\mathcal{H}_\mathcal{M}$ from every initial condition $\varphi \in \mathcal{M}^{C\cup D}$. Moreover, every such maximal solution x satisfies exactly one of the following conditions:

(a) x is complete;

(b) $\operatorname{dom}_{\geq 0}(x)$ is bounded, the interval I^J has nonempty interior, and $\limsup_{t\to T^-} |x(t,J)| = \infty$, where $J = \sup_j \operatorname{dom} x$ and $T = \sup_t \operatorname{dom} x$;

(c) $\varphi(T,J) \notin C \cup D$, where $(T,J) = \sup\operatorname{dom} x$.

Furthermore, if $\mathcal{G}(\varphi) \subseteq C \cup D$ for all $\varphi \in \mathcal{M}^{C\cup D}$ with $\varphi(0,0) \in D$, then (c) above does not occur.

The basic existence theorem above says that, given an initial hybrid memory arc φ that lies within $C \cup D$, the system has a nontrivial solution if either it is allowed to jump immediately, i.e., $\varphi(0,0) \in D$, or the (VC) condition is satisfied within a neighborhood of $\varphi(0,0)$, which allows the system to flow for at least a positive amount of time. Furthermore, if the (VC) condition is satisfied at all points $C\backslash D$, which says that the system should be allowed to flow if it cannot jump, then every maximal solution that is not complete either blows up in finite time or jumps to a point outside of $C \cup D$.

Theorem 3: Let $\mathcal{H}_\mathcal{M} = (C,\mathcal{F},D,\mathcal{G})$ satisfy (A1)–(A3). Then $\mathcal{H}_\mathcal{M}$ is nominally well-posed.

As shown for hybrid systems without memory [11], [12], nominal well-posedness has important consequences, including properties of limit sets, compact attractors, and extensions of invariance principles for hybrid systems. The development of such results for hybrid systems with memory, however, will be left as future work. Here we emphasize that, for hybrid systems with memory, the limit set is given by limits of the hybrid memory arcs induced by a hybrid arc over its hybrid time domain. In our setting, such limits can be well formulated as graphical limits of hybrid memory arcs. This may not be the case in other frameworks, e.g., considering uniform convergence topology in a certain space of piecewise continuous functions.

III. Sketch of the Proofs

Due to lack space, the complete proofs for the main results are omitted. Nonetheless, in this section, we discuss the main ideas for proving the technical results in this paper. Detailed proofs are included in a longer version of the paper [30].

The proof of Proposition 1 is essentially based on relations of the distance functions $\mathbf{d}$, $\mathbf{d}_\rho$, and $d(z,H)$ as established in [28, Lemma 4.34 and Proposition 4.37].

The proof of Lemma 1 essentially follows from (5) in Proposition 1(b).

The proof of the basic existence result (Theorem 2) for hybrid systems with memory essentially relies on some local existence results of viability solutions for functional differential inclusions with memory [31]. To be able to extend local solutions in the hybrid systems setting, the initial data considered should be hybrid in the sense that it can have both jumps and flows. Therefore, the phase space (where the memory lives) should be $\mathcal{M}$ instead of

$C[-r, 0]$, the space of $\mathbb{R}^n$-valued continuous functions on $[-r, 0]$, where $0 < r \leq \infty$. Since the theory in [31] permits $r = \infty$, we are also motivated to consider unbounded hybrid memory domains instead of just compact memory domains. In addition, our definition of hybrid memory domains does not restrict the number of jumps in a bounded delay interval. This allows our formulation to include discrete- or eventually discrete-time systems as special cases.

Sketch of proof for Theorem 2: If $\varphi(0, 0) \in D$, a local solution x exists by jumping according to $x(0, 1) = z$ with any $z \in \mathcal{G}(\varphi)$. Otherwise, the viability condition (VC) is satisfied for some neighborhood U of $\varphi(0, 0)$. Since the flow set C and jump set D are both in $\mathbb{R}^n$, we only need to consider the case when the viability set is in $\mathbb{R}^n$. This allows us to build our proof for the local existence part of Theorem 2 upon the local existence result for differential inclusions without memory (see, e.g., Proposition 3.4.2 of [32]). The key steps of the proof consist of (1) the construction of a sequence of hybrid arcs X_n, whose domains are given by $\operatorname{dom} X_n = \operatorname{dom} \varphi \cup ([0, T_0], 0)$ for some $T_0 > 0$ and are identical to φ on $\operatorname{dom} \varphi$; (2) the extraction of a subsequence $X_n(\cdot, 0)$ (still denoted by X_n) that converges uniformly to $X(\cdot, 0)$ on $[0, T]$; and (3) proving that $X(\cdot, 0)$ satisfies $\dot{X}(t, 0) \in \mathcal{F}(\mathcal{A}_{[t,0]} X)$ for almost all $t \in (0, T_0)$, which shows the case of local existence by flowing first.

The continuation results (a)–(c) can be verified by standard argument on continuation of solutions by flowing based on local boundedness of $\mathcal{F}$ and by jumps on conditions of $\mathcal{G}$.

Finally, the proof of Theorem 3 can be done by properly modifying the proof for well-posedness of hybrid systems without memory (see Theorem 6.16 of [12]).

IV. Conclusions

In this paper, we have considered generalized solution concepts for hybrid systems with delays. We first defined the phase space to be the space all hybrid memory arcs equipped with graphical distance topology. We then formulated regularity conditions on the hybrid data in terms of the graphical topology. Two results have been established, namely the basic existence and nominal well-posedness results. These results are a first step to develop a robust stability theory for hybrid systems with delays, which would extend the robust stability theory that has been developed for hybrid systems using generalized solution concepts.

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References

[1] R. Alur, C. Courcoubetis, T. Henzinger, and P. Ho, "Hybrid automata: An algorithmic approach to the specification and verification of hybrid systems," *Hybrid Systems*, pp. 209–229, 1993.

[2] J. P. Aubin, J. Lygeros, M. Quincampoix, S. S. Sastry, and N. Seube, "Impulse differential inclusions: A viability approach to hybrid systems," *IEEE Trans. Automat. Control*, vol. 47, pp. 2–20, 2002.

[3] A. Back, J. Guckenheimer, and M. Myers, "A dynamical simulation facility for hybrid systems," *Hybrid Systems*, pp. 255–267, 1993.

[4] M. S. Branicky, "Studies in hybrid systems: Modeling, analysis, and control," Ph.D. dissertation, Massachusetts Institute of Technology, 1995.

[5] R. W. Brockett, "Hybrid models for motion control systems," *Progress in Systems and Control Theory*, vol. 14, pp. 29–29, 1993.

[6] A. Nerode and W. Kohn, "Models for hybrid systems: Automata, topologies, controllability, observability," *Hybrid Systems*, pp. 317–356, 1993.

[7] A. van der Schaft, *An Introduction to Hybrid Dynamical Systems*. Springer-Verlag, 2000.

[8] L. Tavernini, "Differential automata and their discrete simulators," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 11, pp. 665–683, 1987.

[9] H. Witsenhausen, "A class of hybrid-state continuous-time dynamic systems," *IEEE Trans. Automat. Control*, vol. 11, pp. 161–167, 1966.

[10] H. Ye, A. N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *IEEE Trans. Automat. Control*, vol. 43, pp. 461–474, 1998.

[11] R. Goebel and A. R. Teel, "Solutions to hybrid inclusions via set and graphical convergence with stability theory applications," *Automatica*, vol. 42, pp. 573–587, 2006.

[12] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[13] R. Goebel, J. Hespanha, A. R. Teel, C. Cai, and R. G. Sanfelice, "Hybrid systems: generalized solutions and robust stability," in *NOLCOS*, 2004, pp. 1–12.

[14] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Generalized solutions to hybrid dynamical systems," *ESAIM Control Optim. Calc. Var.*, vol. 14, pp. 699–724, 2008.

[15] C. Cai, R. Goebel, and A. R. Teel, "Smooth Lyapunov functions for hybrid systems—Part II: (Pre)Asymptotically stable compact sets," *IEEE Trans. Automat. Control*, vol. 53, pp. 734–748, 2008.

[16] C. Cai, A. R. Teel, and R. Goebel, "Smooth lyapunov functions for hybrid systems—Part I: Existence is equivalent to robustness," *IEEE Trans. Automat. Control*, vol. 52, pp. 1264–1277, 2007.

[17] R. Sanfelice, R. Goebel, and A. Teel, "Invariance principles for hybrid systems with connections to detectability and asymptotic stability," *IEEE Trans. Automat. Control*, vol. 52, pp. 2282–2297, 2007.

[18] M. Cloosterman, N. van de Wouw, W. Heemels, and H. Nijmeijer, "Stability of networked control systems with uncertain time-varying delays," *IEEE Trans. Automat. Control*, vol. 54, pp. 1575–1580, 2009.

[19] R. Sipahi, S. I. Niculescu, C. T. Abdallah, W. Michiels, and K. Gu, "Stability and stabilization of systems with time delay," *IEEE Control Syst. Mag.*, vol. 31, pp. 38–65, 2011.

[20] G. Ballinger and X. Liu, "Existence and uniqueness results for impulsive delay differential equations," *Dynam. Contin. Discrete Impuls. Systems*, vol. 5, pp. 579–592, 1999.

[21] X. Liu and G. Ballinger, "Uniform asymptotic stability of impulsive delay differential equations," *Comput. Math. Appl.*, vol. 41, pp. 903–915, 2001.

[22] J. Liu, X. Liu, and W. Xie, "Input-to-state stability of impulsive and switching hybrid systems with time-delay," *Automatica*, vol. 47, pp. 899–908, 2011.

[23] S. Kim, S. A. Campbell, and X. Liu, "Stability of a class of linear switching systems with time delay," *IEEE Trans. Automat. Control*, vol. 53, pp. 384–393, 2006.

[24] J. Liu, X. Liu, and W.-C. Xie, "Generalized invariance principles for switched delay systems," *IMA J. Math. Control Inform.*, vol. 28, pp. 19–39, 2011.

[25] P. Yan and H. Özbay, "Stability analysis of switched time delay systems," *SIAM J. Control Optim.*, vol. 47, pp. 936–949, 2008.

[26] X. Liu and J. Shen, "Stability theory of hybrid dynamical systems with time delay," *IEEE Trans. Automat. Control*, vol. 51, pp. 620–625, 2006.

[27] R. Yuan, Z. Jing, and L. Chen, "Uniform asymptotic stability of hybrid dynamical systems with delay," *IEEE Trans. Automat. Control*, vol. 48, pp. 344–348, 2003.

[28] R. T. Rockafellar and R. J. B. Wets, *Variational Analysis*. Springer-Verlag, 1998, vol. 317.

[29] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*. Springer, 1993, vol. 99.

[30] J. Liu and A. R. Teel, "Generalized solutions to hybrid systems with delays," 2012. [Online]. Available: http://jun-liu.staff.shef.ac.uk/preprints/lt-12-cdc.pdf

[31] G. Haddad, "Monotone viable trajectories for functional differential inclusions," *J. Differential Equations*, vol. 42, pp. 1–24, 1981.

[32] J. P. Aubin, *Viability Theory*. Birkhauser, 1991.