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Distributed stochastic consensus of multi-agent systems with noisy and delayed measurements

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Abstract: Networked systems are often subject to environmental uncertainties and communication delays, which make timely and accurate information exchange among neighbours difficult or impossible. This study investigates the distributed consensus problem of dynamical networks of multi-agents in which each agent can only obtain noisy and delayed measurements of the states of its neighbours. The authors consider consensus protocols that take into account both the noisy measurements and the communication time delays, and introduce the notions of almost sure average-consensus and p th moment average-consensus. Using a convergence theorem for continuous-time semimartingales and moment inequality techniques for stochastic delay differential equations, the authors establish sufficient conditions for both almost sure and moment average-consensus. These results naturally generalise to networks with arbitrary and Markovian switching topologies. The consensus protocol considered here can be applied to networks with arbitrary bounded communication delays, which appears to be the first consensus algorithm that is both average preserving and robust to arbitrarily sized delays. Numerical simulations are also provided to demonstrate the theoretical results.

1 Introduction

Over the last decade, there have been extensive interests in consensus problems related to networked multi-agent systems (see, e.g. [1–15]; see also [16, 17] for recent surveys and extensive references therein). Consensus problems naturally arise when a group of agents, often distributed over a network, are seeking agreement upon a certain quantity of interest, which might be attitude, position, velocity, voltage, direction, temperature and so on, depending on different applications.

1.1 Motivation of the paper

Networked systems are often subject to environmental uncertainties and communication delays, which make it difficult or impossible for a networked agent to obtain timely and accurate information of its neighbours. Moreover, link gains/failures and formation reconfiguration make it necessary to address consensus problems for networks with switching network topology. The recent work of [8] studies stochastic consensus problems of networked agents in the discrete-time setting using algorithms from stochastic approximation. Li and Zhang [10] extended the work of Huang and Manton [8] to the continuous-time setting, and obtained both necessary and sufficient conditions for mean-square stochastic consensus for networks that are

both balanced and containing a spanning tree (equivalent to strongly connected and balanced [10]), whereas Li and Zhang [18] established both mean-square and almost sure consensus for multi-agent systems with time-varying topology. Liu *et al.* [11] investigated stochastic consensus problems with communication time delays. The results are concerned with mean-square consensus and applicable to uniform delays satisfying a certain upper bound. In [19], distributed consensus has been studied for linear discrete-time multi-agent systems with delays and noises in transmission channels.

In view of the consensus results obtained by the authors of [8, 10, 11, 18, 19] on stochastic consensus problems, there are several main motivations for the current paper.

First, in practice, it is generally desirable to examine the almost sure sample convergence in stochastic consensus models; if such convergence can be proved, we can claim that the consensus results hold true with probability 1. In the discrete setting, the work by Huang and Manton [8] (see also [18, 19]) studies such properties under the name strong consensus. It remains unanswered whether almost sure consensus results can be obtained in the continuous-time setting. In fact, almost sure consensus in the continuous-time setting requires studying sample convergence properties of continuous-time stochastic processes, which are generally more challenging to obtain than moment properties.

Second, the above-mentioned work [8, 10, 11, 18] all focuses on mean-square (second moment) consensus, while the studying of general p th moment convergence and moment stability is an important common theme in stochastic stability theory [20–22]. The main practical reason to look at the convergence and stability of a higher moment is that, even though the solutions of a stochastic system converge exponentially to 0 with probability 1 or in the mean-square sense, the p th moment of the system may still diverge for large enough p , as revealed by the large deviations theory [22]. Moreover, general p th moment properties are more difficult to investigate and include mean-square properties as a special case. Therefore it is of both practical importance and theoretical interest to study general p th moment consensus properties of dynamical networks.

Third, most current results on consensus problems with communication delays (e.g. [2, 11, 13, 15, 23–27]) consider consensus protocols that will lead to consensus under bounded delays and quantify explicit delay bounds that guarantee consensus. Note that, although the results by [28, 29] show consensus for arbitrary delays, the consensus value attained there is not the average of the initial conditions, but rather depends on the size of time delays. Therefore it remains open how average-consensus can be reached for communication delays of arbitrary size.

Other work on consensus problems that explicitly takes into account measurement and environmental noises in different contexts includes [1, 4, 5, 7, 15, 30–32], in some of which the noises are modelled as deterministic but unknown disturbances (e.g. [1, 31]). On the other hand, consensus problems with communication delays have also been studied extensively in recent years (see, e.g. [2, 13, 23–29]). However, very little work has been done on stochastic consensus problems of networks with communication delays, either in a discrete- or continuous-time setting, while noise and delays are ubiquitous in communication networks. Notable exceptions include recent work reported in [11, 19], in which distributed consensus has been studied for discrete-time and continuous-time multi-agent systems, respectively, with delays and noises in transmission channels.

1.2 Contribution of the paper

Motivated by the above considerations, the main contribution of this paper is to present several novel technical developments on stochastic consensus problems along the line of work by the authors of [10, 11, 18]. The key improvements compared to previous work in the literature, especially the recent work by Liu *et al.* [11], are summarised as follows.

First, ‘almost sure’ stochastic consensus is formulated and proved in the ‘continuous-time’ setting. By applying a convergence theorem for non-negative semi-martingales [33], we are able to establish almost sure consensus under the same time-varying protocols we consider for moment consensus. Therefore it is guaranteed that the consensus protocol considered in this paper can lead to consensus with probability 1. Note that, as pointed out earlier, almost sure consensus has been considered in the ‘discrete-time’ setting [8] (see also [19]), whereas this paper aims to prove almost sure stochastic consensus in the continuous-time setting.

Second, we introduce the notion of general p th moment average-consensus and investigate consensus under this notion, using different inequality techniques for stochastic differential equations and a Gronwall–Bellman–Halany-type inequality established by Liu *et al.* [11]. We consider

time-varying consensus protocols that take into account both the noisy measurements and the communication time-delays, and establish sufficient conditions under which the considered consensus protocols lead to general p th moment average-consensus. These conditions are non-restrictive in the sense that the same conditions, restricted to mean-square consensus without delays, are shown to be necessary for reaching consensus [10].

Third, both moment and almost sure consensus results obtained in this paper are applicable to arbitrary bounded delays; that is, we establish average-consensus using delayed protocols that work for networks with uniform communication delays of arbitrary size. The reason behind this seemingly surprising result is that the time-varying function introduced in the consensus protocol to attenuate noise bears some nice properties. By carefully exploiting these properties, we are able to show that delays of arbitrary size can be tolerated by the consensus protocol. Note that, although the results by [28, 29] show consensus for arbitrary delays, the consensus value attained there is not the average of the initial conditions, but rather depends on the size of time delays, whereas our results guarantee average-consensus for arbitrarily sized delays.

1.3 Organisation of the paper

The rest of this paper is organised as follows. In Section 2, we formulate the consensus problem, introduce the consensus protocol, and introduce two notions of stochastic consensus. The main consensus results are presented in Section 3, which investigate both moment and almost sure consensus properties, followed by demonstrations through numerical simulations in Section 4 with some discussions. The paper is concluded by Section 5.

2 Problem formulation and consensus protocols

2.1 Network topology

The interaction topology of a network of n -agents is modelled by a weighted digraph (or directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order n with set of nodes $\mathcal{V} = \{v_1, \dots, v_n\}$, set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a weighted adjacent matrix $\mathcal{A} = [a_{ij}]_{n \times n}$ with non-negative elements a_{ij} . An edge of \mathcal{G} is denoted by $e_{ji} = (v_j, v_i)$. An edge e_{ji} exists if and only if $a_{ij} > 0$. It is assumed that $a_{ii} = 0$ for $i = 1, \dots, n$. The set of neighbours of a node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Let $x_i \in \mathbb{R}$ denote the value of node v_i , which is a scalar quantity of interest. Denote the set $\{1, \dots, n\}$ by \mathcal{I} . The ‘graph Laplacian’ \mathcal{L} of the network is defined by

$$\mathcal{L} = D - \mathcal{A} \quad (1)$$

where $D = \text{diag}(d_1, \dots, d_n)$ is the in-degree matrix of \mathcal{G} with elements $d_i = \sum_{j \neq i} a_{ij}$ and \mathcal{A} is the weighted adjacent matrix. A digraph (and the corresponding network) is ‘strongly connected’ if there is a directed path connecting any two arbitrary nodes in the graph. A digraph (and the corresponding network) is said to be ‘balanced’ if $\sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji}$ for all $i \in \mathcal{I}$.

2.2 Consensus protocols

Consider each node of the graph to be a dynamic agent with dynamics

$$\dot{x}_i = u_i, \quad i \in \mathcal{I} \quad (2)$$

where the state feedback $u_i = u_i(x_{i_1}, \dots, x_{i_k})$ is called a *protocol* with topology \mathcal{G} if the set of nodes $\{x_{i_1}, \dots, x_{i_k}\}$ are all taken from the set $\{v_i\} \cup \mathcal{N}_i$, i.e. only the information of v_i itself and its neighbours are available in forming the state feedback for the node v_i .

We consider the following consensus protocol by Olfati-Saber and Murray [13]

$$u_i = \sum_{v_j \in \mathcal{N}_i} a_{ij}(x_j - x_i), \quad i \in \mathcal{I} \quad (3)$$

The above protocol requires that agent i can obtain information from its neighbours in \mathcal{N}_i timely and accurately, that is, it assumes zero communication time-delay and accurate information exchange among agents. Let y_{ji} be a measurement of x_j by x_i given by

$$y_{ji} = x_j + \sigma_{ji}\dot{w}_{ji}(t), \quad i \in \mathcal{I} \quad (4)$$

where $\{\dot{w}_{ji}(t): i, j = 1, \dots, n\}$ are independent standard white noises and $\sigma_{ji} \geq 0$ represent the noise intensity. Replacing x_j in (3) with the noisy measurement y_{ji} gives the following stochastic consensus protocol

$$u_i = \sum_{v_j \in \mathcal{N}_i} a_{ij}(y_{ji} - x_i), \quad i \in \mathcal{I} \quad (5)$$

If, in addition, communication delays are considered, we consider the following delayed stochastic consensus protocol:

$$u_i(t) = c(t) \sum_{v_j \in \mathcal{N}_i} a_{ij}[y_{ji} - x_i(t - \tau_{ij}(t))], \quad i \in \mathcal{I} \quad (6)$$

where

$$y_{ji} = x_j(t - \tau_{ij}(t)) + \sigma_{ji}\dot{w}_{ji}(t), \quad i \in \mathcal{I}$$

and the time-varying delays $\tau_{ij}(t)$ lie in $[0, \tau]$ for some $\tau > 0$ and are assumed to be continuous in t . The function $c: \mathbb{R}^+ \mapsto \mathbb{R}^+$ in (6) is a piecewise continuous, monotonically decreasing function satisfying

$$\int_0^\infty c(s)ds = \infty \quad \text{and} \quad \int_0^\infty c^2(s) ds < \infty \quad (7)$$

The role of the function $c(t)$ is to attenuate the noise effects as $t \rightarrow \infty$. Condition (7) and the assumption that $c(t)$ is monotonically decreasing, on the one hand, implies that $c(t)$ is vanishing as $t \rightarrow \infty$, but, on the other hand, not too fast because of the restriction $\int_0^\infty c(s) ds = \infty$. Without loss of generality, we can assume that $\sup_{t \geq 0} c(t) \leq 1$. It is easy to verify that the class of functions $c(t) := [1]/[(t+a)^\alpha]$ defined for $t \geq 0$, where $\frac{1}{2} < \alpha \leq 1$ and $a > 0$ all satisfy these conditions.

If \mathcal{N}_i is fixed, (6) gives a ‘fixed topology protocol’. If \mathcal{N}_i is time-varying, we have a ‘switching topology protocol’. The communication delays and noisy measurements in the protocol (6) are illustrated by Fig. 1.

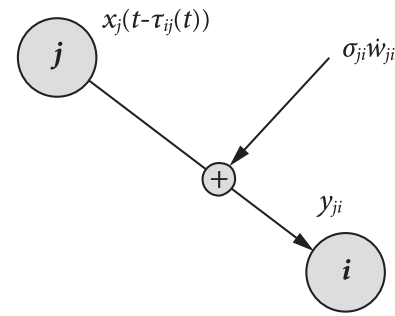


Fig. 1 Delayed measurements with additive noises

2.3 Network dynamics

If the time-delays are uniform, that is, $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$, the collective dynamics of system (2) under the consensus protocol (6) can be written in a compact form of a stochastic delay differential equation (SDDE) as

$$dx(t) = c(t)[-Lx(t - \tau(t)) dt + \Theta dW(t)] \quad (8)$$

where $x(t) = (x_1(t), \dots, x_n(t))$; $W(t)$ is an n^2 -dimensional standard Wiener process; L is the graph Laplacian of the network; and $\Theta \in \mathbb{R}^{n \times n^2}$ is a constant matrix defined by $\Theta = \text{diag}(\Theta_1, \dots, \Theta_n)$, where Θ_i is an n -dimensional row vector given by $\Theta_i = [a_{i1}\sigma_{1i} \ a_{i2}\sigma_{2i} \ \dots \ a_{in}\sigma_{ni}]$.

Remark 1: In this paper, we shall focus on the case where the time-varying delays are uniform, i.e. $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$. In addition, we assume that the derivative $\dot{\tau}(t)$ exists for all $t \geq 0$. This is for simplicity of exposition, so that we can focus on showing both almost sure and p th moment stochastic consensus results. As the main results of this paper will show, the consensus results hold for delays of arbitrary size.

2.4 Consensus notions

We introduce the following notions of consensus for the multi-agent systems (2) under the consensus protocol (6) in an uncertain environment.

Definition 1: Given $p > 0$, the agents in (2) are said to reach ‘average-consensus in the p th moment’ if $E|x_i(t)|^p < \infty$ for all $t \geq 0$ and $i \in \mathcal{I}$ and there exists a random variable x^* such that $E(x^*) = \text{avg}(x(0))$, where $\text{avg}(x) = \sum_{i=1}^n x_i/n$ for $x = (x_1, x_2, \dots, x_n)$, and $\lim_{t \rightarrow \infty} E|x_i(t) - x^*|^p = 0$ for all $i \in \mathcal{I}$. Particularly, if $p = 2$, the agents are said to reach ‘mean-square average-consensus’.

Definition 2: The agents in (2) are said to reach ‘almost sure average-consensus’ if there exists a random variable x^* such that $E(x^*) = \text{avg}(x(0)) = \sum_{i=1}^n x_i(0)/n$ and $\lim_{t \rightarrow \infty} x_i(t) = x^*$ almost surely for all $i \in \mathcal{I}$.

Remark 2: Both mean-square consensus and almost sure consensus (or called strong consensus) are defined and investigated by [8, 18] in the discrete-time setting, without considering communication delays. In addition, continuous-time mean-square average-consensus has been defined and studied by the authors [10, 11], with and without considering communication delays, respectively. Here, we formulate general p th moment and almost sure average-consensus and obtain results on both of them in the continuous-time setting.

3 Main results

In this section, we analyze both the p th moment and almost sure consensus properties of the dynamics of system (2).

3.1 Networks with fixed topology

We start by analysing networks with fixed topology, that is, the weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is time-invariant. The first theorem provides sufficient conditions for the dynamic agents in (2) to reach general p th moment consensus under the consensus protocol (6) with arbitrary-bounded delays.

Theorem 1: If \mathcal{G} is a strongly connected and balanced digraph and the time-delays are uniform, that is, $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$, then the consensus protocol (6) leads to

- (i) p th ($p \geq 2$) moment average-consensus for the agents in (2).
- (ii) Almost sure average-consensus for the agents in (2), if, in addition, the time-varying delay satisfies $\dot{\tau}(t) \leq d < 1$ for all $t \geq 0$ and some constant d .

Both results hold for networks with uniform communication delays of arbitrary size.

Before we prove this theorem, a few remarks are in order.

Remark 3: Theorem 1 requires that the network graph is both strongly connected and balanced, which is equivalent to that the network graph has a spanning tree and is balanced at the same time. This seemingly strong condition, however, is both necessary and sufficient for reaching average consensus for multi-agent systems in networks with or without communication noises [10, 13]. Here, under the same condition, we have shown that the time-varying consensus protocol (6) can achieve both general moment and almost sure average-consensus despite arbitrary bounded communication delays. This consensus protocol leads to delay-independent consensus while still ‘preserving average’. This is in contrast with consensus protocols that allow each agent to use the current version (instead of a delayed version) of its own state can, which turn out to be delay-independent but without preserving average [12, 25, 34].

To prove Theorem 1, we introduce a so-called displacement vector as considered by authors of [10, 11, 13]

$$\delta(t) = x(t) - \mathbf{1}\alpha(t) = (I - J)x(t) \tag{9}$$

where $\mathbf{1}$ stands for the column n -vector with all ones, $\alpha(t) = \text{avg}(x(t)) = \frac{1}{n}\mathbf{1}^T x(t) = \frac{1}{n} \sum_{i=1}^n x_i(t)$, I is the $n \times n$ identity matrix, and $J = \frac{1}{n}\mathbf{1}\mathbf{1}^T$. It is easy to see that

$$\mathbf{1}^T \delta(t) = \sum_{i=1}^n x_i(t) - n\alpha(t) = 0, \quad t \geq 0 \tag{10}$$

The dynamics of $\delta(t)$ are given by

$$d\delta(t) = c(t)[-L\delta(t - \tau(t)) dt + (I - J)\Theta dW(t)] \tag{11}$$

where we have used the fact that both $\mathbf{1}^T L$ and $L\mathbf{1}$ are zero vectors. For $p \geq 2$, the p th moment consensus analysis relies

on the Lyapunov function candidate

$$V(t) = (\delta^T(t)\delta(t))^{\frac{p}{2}} = |\delta(t)|^p, \quad t \geq 0.$$

The second smallest eigenvalue of $\hat{\mathcal{L}} = (\mathcal{L} + \mathcal{L}^T)/2$, denoted by $\lambda_2(\hat{\mathcal{L}})$ and called the ‘algebraic connectivity’ of the graph \mathcal{G} , was originally introduced by Fiedler [35] for undirected graphs and later extended to digraphs by Olfati-Saber and Murray [13]. The following property of the graph Laplacian \mathcal{L} for strongly connected and balanced digraphs (see Theorem 7 of [13]),

$$\delta^T \mathcal{L} \delta \geq \lambda_2(\hat{\mathcal{L}})|\delta|^2, \quad \forall \mathbf{1}^T \delta = 0 \tag{12}$$

plays an important role in ensuring that the protocol (6) leads to both p th moment and almost sure consensus of the agents in (2). The detailed proof for the p th moment consensus part in Theorem 1 relies heavily on moment inequality techniques for SDDEs and the following general Gronwall–Bellman–Halalay type inequality established by Liu *et al.* [11].

Lemma 1 [11]: Let t_0 and r be non-negative constants. Let $m: [t_0 - r, \infty) \mapsto \mathbb{R}^+$ be continuous and satisfy

$$D^+ m(t) := \limsup_{h \rightarrow 0^+} \frac{m(t+h) - m(t)}{h} \leq \gamma(t) - \mu c(t)m(t) + \lambda c(t) \sup_{-r \leq s \leq 0} m(t+s) \tag{13}$$

on $[t_0, \infty)$, where γ and c are piecewise continuous functions with $\gamma(t) \geq 0$ and $c(t) \in (0, 1]$ for all $t \geq 0$, and μ and λ are constants satisfying $\mu > \lambda > 0$. Then

$$m(t) \leq m_0 \exp \left\{ -\rho \int_{t_0}^t c(s) ds \right\} + \int_{t_0}^t \exp \left\{ -\rho \int_s^t c(r) dr \right\} \gamma(s) ds \tag{14}$$

holds on $[t_0, \infty)$, where $\rho > 0$ is the root of $-\rho = -\mu + \lambda e^{\rho r}$ and $m_0 = \sup_{-r \leq s \leq 0} m(t_0 + s)$.

We will apply the following result on almost sure convergence of non-negative semimartingales to prove the almost sure consensus part in Theorem 1.

Lemma 2 [20, 33]: Let (Ω, \mathcal{F}, P) be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Let $\{A(t)\}_{t \geq 0}$ and $\{U(t)\}_{t \geq 0}$ be two continuous $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted increasing processes with $A(0) = U(0) = 0$ a.s. Let $\{M(t)\}_{t \geq 0}$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ξ be a non-negative \mathcal{F}_0 -measurable such that $E\xi < \infty$. Define

$$X(t) = \xi + A(t) - U(t) + M(t), \quad t \geq 0.$$

If $X(t)$ is non-negative for all $t \geq 0$, then

$$\left\{ \lim_{t \rightarrow \infty} A(t) < \infty \right\} \subset \left\{ \lim_{t \rightarrow \infty} X(t) \text{ exists and is finite} \right\} \cap \left\{ \lim_{t \rightarrow \infty} U(t) < \infty \right\} \quad \text{a.s.}$$

where $B \subset D$ a.s. means $P(B \cap D^c) = 0$. In particular, if $\lim_{t \rightarrow \infty} A(t) < \infty$ a.s., then, for almost all $\omega \in \Omega$,

$$\lim_{t \rightarrow \infty} X(t, \omega) < \infty, \quad \lim_{t \rightarrow \infty} U(t, \omega) < \infty, \quad -\infty < \lim_{t \rightarrow \infty} M(t, \omega) < \infty.$$

Proof of Theorem 1: Applying Itô's formula (see, e.g. [36], Theorem 3.3, Chapter IV) to $V(t)$ in view of (9), we have

$$dV(t) = d[\delta^T(t)\delta(t)]^2 = \mathbf{LV}dt + pc(t)|\delta(t)|^{p-2}\delta^T(t)(I - J)\Theta dW(t) \quad (15)$$

where the drift term \mathbf{LV} is given by

$$\begin{aligned} \mathbf{LV} = & -pc(t)|\delta(t)|^{p-2}\delta^T(t)\mathcal{L}\delta(t) \\ & + pc(t)|\delta(t)|^{p-2}\delta^T(t)\mathcal{L}[\delta(t) - \delta(t - \tau(t))] \\ & + \frac{p}{2}C_0c^2(t) + \frac{p(p-2)}{2}c^2(t)|\delta(t)|^{p-4}|\delta^T(t)(I - J)\Theta|^2 \end{aligned} \quad (16)$$

where

$$C_0 = |(I - J)\Theta|^2 = \text{trace}[(I - J)^2\Theta\Theta^T]$$

Integrating equation (15) gives

$$E(V(t)) - E(V(\tau)) = \int_{\tau}^t E[\mathbf{LV}] ds \quad (17)$$

since $E[\int_{\tau}^t c(s)|\delta(s)|^{p-2}\delta^T(s)(I - J)\Theta dW(s)] = 0$. We shall first show almost sure average-consensus and then p th moment average-consensus.

Proof of almost sure consensus: Let c be a fixed function in the consensus protocol (6) that satisfies the conditions in (7). Since $c(t) \rightarrow 0$ as $t \rightarrow \infty$, for each $\varepsilon \in (0, 1)$, there exist some $T_{\varepsilon} = T(\varepsilon) \geq 2\tau$ such that $c(t) < \varepsilon$ for all $t \geq T_{\varepsilon} - \tau$. Letting $p = 2$, (15) and (16) obtained by Itô's formula become

$$\begin{aligned} dV(t) = & d[\delta^T(t)\delta(t)] \\ = & -2c(t)\delta^T(t)\mathcal{L}\delta(t) dt \\ & + 2c(t)\delta^T(t)\mathcal{L}[\delta(t) - \delta(t - \tau(t))] dt \\ & + C_0c^2(t) dt + 2c(t)\delta^T(t)(I - J)\Theta dW(t) \end{aligned} \quad (18)$$

which implies

$$\begin{aligned} V(t) = & V(T_{\varepsilon}) - 2 \int_{T_{\varepsilon}}^t c(s)\delta^T(s)\mathcal{L}\delta(s) ds \\ & + 2 \int_{T_{\varepsilon}}^t c(s)\delta^T(s)\mathcal{L}[\delta(s) - \delta(s - \tau(s))] ds \\ & + C_0 \int_{T_{\varepsilon}}^t c^2(s) ds + 2 \int_{T_{\varepsilon}}^t c(s)\delta^T(s)(I - J)\Theta dW(s) \\ \leq & V(T_{\varepsilon}) - 2\lambda_2(\hat{\mathcal{L}}) \int_{T_{\varepsilon}}^t c(s)|\delta(s)|^2 ds + C_0 \int_{T_{\varepsilon}}^t c^2(s) ds \\ & + 2 \int_{T_{\varepsilon}}^t c(s)\delta^T(s)(I - J)\Theta dW(s) \\ & + 2 \int_{T_{\varepsilon}}^t c(s)\delta^T(s)\mathcal{L} \left[\int_{s-\tau(s)}^s c(r)\mathcal{L}\delta(r - \tau(r)) dr \right. \\ & \left. + \int_{s-\tau(s)}^s c(r)(I - J)\Theta dW(r) \right] ds \\ \leq & V(T_{\varepsilon}) - 2\lambda_2(\hat{\mathcal{L}}) \int_{T_{\varepsilon}}^t c(s)|\delta(s)|^2 ds \\ & + \mathcal{I}_1(t) + \mathcal{I}_2(t) + M(t) \end{aligned} \quad (19)$$

where $M(t) = 2 \int_{T_{\varepsilon}}^t c(s)\delta^T(s)(I - J)\Theta dW(s)$ is a continuous local martingale satisfying $M(T_{\varepsilon}) = 0$,

$$\begin{aligned} \mathcal{I}_1(t) = & C_0 \int_{T_{\varepsilon}}^t c^2(s) ds \\ & + 2 \int_{T_{\varepsilon}}^t \int_{s-\tau(s)}^s c(s)c(r)|\delta^T(s)\mathcal{L}^2\delta(r - \tau(r))| dr ds \end{aligned}$$

and

$$\mathcal{I}_2(t) = 2 \int_{T_{\varepsilon}}^t \left| \int_{s-\tau(s)}^s c(s)c(r)\delta^T(s)\mathcal{L}(I - J)\Theta dW(r) \right| ds.$$

Regarding T_{ε} as the starting time, both $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ are $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted and continuous increasing processes satisfying $\mathcal{I}_1(T_{\varepsilon}) = \mathcal{I}_2(T_{\varepsilon}) = 0$, and $\int_{T_{\varepsilon}}^t c(s)\delta^T(s)(I - J)\Theta dW(s)$ in (18) is a continuous local martingale null at T_{ε} . We want to estimate the two integrals $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ as $t \rightarrow \infty$. We start with $\mathcal{I}_1(t)$ and compute

$$\begin{aligned} \mathcal{I}_1(t) \leq & C_0 \int_{T_{\varepsilon}}^{\infty} c^2(s) ds + \varepsilon \int_{T_{\varepsilon}}^t c(s)|\delta(s)|^2 ds \\ & + \varepsilon \|\mathcal{L}\|^4 \int_{T_{\varepsilon}}^t \int_{s-\tau(s)}^s c(r)|\delta(r - \tau(r))|^2 dr ds \end{aligned} \quad (20)$$

Since $d[r - \tau(r)]/dr = 1 - \dot{\tau}(r) \geq 1 - d > 0$ by the theorem condition, we can use a change of variable $\xi = r - \tau(r)$ and show that

$$\begin{aligned} & \int_{T_{\varepsilon}}^t \int_{s-\tau(s)}^s c(r)|\delta(r - \tau(r))|^2 dr ds \\ & \leq \frac{1}{1-d} \int_{T_{\varepsilon}}^t \int_{s-2\tau}^s |\delta(\xi)|^2 d\xi ds \\ & \leq \frac{1}{1-d} \int_{T_{\varepsilon}-2\tau}^t \int_{\xi}^{\xi+2\tau} |\delta(\xi)|^2 ds d\xi \\ & = \frac{2\tau}{1-d} \int_{T_{\varepsilon}-2\tau}^t |\delta(\xi)|^2 d\xi \end{aligned}$$

which leads to

$$\begin{aligned} \mathcal{I}_1(t) \leq & C_0 \int_{T_{\varepsilon}}^{\infty} c^2(s) ds + \left(\varepsilon + \frac{2\varepsilon\tau\|\mathcal{L}\|^4}{1-d} \right) \int_{T_{\varepsilon}}^t |\delta(s)|^2 ds \\ & + \frac{2\varepsilon\tau\|\mathcal{L}\|^4}{1-d} \int_{T_{\varepsilon}-2\tau}^{T_{\varepsilon}} |\delta(\xi)|^2 d\xi \end{aligned} \quad (21)$$

On the other hand

$$\begin{aligned} \mathcal{I}_2(t) \leq & \int_{T_{\varepsilon}}^t c^2(s)|\delta(s)|^2 ds \\ & + \int_{T_{\varepsilon}}^t \left| \int_{s-\tau(s)}^s c(r)\mathcal{L}(I - J)\Theta dW(r) \right|^2 ds \\ \leq & \varepsilon \int_{T_{\varepsilon}}^{\infty} |\delta(s)|^2 ds + \mathcal{I}_3(t) \end{aligned} \quad (22)$$

where

$$\mathcal{I}_3(t) = \int_{T_{\varepsilon}}^t \left| \int_{s-\tau(s)}^s c(r)\mathcal{L}(I - J)\Theta dW(r) \right|^2 ds$$

Clearly, $\mathcal{I}_3(t)$ is also an $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted and continuous increasing process null at T_ε and, by using Itô isometry (see, e.g. Theorem 1.5.21 of [20]),

$$E[\mathcal{I}_3(t)] = \hat{C}_0 \int_{T_\varepsilon}^t \int_{s-\tau(s)}^s c^2(r) dr ds \leq \tau \hat{C}_0 \int_{T_\varepsilon-\tau}^\infty c^2(r) dr < \infty \tag{23}$$

where

$$\hat{C}_0 = |\mathcal{L}(I - J)\Theta|^2 = \text{trace}[\mathcal{L}(I - J)^2\Theta\Theta^T]$$

We claim that the $\lim_{t \rightarrow \infty} \mathcal{I}_3(t)$ exists and is finite a.s. The existence of the limit follows from the fact that $\mathcal{I}_3(t)$ is an increasing process, and $\lim_{t \rightarrow \infty} \mathcal{I}_3(t) < \infty$ a.s. follows from

$$\begin{aligned} P\left(\lim_{t \rightarrow \infty} \mathcal{I}_3(t) = \infty\right) &= P\left(\bigcap_{n=1}^\infty \{\exists T_n \text{ s.t. } \mathcal{I}_3(T_n) \geq n\}\right) \\ &= P\left(\lim_{n \rightarrow \infty} \{\exists T_n \text{ s.t. } \mathcal{I}_3(T_n) \geq n\}\right) \\ &= \lim_{n \rightarrow \infty} P(\{\exists T_n \text{ s.t. } \mathcal{I}_3(T_n) \geq n\}) \\ &\leq \lim_{n \rightarrow \infty} \frac{E[\mathcal{I}_3(T_n)]}{n} = 0 \end{aligned}$$

where the last line follows from Markov's inequality and (23). Substituting (21) and (22) back into (19) gives

$$V(t) \leq \zeta + \mathcal{I}_3(t) - \mu_\varepsilon \int_{T_\varepsilon}^t c(s)|\delta(s)|^2 ds + M(t) \tag{24}$$

where $M(t)$ is a continuous local martingale null at T_ε

$$\begin{aligned} \zeta &= V(T_\varepsilon) + \frac{2\varepsilon\tau\|\mathcal{L}\|^4}{1-d} \int_{T_\varepsilon-2\tau}^{T_\varepsilon} |\delta(\xi)|^2 d\xi \\ &\quad + C_0 \int_{T_\varepsilon}^\infty c^2(s) ds < \infty \end{aligned} \tag{25}$$

is an $\mathcal{F}_{T_\varepsilon}$ -measurable random variable, and we can choose $\varepsilon > 0$ sufficiently small such that

$$\mu_\varepsilon = 2\lambda_2(\hat{\mathcal{L}}) - 2\varepsilon - \frac{2\varepsilon\tau\|\mathcal{L}\|^4}{1-d} > 0 \tag{26}$$

We have shown earlier that $\mathcal{I}_3(t)$ is a non-negative, $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted, and continuous increasing process null at T_ε with $\lim_{t \rightarrow \infty} \mathcal{I}_3(t) < \infty$ a.s. By Lemma 2, we obtain

$$\int_{T_\varepsilon}^\infty |\delta(\xi)|^2 d\xi < \infty \quad \text{a.s.}$$

It follows from the sample continuity of $\delta(t)$ for all $t \geq 0$ that

$$\lim_{t \rightarrow \infty} |\delta(t)| = 0 \quad \text{a.s.}$$

which implies almost sure average-consensus for the agents in (2).

Proof of pth moment consensus: Let $m(t) = E(V(t))$ for $t \geq 0$. By (16) and (17),

$$\begin{aligned} D^+m(t) &= E[\text{LV}] \\ &\leq -pc(t)E\{|\delta(t)|^{p-2}\delta^T(t)\mathcal{L}\delta(t)\} + \frac{p}{2}C_0c^2(t) \\ &\quad + pc(t)E\{|\delta(t)|^{p-2}\delta^T(t)\mathcal{L}[\delta(t) - \delta(t - \tau(t))]\} \\ &\quad + \frac{p(p-2)}{2}c^2(t)E\{|\delta(t)|^{p-4}|\delta^T(t)(I - J)\Theta|^2\} \end{aligned} \tag{27}$$

We aim to obtain an estimate of the right-hand side of (27). Note that

$$E\{|\delta(t)|^{p-2}\delta^T(t)\mathcal{L}\delta(t)\} \geq \lambda_2(\hat{\mathcal{L}})m(t) \tag{28}$$

and

$$c^2(t)E\{|\delta(t)|^{p-4}|\delta^T(t)(I - J)\Theta|^2\} \leq C_0E\{c^2(t)|\delta(t)|^{p-2}\} \tag{29}$$

Using Young's inequality produces

$$ab \leq \frac{a^q}{q} + \frac{b^r}{r}$$

which holds for all $a, b \geq 0$ and positive numbers q and r such that $1/q + 1/r = 1$, we can show that

$$\begin{aligned} &E\{c^2(t)|\delta(t)|^{p-2}\} \\ &= E\left\{\left\{|\delta(t)|^{p-2}|c(t)|^{\frac{p-2}{p}}\right\} \cdot \left\{|\varepsilon^{-\frac{p-2}{p+2}}c(t)|^{\frac{p+2}{p}}\right\}\right\} \\ &\leq \varepsilon^{\frac{p-2}{p}}c(t)m(t) + \frac{2}{p}\varepsilon^{-\frac{p-2}{p+2}\frac{p}{2}}|c(t)|^{\frac{p+2}{2}} \\ &\leq \varepsilon c(t)m(t) + \hat{\varepsilon}|c(t)|^2 \end{aligned} \tag{30}$$

where $\hat{\varepsilon}$ is a finite positive constant that depends on ε and p . While the first inequality above is a direct application of Young's inequality, the second inequality has used the fact that $p \geq 2$ and $c(t) \leq 1$ for all $t \geq 0$. Using Young's inequality again, we have

$$\begin{aligned} &E\{|\delta(t)|^{p-2}\delta^T(t)\mathcal{L}[\delta(t) - \delta(t - \tau(t))]\} \\ &\leq E\{|\delta(t)|^{p-1}|\mathcal{L}[\delta(t) - \delta(t - \tau(t))]\} \\ &\leq \beta(p-1)m(t) + \frac{1}{\beta^{p-1}}E\{|\mathcal{L}[\delta(t) - \delta(t - \tau(t))]|^p\} \end{aligned} \tag{31}$$

where $\beta > 0$ is a constant to be chosen later. By (11),

$$\begin{aligned} &E|\mathcal{L}[\delta(t) - \delta(t - \tau(t))]|^p \\ &= E\left|\int_{t-\tau(t)}^t [c(s)\mathcal{L}^2\delta(s - \tau(s))ds + c(s)\mathcal{L}(I - J)\Theta dW(s)]\right|^p \\ &\leq 2^{p-1}E\left|\int_{t-\tau(t)}^t c(s)\mathcal{L}^2\delta(s - \tau(s)) ds\right|^p \\ &\quad + 2^{p-1}E\left|\int_{t-\tau(t)}^t c(s)\mathcal{L}(I - J)\Theta dW(s)\right|^p \\ &\leq 2^{p-1}\tau^{p-1}\|\mathcal{L}^2\|^p \int_{t-\tau(t)}^t |c(s)|^p |\delta(s - \tau(s))|^p ds \\ &\quad + \hat{C} \int_{t-\tau}^t |c(s)|^2 ds \end{aligned} \tag{32}$$

where we have used Hölder's inequality twice and a moment inequality for stochastic integrals (see, e.g. Theorem 1.7.1 of

[20]), with

$$\hat{C} := \|\mathcal{L}(I - J)\Theta\|^p \tau^{\frac{p-2}{2}} \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}$$

being a constant depending on τ , p , and $\mathcal{L}(I - J)\Theta$. Since $c(t) \rightarrow 0$ as $t \rightarrow \infty$, for any $\kappa < 1$, there exists $T = T(\kappa) \geq \tau$, such that $c(t) \leq \kappa$ for all $t \geq T - \tau$. For $t \geq T$, inequality (32) gives

$$\begin{aligned} & E |\mathcal{L}[\delta(t) - \delta(t - \tau(t))]|^p \\ & \leq \kappa 2^{p-1} \tau^{p-1} \|L^2\|^p \int_{t-\tau(t)}^t E |\delta(s - \tau(s))|^p ds \\ & \quad + \hat{C} \int_{t-\tau}^t |c(s)|^2 ds \end{aligned} \quad (33)$$

Now, putting all the above inequalities (28)–(33) together into (27), we obtain

$$\begin{aligned} D^+ m(t) & \leq -p\lambda_2(\hat{\mathcal{L}})c(t)m(t) + \mu c(t) \sup_{-2r \leq s \leq 0} m(t+s) \\ & \quad + \gamma(t), \quad \forall t \geq T \end{aligned} \quad (34)$$

where

$$\mu = \varepsilon \frac{p(p-2)}{2} C_0 + \beta p(p-1) + \kappa p \beta^{1-p} 2^{p-1} \tau^{p-1} \|L^2\|^p \quad (35)$$

and

$$\begin{aligned} \gamma(t) & = p\beta^{1-p} \hat{C} \int_{t-\tau}^t |c(s)|^2 ds \\ & \quad + \frac{p(p-2)}{2} \hat{\varepsilon} C_0 c^2(t) + \frac{p}{2} C_0 c^2(t) \\ & := C_1 \int_{t-\tau}^t |c(s)|^2 ds + C_2 c^2(t) \end{aligned} \quad (36)$$

Note that, in view of (35), we can choose ε , β and κ sufficiently small such that $p\lambda_2(\hat{\mathcal{L}}) > \mu$, for any given p , τ and \mathcal{L} . Therefore there exists $\rho > 0$ such that $-p\lambda_2(\hat{\mathcal{L}}) + \mu e^{2\rho\tau} = -\rho$. Lemma 1 shows that

$$\begin{aligned} m(t) & \leq m_0 \exp \left\{ -\rho \int_T^t c(s) ds \right\} \\ & \quad + \int_T^t \exp \left\{ -\rho \int_s^t c(r) dr \right\} \gamma(s) ds \end{aligned} \quad (37)$$

on $[\tau, \infty)$ with $m_0 = \sup_{T-2\tau \leq s \leq T} m(s) < \infty$ and $\gamma(t)$ defined in (36). Based on properties of the function $c(t)$ in (7), it can be shown that the right-hand side of (37) goes to 0 as $t \rightarrow \infty$ (see [11] for details). Therefore $m(t) \rightarrow 0$ as $t \rightarrow \infty$. Now note that

$$\alpha(t) = \frac{1}{n} \mathbf{1}^T x(t) = \frac{1}{n} \mathbf{1}^T x(0) + \int_0^t \frac{1}{n} c(s) \mathbf{1}^T (I - J)\Theta dW(s).$$

Using a moment inequality for stochastic integrals (see, e.g. Theorem 1.7.1 of [20])

$$\begin{aligned} E \left| \int_0^t \frac{1}{n} c(s) \mathbf{1}^T (I - J)\Theta dW(s) \right|^p & \leq C \int_0^t c^2(s) ds \\ & < C \int_0^\infty c^2(s) ds < \infty \end{aligned}$$

where C is a finite constant, it follows that α is an L^p -bounded martingale. Doob's martingale convergence

theorem (see, e.g. [36], Theorem 2.10, Chapter II) guarantees there exists a random variable x^* with $E|x^*|^p < \infty$, such that

$$\lim_{t \rightarrow \infty} \alpha(t) = x^* \text{ a.s.} \quad \text{and} \quad \lim_{t \rightarrow \infty} E|\alpha(t) - x^*|^p = 0.$$

Moreover, $E(x^*) = E(\alpha(t)) = E(\alpha(0)) = \sum_{i=1}^n x_i(0)/n$. On the other hand, for all $i \in \mathcal{I}$,

$$\begin{aligned} |x_i(t) - x^*|^p & \leq 2^{p-1} |x_i(t) - \alpha(t)|^p + 2^{p-1} |\alpha(t) - x^*|^p \\ & = 2^{p-1} |\delta_i(t)|^p + 2^{p-1} |\alpha(t) - x^*|^p \\ & \leq 2^{p-1} V(t) + 2^{p-1} |\alpha(t) - x^*|^p \end{aligned} \quad (38)$$

It follows from both $m(t) = E(V(t)) \rightarrow 0$ and $E|\alpha(t) - x^*|^p \rightarrow 0$, that

$$E|x_i(t) - x^*|^p \rightarrow 0,$$

as $t \rightarrow \infty$. Moreover, it is easy to check that

$$E|x_i(t)|^p \leq 2^{p-1} EV((t)) + 2^{p-1} E\{\alpha^p(t)\} < \infty, \quad t \geq 0.$$

Therefore p th moment consensus is reached. The proof is complete. \square

Remark 4: Compared with [11], the theoretical results in this paper are clearly sharpen. In fact, the results obtained by [11] are delay-dependent and the results in this paper are delay-independent. Moreover, in the current paper, we are able to establish both almost sure consensus and general p th moment consensus in the continuous-time setting. It is clear from the proof of Theorem 1 that the role of the function $c(t)$ is to attenuate the noise effects as $t \rightarrow \infty$. Condition (7) and the assumption that $c(t)$ is monotonically decreasing implies that $c(t)$ is vanishing as $t \rightarrow \infty$. This is also used in the proof to show that the convergence results hold for arbitrarily bounded delays, a result that is not established in [11].

3.2 Networks with arbitrary switching topology

In general, the network topology specified by the weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ can be time-varying because of node and link failures/creations, packet-loss, asynchronous consensus, formation reconfiguration, evolution and flocking as pointed out by Olfati-Saber *et al.* [16]. To effectively model the dynamic changing of the network structures, we consider a collection of digraphs and introduce a general time-dependent switching signal, either deterministic or stochastic, to model the switching of the network structures among the collection of digraphs.

We first consider deterministic time-dependent switching. Let \mathcal{Q} denote a finite index set and $\{\mathcal{G}_q: q \in \mathcal{Q}\}$ a family of digraphs. Let $\sigma: \mathbb{R}^+ \rightarrow \mathcal{Q}$ be a piecewise constant and right-continuous function called a 'switching signal'. The collective dynamics (8) can be written as a switched system

$$dx(t) = c(t)[-L_\sigma x(t - \tau(t)) dt + \Theta dW(t)] \quad (39)$$

where L_q ($q \in \mathcal{Q}$) is the corresponding graph Laplacian of \mathcal{G}_q .

Theorem 2: Suppose that each digraph \mathcal{G}_q in $\{\mathcal{G}_q: q \in \mathcal{Q}\}$ is strongly connected and balanced and the time-delays are

uniform, that is, $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$. The consensus protocol (6) leads to

- (i) p th ($p \geq 2$) moment average-consensus for the agents in (2) under arbitrary deterministic switching signals.
- (ii) Almost sure average-consensus for the agents in (2) under arbitrary deterministic switching signals, if, in addition, the time-varying delay satisfies $\dot{\tau}(t) \leq d < 1$ for all $t \geq 0$ and some constant d .

Both results hold for networks with uniform communication delays of arbitrary size.

Proof: Let $\sigma(t)$, $t \geq 0$, be a given switching signal. Then $V(t)$ can serve as a common Lyapunov function for the displacement dynamics

$$d\delta(t) = c(t)[-L_{\sigma(t)}\delta(t - \tau(t)) dt + (I - J)\Theta dW(t)] \quad (40)$$

which follows from (39).

Proof of almost sure consensus: With the switching signal $\sigma(t)$, the estimate in (24) can be replaced by

$$V(t) \leq \zeta + I_3(t) - \mu_\varepsilon \int_{T_\varepsilon}^t c(s)|\delta(s)|^2 ds + M(t) \quad (41)$$

where $M(t)$ is the same continuous local martingale null at T_ε , ζ and μ_ε are of the same form in (25) and (26) with $\|\mathcal{L}\|^4$ replaced by $\max_{q \in \mathcal{Q}} \|\mathcal{L}_q\|^4$, and

$$I_3(t) = \int_{T_\varepsilon}^t \left| \int_{s-\tau(s)}^s c(r) \mathcal{L}_{\sigma(s)}(I - J)\Theta dW(r) \right|^2 ds$$

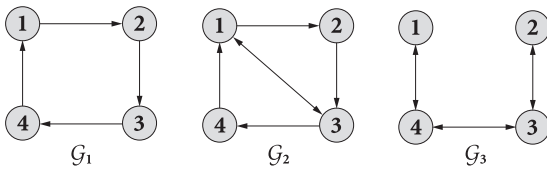
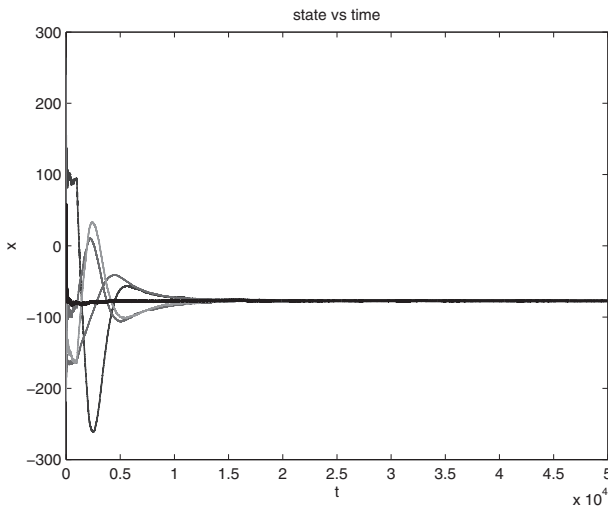


Fig. 2 Three different network topologies of 4-agents



We can again choose $\varepsilon > 0$ sufficiently small such that

$$\mu_\varepsilon = 2 \min_{q \in \mathcal{Q}} \lambda_2(\hat{\mathcal{L}}_q) - 2\varepsilon - \frac{2\varepsilon \tau \max_{q \in \mathcal{Q}} \|\mathcal{L}_q\|^4}{1 - d} > 0$$

and similarly show that $\mathcal{I}_3(t)$ is a non-negative, $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted, and continuous increasing process satisfying $I_3(T_\varepsilon) = 0$ with $\lim_{t \rightarrow \infty} \mathcal{I}_3(t) < \infty$ a.s. The rest of the proof is essentially the same as the proof of Theorem 1.

Proof of pth moment consensus: Following similar argument as in the proof of Theorem 1, we can obtain

$$D^+ m(t) \leq -p \min_{q \in \mathcal{Q}} \lambda_2(\hat{\mathcal{L}}_q) c(t) m(t) + \mu c(t) \sup_{-2r \leq s \leq 0} m(t+s) + \gamma(t), \quad \forall t \geq T \quad (42)$$

where

$$\mu = \varepsilon \frac{p(p-2)}{2} C_0 + \beta p(p-1) + \kappa p \beta^{1-p} 2^{p-1} \tau^{p-1} \max_q \|\mathcal{L}_q\|^{2p}$$

and $\gamma(t)$ is still of the form

$$\gamma(t) = C_1 \int_{t-\tau}^t |c(s)|^2 ds + C_2 e^2(t) \quad (43)$$

with a modification of the constant C_1 . Note that we can again choose ε , β , and κ sufficiently small such that $p \min_{q \in \mathcal{Q}} \lambda_2(\hat{\mathcal{L}}_q) > \mu$, for any given p , τ , and set of $\{\mathcal{L}_q: q \in \mathcal{Q}\}$. Therefore there exists $\rho > 0$ such that $-p \min_{q \in \mathcal{Q}} \lambda_2(\hat{\mathcal{L}}_q) + \mu e^{2\rho\tau} = -\rho$. Lemma 2 implies that the same estimate (37) holds and the rest of the proof is essentially the same as the proof of Theorem 1. \square

3.3 Networks with Markovian switching topology

Hybrid systems driven by continuous-time Markov chains have long been used to model many practical systems where abrupt changes in their structures and parameters caused

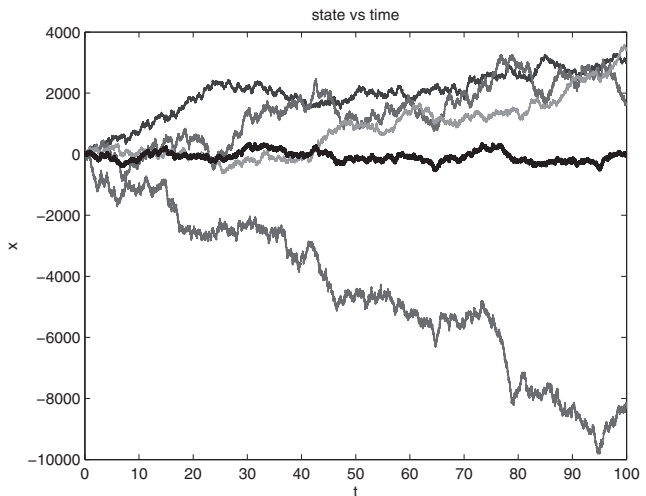


Fig. 3 Simulation results of sample (almost sure) consensus for fixed topology \mathcal{G}_1 : the left figure shows results for $c(t) = 1/(t+1)$ and the right figure shows results for $c(t) = 1$, both with $\tau = 1000$ Coloured lines represent individual agent states and the thick black line indicates the averaged state

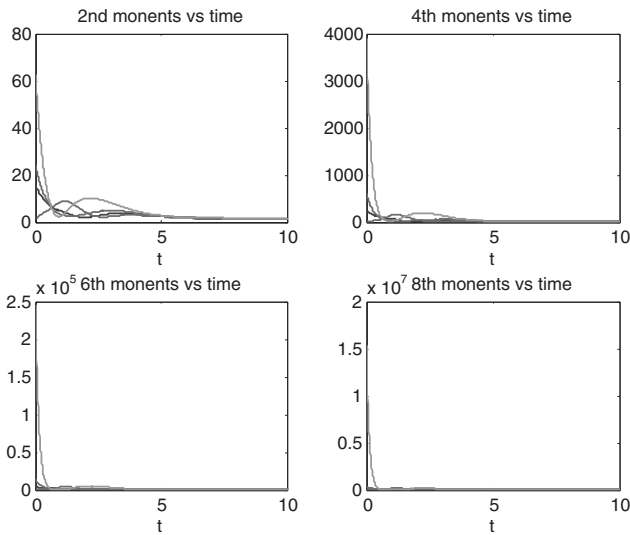


Fig. 4 Simulation results of moment consensus for fixed topology \mathcal{G}_1 : the four different panels show convergence results for different moments, with differently coloured lines represent moments of different individual agents. We have chosen $c(t) = 1/(t + 1)$, delay size $\tau = 1$, (deterministic) initial conditions $[4 \ 5 \ -1 \ 8]$, and noise intensity $\Theta = 2$

by phenomena such as component failures and repairs as pointed out by [21]. In this subsection, we consider the case where the switching signal is modelled by a continuous-time Markov chain. More specifically, let $\sigma: \mathbb{R}^+ \rightarrow \mathcal{P}$ be a right-continuous Markov chain with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P(\sigma(t + \Delta) = j | \sigma(t) = i) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta), & i = j \end{cases}$$

where $\Delta > 0$, N is the cardinality of \mathcal{P} , $\gamma_{ij} \geq 0$ for $i \neq j$, and $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. Such switching signals are called ‘Markovian switching signals’.

Theorem 3: Suppose that each digraph \mathcal{G}_q in $\{\mathcal{G}_q: q \in \mathcal{Q}\}$ is strongly connected and balanced and the time-delays are

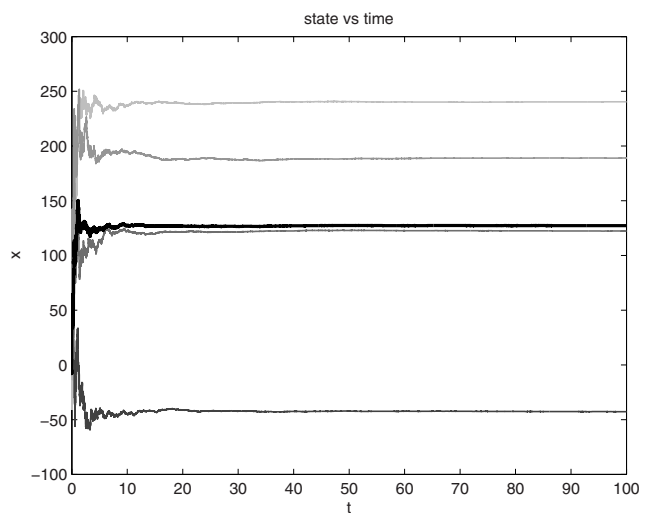
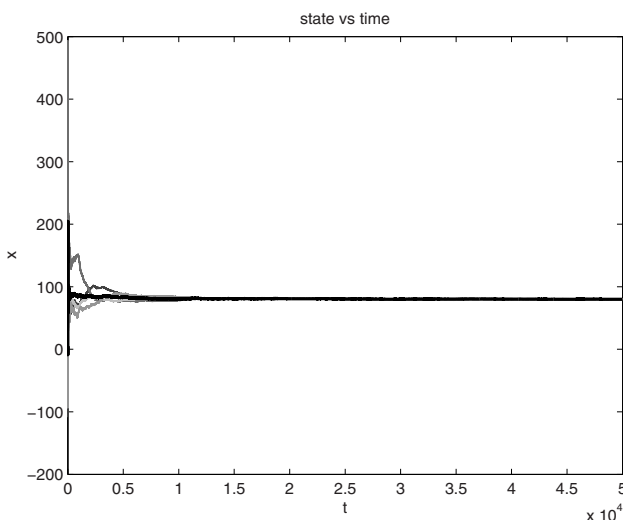


Fig. 5 Simulation results of sample (almost sure) consensus for a Markovian switching topology among $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ driven by the generator $\Gamma = [-0.45 \ 0.15 \ 0.4; 0.2 \ -0.4 \ 0.4; 0.5 \ 0.25 \ -0.25]$: the left figure shows results for $c(t) = 1/(t + 1)$ and the right figure shows results for $c(t) = 1/(t + 1)^2$, both with $\tau = 1000$ Coloured lines represent individual agent states and the thick black line indicates the averaged state

uniform, that is, $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$. The consensus protocol (6) leads to

- (i) p th moment average-consensus for the agents in (2) under arbitrary Markovian switching signals.
- (ii) Almost sure average-consensus for the agents in (2) under arbitrary Markovian switching signals, if, in addition, the time-varying delay satisfies $\dot{\tau}(t) \leq d < 1$ for all $t \geq 0$ and some constant d .

Both results hold for networks with uniform communication delays of arbitrary size.

Proof: The proof is essentially the same as that of Theorem 2 except that we should apply the generalized Itô’s formula [21, Theorem 1.45] because of the Markovian switching. Since a common Lyapunov function $V = |\delta|^p$, $p > 0$, is used, we obtain the same estimates for both $m(t)$ and $V(t)$ as in the proof of Theorem 2. The rest of the proof is the same. \square

Remark 5: Both Theorem 2 and 3 require that each of the digraphs is strongly connected and balanced digraph, while the switching can be arbitrary. This is in accordance with the stability theory for switched systems, where stability under arbitrary switching must imply each of the subsystems itself is stable. Relaxed conditions such as joint connectivity together with some constraints on the switching signals can also lead to consensus under switching topology (see, e.g. [9, 37]). The focus here is on deriving both p th moment and almost sure average-consensus under arbitrary switching signals.

4 Simulation results and discussions

In this section, we conduct numerical simulations to demonstrate our main results, mainly to show that the consensus protocol (6) can lead to consensus for networks of agents with arbitrary sized communication delays and measurement noises.

Consider dynamical networks of four agents. Fig. 2 shows three different topologies denoted by the family $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$. While all digraphs in the figure have 0–1 weights, they are also all strongly connected and balanced. The intensity of the measurement noises σ_{ij} can be of arbitrarily fixed size. We simulate two different situations. First, we consider a fixed network topology given by \mathcal{G}_1 . According to Theorem 1, for communication delays of bounded but otherwise arbitrary size, the stochastic consensus protocol (6) will lead to both moment and almost sure average-consensus for the network \mathcal{G}_1 . To illustrate this, we simulate a somewhat extreme case, where the communication delay is chosen to be 1000. With $c(t) = 1/(t + 1)$, average-consensus is confirmed by simulation as shown in Fig. 3, although, because of the large size of delay, it takes a large amount of time to actually reach consensus. As we can see, choosing the right function $c(t)$ as shown in the main results of this paper is critical to achieve consensus, both attenuate the noise and to overcome the destabilisation induced by large size communication delays. It is shown in Fig. 3 that if we choose $c(t) \equiv 1$, the noises cannot be attenuated. Actually, the states tend to diverge from each other rather quickly, and average-consensus is not reached. Furthermore, Fig. 4 illustrates consensus results of different moments for fixed topology \mathcal{G}_1 . Second, we consider the situation where the network topologies are randomly switching among the three different configurations in $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ according to a continuous-time Markov chain. It follows from Theorem 3 that, for communication delays of bounded but otherwise arbitrary size, both moment and almost sure average-consensus are reached. This is confirmed by simulation as shown in Fig. 5, where we choose $c(t) = 1/(t + 1)$. Again, the importance of the function $c(t)$, is evident here. If we choose $c(t) = 1/(t + 1)^2$, then, while the noises seem to be over attenuated, the states are trapped at different values, and consensus cannot be reached.

5 Conclusions

We have investigated the average-consensus problem of networked multi-agents systems subject to measurement noises. A time-varying consensus protocol that takes into the account both the measurement noises and general time-varying communication delays has been considered. We have considered general networks with fixed topology, with arbitrary deterministic switching topology, and with Markovian switching topology. For each of these three cases, we have obtained sufficient conditions under which the considered consensus protocol leads to both p th moment and almost sure average-consensus.

To conclude the paper, we highlight two key contributions of this paper and the techniques used here to achieve them. First, both almost sure consensus and general p th moment consensus are formulated and derived in the continuous-time setting, using a semimartingale convergence theorem and moment inequality techniques, respectively, from stochastic calculus. Second, the consensus results are delay-independent in the sense that they can be applied to networks of arbitrarily sized communication delays. This non-conservatism is achieved by carefully analysing the noise attenuation function. It remains an open question whether similar results can be proved for networks with heterogeneous communication delays.

Future work along the line of this research could also include its generalisation to double-integrator multi-agent systems, as recently studied by Cheng *et al.* [5], where

only mean-square consensus has been studied for networks without communication delays, and the consensus and synchronisation of non-linear coupled oscillators, which has recently found applications in power networks [38].

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7 References

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