Invariance principles for delay differential inclusions

Kun-Zhi Liu[1], Xi-Ming Sun[1], Wei Wang[1], Jun Liu[2]

1. School of Control Science and Engineering, Dalian University of Technology, Dalian 116024, China
E-mail: kunzhiliu1989@mail.dlut.edu.cn,sunxm@dlut.edu.cn,wangwei@dlut.edu.cn

2. Department of Automatic Control and Systems Engineering, University of Sheffield, Sheffield S1 3JD, U.K.
E-mail: j.liu@sheffield.ac.uk

Abstract: This paper establishes two invariance principles for delay differential inclusions. The delay differential inclusions are required to satisfy the basic assumptions: the right-hand sides are upper semicontinuous and take nonempty compact and convex values on the domains. The classical LaSalle's invariance principle for delay differential inclusions is established successfully by locally Lipschitz Lyapunov-Krasovskii functionals and several stability corollaries are developed. Besides, the concept of limit delay differential inclusions is proposed to generalize the invariance principle to time-varying delay differential inclusions. Some numerical examples are given to show the effectiveness of the proposed results.

Key Words: Delay differential inclusions, invariance principle, Lyapunov-Krasovskii functional, limit delay differential inclusions

1 Introduction

LaSalle's invariance principle is one of the most important theoretical tools to analyze the convergence and asymptotic stability of autonomous ordinary differential equations and difference equations ([12], [11]). Delay differential inclusions can describe a wide variety of dynamic systems affected by delays and can also be used to analyze stability and robustness of delay differential equations with discontinuous right-hand sides [17]. This paper is aimed at developing the classical invariance principle [12], [10] for delay differential inclusions.

Lyapunov's second theorem is the most powerful weapon to analyze the stability of general nonlinear systems ([11]). It demands that the derivative of a Lyapunov function is nonzero beyond the origin. However, such a strict Lyapunov function is not easy to construct and a weaker Lyapunov function may be obtained relatively easily, which implies that the derivative of a Lyapunov function can be zero beyond the origin. LaSalle's invariance principle is a sharp tool to analyze the convergence and asymptotic stability of autonomous systems with a weak Lyapunov function. This result has many important extensions, such as [10, 14, 15, 20, 21, 1]. More specifically, LaSalle's invariance principle is generalized to time-varying differential equations in [1] by introducing the concept of limit equations. Such an invariance principle concludes that the ω-limit set of a solution of time-varying system is an invariant set of its limit equation. The author of [20] extends the integral invariance principle to differential inclusions. In [14], LaSalle's invariance principle is extended to impulsive switched systems that admit one jump at a jumping instant. For differential equations with discontinuous right-hand side, LaSalle's invariance principle is also developed under Filippov solutions in [21].

Time delays are often inevitable in engineering and have attracted a lot of attention [18], [5], [23], [22]. The author of [10] extends the LaSalle's invariance principle to functional differential equations. It is asserted in [10] that any trajectory approaches the largest invariant subset of a set where the derivative of a Lyapunov functional along the solution starting from any initial function belonging to this set is zero. On the other hand, it has been shown that delay differential inclusions can describe a wide variety of dynamic systems affected by delays which are often inevitable in practical applications ([9], [6]). However, stability criteria are still not well developed for such a class of widely used systems. Some stability theorems for delay differential inclusions have been developed in [17] and [24] where [17] admits a more general functional. Only invariantly differentiable functional is used in [24] and the results in [24] cannot be applied to nonautonomous delay differential inclusions with only weak Lyapunov functional. The key reason why the results in [24] do not apply to time-varying delay differential inclusions is that the ω-limit set of a solution of time-varying systems is no longer an invariant set. Since the general invariance principles are very important in stability analysis of delayed systems while remain unsolved, this motivates the research of this paper.

The contribution of the paper consists of two respects. The first one is that the classical LaSalle's invariance principle is established for autonomous delay differential inclusions by general locally Lipschitz functionals. To establish such an invariance principle, the weakly invariant property of the ω-limit set of a precompact solution for an autonomous delay differential inclusion is given. The introduced challenge is that the composite function of a locally Lipschitz functional and a locally absolutely continuous

This work was supported by the National Natural Science Foundation of China under Grant Nos 61174058 and 61325014. Corresponding author: sunxm@dlut.edu.cn

function may be not absolutely continuous. Besides, it is not trivial to establish the invariance principle by a locally Lipschitz functional, which does not depend on specific solutions. To overcome these obstacles, we adopt a lemma related to the properties of the locally Lipschitz functional such that the invariance principle can be formulated independent of specific solutions. As is known, the classical LaSalle's invariance principle can not be applied to nonautonomous systems. The second contribution in this paper is that the concept of limit delay differential inclusions is proposed for the first time. With this new concept in time-varying delay differential inclusions, the LaSalle's invariance principle is generalized to time-varying delay differential inclusions. The obtained results can be used to analyze the stability of nonlinear time-varying delayed system such as adaptive control systems.

This paper is organized as follows. Notation and preliminaries for delay differential inclusions are given in Section 2. The main results are presented in Section 3 and 4. A case study is shown in Section 5. Section 6 draws the conclusions of this paper.

2 Preliminaries

The following notation is adopted in this paper.

$\mathbb{R}^n$ is the n-dimensional Euclidean space. For $x \in \mathbb{R}^n$, $|x|$ denotes the 2-norm of x. $C([-r, 0]; \mathbb{R}^n)$ denotes the collection of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$, equipped with the norm $||\phi||_r = \max_{-r \le s \le 0} |\phi(s)|$, where $\phi \in C([-r, 0]; \mathbb{R}^n)$. Without ambiguity, C_r will often denote $C([-r, 0]; \mathbb{R}^n)$. For a continuous function x defined at least on $[-r + t, t]$, let $A(t)x = x(t + \theta), \theta \in [-r, 0]$. For a set $Y \subset C_r$, $\bar{Y}$ denotes the closure of Y in the space C_r. A metric space $\mathbb{R}_{\ge 0} \times C_r$ is endowed with the metric $||y - z|| = \max\{|t - s|, ||\phi - \varphi||_r\}$, where $y = (t, \phi)$ and $z = (s, \varphi)$ belong to $\mathbb{R}_{\ge 0} \times C_r$. A set-valued functional $\mathcal{F} : \mathbb{R}_{\ge 0} \times C_r \rightrightarrows \mathbb{R}^n$ is upper semicontinuous on $\mathbb{R}_{\ge 0} \times D \subseteq \mathbb{R}_{\ge 0} \times C_r$ if, given $\bar{y} \in \mathbb{R}_{\ge 0} \times D$, for each $\epsilon > 0$ there exists $\delta > 0$ such that, for all $\bar{z} \in \mathbb{R}_{\ge 0} \times D$ satisfying $||\bar{z} - \bar{y}|| < \delta$, we have $\mathcal{F}(\bar{z}) \subseteq \mathcal{F}(\bar{y}) + \epsilon B$, where B is the unit open ball in $\mathbb{R}^n$.

Consider the following autonomous delay differential inclusions

$$\dot{x} \in \mathcal{F}(A(t)x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state. $D \subset C_r$ is an open set containing the origin. $\mathcal{F} : D \rightrightarrows \mathbb{R}^n$ is a set-valued functional satisfying the basic assumptions: it is upper semicontinuous and $\mathcal{F}(\phi)$ is nonempty, compact and convex for each $\phi \in D$. Throughout this paper, $0 \in \mathcal{F}(0)$ and the set-valued functional $\mathcal{F}$ maps bounded sets of D into bounded sets of $\mathbb{R}^n$. For simplicity, the delay differential inclusions (1) will be denoted by $\mathcal{DI}$.

Let us review some basic facts about $\mathcal{DI}$.

Definition 1 *[7] A continuous function $x : [-r, T) \to \mathbb{R}^n$ with $0 < T \le \infty$ is said to be a solution of $\mathcal{DI}$ if it satisfies the following conditions:*
(1) it is absolutely continuous on each compact subset of $[0, T)$;
(2) $A(t)x \in D$ for all $t \in [0, T)$;
(3) it satisfies $\mathcal{DI}$ almost everywhere on $[0, T)$.

On the existence and continuation of solutions of $\mathcal{DI}$, details can be found in [8], [17].

Given a solution x of $\mathcal{DI}$, $A(t)x$ is the trajectory of x in the space C_r. The ω-limit set $\omega(x)$ of $A(t)x$ is a set of all functions $\varphi \in C_r$ for which there exists a sequence $A(t_n)x$, with $t_n \to \infty$ as $n \to \infty$, such that $\lim_{n\to\infty} A(t_n)x = \varphi$. A set $H \subset D$ is called a weakly invariant set of $\mathcal{DI}$ if for any $\phi \in H$, there exists a continuous function $x(t)$ defined on $(-\infty, \infty)$ satisfying $A(t)x \in H$ for all $t \in (-\infty, \infty)$ and $A(0)x = \phi$, such that for any $\sigma \in \mathbb{R}$, the function x^* defined on $[0, \infty)$ with $A(0)x^* = A(\sigma)x$ and $x^*(t) = x(t + \sigma)$ for all $t \ge 0$ is a solution of $\mathcal{DI}$. A solution $x(t)$ of $\mathcal{DI}$ is said to be precompact if it is defined on $[-r, \infty)$ and satisfies that the set $\overline{\{A(t)x : t \in [0, \infty)\}}$ is compact and $\overline{\{A(t)x : t \in [0, \infty)\}} \subset D$.

Given a continuous function $w : [t_0, b) \to \mathbb{R}$, $D^+w(t) = \limsup_{s\to 0^+} \frac{w(t+s)-w(t)}{s}$.

Given a continuous functional V defined on C_r, the upper right-hand derivative of functional V for some $\upsilon \in \mathbb{R}^n$ is given, in the constructive way ([19], [3], [17]), as

$$D^+V(\phi, \upsilon) = \limsup_{h\to 0^+} \frac{V(\phi_h^*) - V(\phi)}{h}, \tag{2}$$

where

$$\phi_h^*(s) = \begin{cases} \phi(s + h), s \in [-r, -h] \\ \phi(0) + \upsilon(s + h), s \in (-h, 0]. \end{cases} \tag{3}$$

3 Main results

In this section, the classical invariance principle for functional differential equations [10], [12] will be generalized to delay differential inclusions.

Lemma 1 *If x is a precompact solution of $\mathcal{DI}$, then the ω-limit set $\omega(x)$ of $A(t)x$ is nonempty, compact, connected and weakly invariant. Moreover, $A(t)x$ approaches $\omega(x)$ as $t \to \infty$.*

The above lemma is a direct result of the properties of solutions of $\mathcal{DI}$ under the basic assumptions (see Theorem 3 in [16] and [24]).

Remark 1 *Due to the nonuniqueness of solutions of $\mathcal{DI}$, only weak invariant properties of the ω-limit set $\omega(x)$ can be asserted. This is the main difference from the invariant properties of the ω-limit set of solutions for autonomous functional differential equations [10], [12].*

A scalar functional V defined on $\phi \in C_r$ is said to be locally Lipschitz if, for any $\varphi \in C_r$, there exists $\ell_\varphi > 0$ such that the following inequality holds in some neighborhood N_φ of φ:

$$|V(\varphi_1) - V(\varphi_2)| \le \ell_\varphi ||\varphi_1 - \varphi_2||_r, \forall \varphi_1, \varphi_2 \in N_\varphi. \tag{4}$$

Definition 2 *[17] A continuous functional* $V : D \to \mathbb{R}$ *is said to satisfy the basic properties if for any continuous function* $x : [t_0 - r, T) \to \mathbb{R}^n$ *that is locally absolutely continuous in* $[t_0, T)$ *and satisfies* $A(t)x \in D$ *of all* $t \in [t_0, T)$*, the composite function* $V(A(t)x)$ *is locally absolutely continuous in* $[t_0, T)$.

Remark 2 *A functional* V *possessing the basic properties implies that the composite function* $V(A(t)x)$ *is differentiable almost everywhere on* $[0, T)$*. A class of widely used functional* $V(\phi) = R(\phi(0)) + \int_{-r}^{0} S(\phi(s))ds$ *with* $R : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *locally Lipschitz continuous and* $S : C_r \to \mathbb{R}_{\geq 0}$ *continuous satisfy the basic properties.*

The following definition is a straitforward extension of Lyapunov functional for delay differential equations (see [12]) to delay differential inclusions.

Definition 3 *Let* $G \subset D$*.* V *is said to be a Lyapunov functional of* $\mathcal{DI}$ *on* G *if (i) it is locally Lipschitz continuous and possesses the basic properties on* $\overline{G}$*, (ii)* $D^+V(\phi, f) \leq 0$ *for all* $\phi \in G$ *and* $f \in \mathcal{F}(\phi)$.

Before giving one of our main theorems, denote $E_V = \{\phi : \phi \in \bar{G}, \exists f \in F(\phi) \ s.t. \ D^+V(\phi, f) = 0\}$ and let M_V be the largest weakly invariant subset of $\mathcal{DI}$ in $\bar{E}_V$.

Lemma 2 *[17] Suppose that a functional* $V : C_r \to \mathbb{R}$ *is locally Lipschitz in* ϕ*. Then for any solution* $x : [t_0 - r, T) \to \mathbb{R}^n$ *of* $\mathcal{DI}$*, it holds that*

$$D^+w(t) = D^+V(A(t)x, \dot{x}(t))$$

for almost all $t \in [t_0, T)$ *where* $w(t)$ *is the composite function* $V(A(t)x)$.

Theorem 1 *Let* V *be a Lyapunov functional of* $\mathcal{DI}$ *on* G*, and let* $x(t)$ *be a solution of* $\mathcal{DI}$ *that is precompact and such that* $A(t)x$ *remains in* G *for all* $t \geq 0$*. Then, for some* c*,* $A(t)x$ *approaches* $M_V \bigcap V^{-1}(c)$.

Proof of Theorem 1: From the properties of the functional V, the composite function $V(A(t)x)$ is locally absolutely continuous and thus $\dot{V}(A(t)x)$ exists almost everywhere. Lemma 2 yields

$$\begin{aligned} \dot{V}(A(t)x) &= D^+V(A(t)x) \\ &= D^+V(A(t)x, \dot{x}) \end{aligned} \tag{5}$$

almost everywhere. As a consequence, it follows that, for any $t \geq \tau \geq 0$,

$$\begin{aligned} V(A(t)x) &= V(A(\tau)x) + \int_{\tau}^{t} \dot{V}(A(s)x)ds \\ &= V(A(\tau)x) + \int_{\tau}^{t} D^+V(A(s)x, \dot{x})ds. \end{aligned} \tag{6}$$

Combining condition (ii) in Definition 3 and inequality (6) results in

$$V(A(t)x) \leq V(A(\tau)x) \tag{7}$$

for any $t \geq \tau \geq 0$. Therefore, $V(A(t)x)$ is monotonously decreasing. Since V is continuous and $\overline{\{A(t)x : t \in [0, \infty)\}}$ is a compact set contained in D (this results from the precompact property of x), $V(A(t)x)$ is bounded below. Therefore, $\lim_{t\to\infty} V(A(t)x) = c$ for some constant $c \in \mathbb{R}$. Let $\varphi \in \omega(x)$ be arbitrary, then there exists a sequence $A(t_n)x$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} A(t_n)x = \varphi$. By the continuity of V, one has

$$\lim_{n\to\infty} V(A(t_n)x) = V(\varphi) = c. \tag{8}$$

This implies that V is constant on $\omega(x)$. Let $y(t)$ be a solution of $\mathcal{DI}$ starting from φ and $A(t)y$ remains in $\omega(x)$. Then $\frac{dV(A(t)y)}{dt} = 0$ for all $t \geq 0$. From Lemma 2, we have

$$\frac{dV(A(t)y)}{dt} = D^+V(A(t)y, \dot{y}) = 0 \tag{9}$$

almost everywhere. As a result, there exists a sequence $\{\tau_i\}$ with $\lim_{i\to\infty} \tau_i = 0$ such that

$$D^+V(A(\tau_i)y, \dot{y}) = 0 \tag{10}$$

and $\lim_{i\to\infty} A(\tau_i)y = \varphi$. Thus, $\varphi \in \bar{E}_V$ and $A(t)x$ approaches $M_V \bigcap V^{-1}(c)$ by Lemma 1.

Remark 3 *Theorem 1 establishes the LaSalle's invariance principle for delay differential inclusions by locally Lipschitz functional. This theorem can be used to analyze the convergence and asymptotic stability of discontinuous delayed systems.*

Note that, in order to apply Theorem 1, we have to identify the precompactness of a solution. Next, a mathematical lemma is introduced similar to Lemma 4.8 in Chapter 3 of [12].

Lemma 3 *Let* x *be a bounded solution of* $\mathcal{DI}$ *defined on* $[-r, \infty)$*. If* $A(t)x$ *has no positive limit points on the boundary of* D*, then* $x(t)$ *is precompact.*

Proof: From the definition of solutions $\mathcal{DI}$, one has, for any $t > 0$ and $t + \theta > 0$,

$$x(t+\theta) - x(t) = \int_{t}^{t+\theta} \dot{x}(s)ds. \tag{11}$$

By applying the properties of the set-valued functional $\mathcal{F}$ that maps bounded sets of D into bounded sets $\mathbb{R}^n$, it follows from relation (11) that, for any $t \geq 0$ and $t + \theta > 0$,

$$|x(t+\theta) - x(t)| \leq K|\theta| \tag{12}$$

for some constant $K > 0$. This implies that $x(t)$ is uniformly continuous on $[-r, \infty)$ and that the family $\{A(t)x : t \in [0, \infty)\}$ of functions are equicontinuous. Hence, $\overline{\{A(t)x : t \in [0, \infty)\}}$ is a nonempty compact set. Since $A(t)x$ has no positive limit points on the boundary of D, $\overline{\{A(t)x : t \in [0, \infty)\}} \subset D$. Therefore, $x(t)$ is precompact.□

We will give some stability corollaries for delay differential inclusion. A lemma on the solution properties is presented firstly as follows.

Lemma 4 *[17] Suppose that $\mathcal{F}: D \rightrightarrows \mathbb{R}^n$ is bounded on closed bounded set of D and satisfies basic assumptions on D. Then, for any initial values $\phi \in D$, there exists a solution of $\mathcal{DI}$ satisfying $A(0)x = \phi$. If a solution $x(t, \phi)$ is defined on a finite interval $[-r, T)$ and cannot be continued, then, for any bounded closed set U in C_r, U in D, there is a t_U such that $A(t)x \notin U$ for $t_U \leq t < T$.*

Remark 4 *If $D = C_r$, any solution of $\mathcal{DI}$ will diverge to ∞ in finite time or will be defined on $[-r, \infty)$.*

Corollary 1 *Consider the delay differential inclusion $\mathcal{DI}$. Suppose $D = C_r$ and there exists a functional $V : C_r \to \mathbb{R}$ locally Lipschitz continuous and possessing the basic properties on C_r such that*
$D^+V(\phi, f) \leq 0$ for all $\phi \in C_r$ and $f \in \mathcal{F}(\phi)$.
If $E_V = \{\phi : \phi \in C_r, \exists f \in F(\phi)\ s.t.\ D^+V(\phi, f) = 0\}$ and M_V is the largest weakly invariant subset of $\mathcal{DI}$ in $\bar{E}_V$, then, for any bounded solution x of $\mathcal{DI}$, $A(t)x$ approaches M_V as $t \to \infty$.

Corollary 2 *Consider the delay differential inclusion $\mathcal{DI}$. Suppose that there exists a functional $V : D \to \mathbb{R}_{\geq 0}$ locally Lipschitz continuous and possessing the basic properties on D, $\alpha_i \in \mathcal{K}_\infty (i = 1, 2)$ such that*
(1) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(||\phi||_r)$ for all $\phi \in D$;
(2) $D^+V(\phi, f) \leq 0$ for all $\phi \in D$ and $f \in \mathcal{F}(\phi)$.
Then the solution $x(t) = 0$ is stable. Moreover, if the largest weakly invariant set in D is the origin, then $x(t) = 0$ is asymptotically stable.

Proof: Stability is a direct result in [17]. Attractivity follows from the invariance principle in Theorem 1.□

4 Generalization to general time-varying delay differential inclusions

As can be seen from the analysis from Section 3, the key point of the LaSalle's invariance principle lies in the fact that the ω-limit set of a precompact solution of the autonomous delay differential inclusions is an invariance set. Obviously, this property does not hold for the time-varying case. In the following, we will consider a more general invariance principle that can be applied to time-varying delay differential inclusions.
Consider the following time-varying delay differential inclusion

$$\dot{x}(t) \in \mathcal{F}(t, A(t)x), \tag{13}$$

where $\mathcal{F} : \mathbb{R}_{\geq 0} \times D \rightrightarrows \mathbb{R}^n$ is a set-valued functional that is bounded on closed bounded subsets of D and satisfies the basic conditions (see [17]). Denote the collection of all such set-valued functionals by M.
Since the ω-limit set of $A(t)x$ for a solution x of (13) is no longer a weak invariant set of system (13), we will generalize the idea of limit equations proposed by Artstein (see [12],[1]) to overcome this obstacle.
Let $X \subset [0, \infty) \times C([-r, 0]; \mathbb{R}^n)$ and $\mathcal{F}_i : X \rightrightarrows \mathbb{R}^n, i = 1, 2, \cdots, \infty$, be a set-valued functional sequence defined on X. In the following, the concept of uniform convergence for a set-valued functional sequence is introduced.

Definition 4 *The sequence of set-valued functional $\mathcal{F}_i$: $X \rightrightarrows \mathbb{R}^n, i = 1, 2, \cdots, \infty$, converges uniformly to $\mathcal{F}$ on the set X if, for every $\epsilon > 0$ and $\rho > 0$, there exists a positive integer N such that, for any $i \geq N$,*

$$\left.\begin{array}{l} \mathcal{F}_i(z) \bigcap \rho B \subset \mathcal{F}(z) + \epsilon B \\ \mathcal{F}(z) \bigcap \rho B \subset \mathcal{F}_i(z) + \epsilon B \end{array}\right\} \; for\ all\ z \in X.$$

The translate by t of the set-valued functional $\mathcal{F}(s, \phi)$ is the functional $\mathcal{F}^t$ defined by $\mathcal{F}^t(s, \phi) = \mathcal{F}(t+s, \phi)$. Next, the main result in this section is given as follows.

Theorem 2 *Let $x : [t_0 - T, \infty)$ be a precompact solution of system (13). If, for a sequence $t_k \to \infty$, $A(t_k) \to \varphi \in D$, and the set-valued functional sequence $\mathcal{F}^{t_k}$ converges uniformly to a set-valued functional $\mathcal{G} \in M$ on each bounded closed subset of D, then there exists a solution $y : [t_0 - r, \infty)$ with $A(0)y = \varphi$ of $\dot{x} \in \mathcal{G}(s, A(s)x)$ such that $A(t)y \in \omega(x)$.*

Sketch of proof: Let $x_k = x_k(s)$ be defined by $x_k(s) = x(t_k + s)$. Then $x_k : [-r, \infty)$ with $A(0)x_k = A(t_k)x$ is a precompact solution of $\dot{x} \in \mathcal{F}^{t_k}(s, A(s)x)$. On each compact interval $[0, T]$, the family $\{x_k(s), s \in [0, T]\}$ are equicontinuous since there exists $B > 0$ such that

$$|\dot{x_k}(s)| \leq B \tag{14}$$

for almost all $s \in [0, T]$. Hence there exists a subsequence x_{k_i} of the function sequence x_k that converges uniformly to some absolutely continuous function $\bar{y} : [0, T] \to \mathbb{R}^n$. Let $y : [-r, T]$ satisfy $A(0)y = \varphi$ and $y(s) = \bar{y}(s), s \in [0, T]$. The sequence of function $x_{k_i}(s) : s \in \underline{[-r, T]}$ converges to $y : [-r, T]$ as $t \to \infty$ and $A(t)y \in \overline{\{A(t)x : t \geq t_0\}}$, which belongs to a bounded closed set contained in D by precompactness of the solution x.
We now show that y is a solution of $\dot{x} \in \mathcal{G}(s, A(s)x)$ in brief. Since the sequence of function $x_{k_i} : [-r, T]$ converges uniformly to $y : [-r, T]$, then $A(s)x_{k_i}$ converges uniformly to $A(s)y$. Combining upper semicontinuity of $\mathcal{F}$ and Lemma 13 in [4], this assertion can be given by following similar arguments of the proof for Lemma 1 in [4] (page 76).□
In fact, the above lemma states that the ω-limit set $\omega(x)$ of $A(s)x$ is a weakly invariant set of the limit differential inclusion $\dot{x} \in \mathcal{G}(s, A(s)x)$. This result can be used to analyze the stability of time-varying differential inclusions, where only weak Lyapunov functionals can be found.
Consider the following time-varying delayed systems

$$\dot{x} = f(t, A(t)x), \tag{15}$$

where $f : \mathbb{R}_{\geq 0} \times C_r \to \mathbb{R}^n$ is a continuous functional and maps bounded sets of C_r into bounded sets of $\mathbb{R}^n$.

Corollary 3 *Let $x : [t_0 - T, \infty)$ be a precompact solution of system (15). If, for a sequence $t_k \to \infty$, $A(t_k) \to \varphi \in C_r$, and the functional sequence f^{t_k} converges uniformly to a functional $g \in C_r$ on each bounded closed subset of C_r, then there exists a solution $y : [t_0 - r, \infty)$ with $A(0)y = \varphi$ of $\dot{x} = g(s, A(s)x)$ such that $A(t)y \in \omega(x)$.*

Remark 5 *The above results are a delay version of Artstein's invariance principle. Stability theorems for time-varying differential equations by applying the idea of limit equations have attracted a lot of attention and been developed in the recent decade such as [13]. While stability theorems for delay systems have received less attention and not well developed by adopting the powerful idea initially proposed in [1]. By the results in Theorem 2 and Corollary 3, we can also develop parallel stability theorems for delay systems.*

Example 1 *Consider the following system*

$$\begin{aligned}\dot{x}_1 &= e^{-2t}x_2(t),\\ \dot{x}_2 &= -e^{-2t}x_1(t) - 2x_2(t) + x_2(t-\tau).\end{aligned} \quad (16)$$

Choose a Lyapunov functional V of the following form

$$V(\phi) = \frac{1}{2}(\phi_1^2(0) + \phi_2^2(0)) + \int_{-\tau}^{0} \phi_2^2(s)ds. \quad (17)$$

Taking the derivative of V along the solution of system (16) yields that

$$\dot{V}(A(t)x) = -x_2^2(t) + x_2(t)x_2(t-\tau) - x_2^2(t-\tau).$$

Obviously $\dot{V}(A(t)x) \leq 0$. Let x be any solution of system (16). Then x will be bounded, that is $|x(t)| \leq B$ and $A(t)x$ converges to the following set

$$\{\phi \in C_\tau : \phi_2(0) = \phi_2(-\tau) = 0\}. \quad (18)$$

Since system (16) is a time-varying system, then LaSalle's invariance principle fails for this case. It is easy to confirm that the limit delay differential equation is as follows

$$\begin{aligned}\dot{x}_1 &= 0,\\ \dot{x}_2 &= -2x_2(t) + x_2(t-\tau).\end{aligned} \quad (19)$$

The largest invariant set in $\{\phi : \phi \in C_\tau, ||\phi||_\tau \leq B\}$ for system (19) belongs to the following set

$$\begin{aligned}\{\phi \in C_\tau : \exists c \in \mathbb{R}\ s.t.\ \phi_1(s) = c,\ \forall s \in [-\tau, 0],\\ \phi_2(s) = 0, \forall s \in [-\tau, 0], ||\phi||_\tau \leq B\}.\end{aligned} \quad (20)$$

Therefore, the solution x satisfies $A(t)x \in \{\phi \in C_\tau : \phi_1(s) = c, \forall s \in [-\tau, 0]$ for some $c \in \mathbb{R}, \phi_2(s) = 0, \forall s \in [-\tau, 0], ||\phi||_\tau \leq B\}$, which implies that $x_2(t)$ converges to zero and $x_1(t)$ converges to a constant by Corollary 3. Figure 1 is a simulation for system (16). The state trajectories in Figure 1 show the effectiveness of proposed theorems.

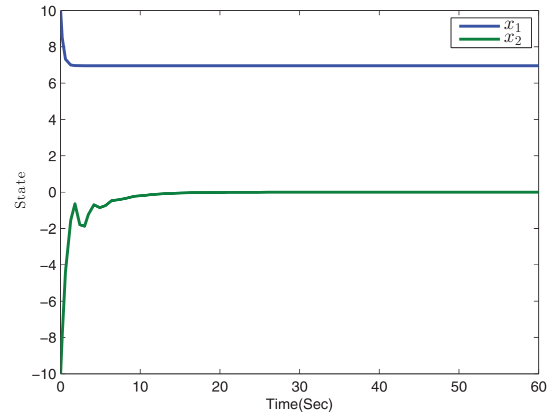

Figure 1: State trajectories for system (16) with initial values $\phi_1(s) = 10, \phi_2(s) = -10, s \in [-2, 0], \tau = 2$.

5 A case study

Consider the interconnected system of a smooth oscillator with nonsmooth friction and uncertain coefficients [2] and a first-order delayed system

$$\begin{aligned}\dot{x}_1 &= x_2(t),\\ \dot{x}_2 &\in \mathcal{F}_2(x_1(t), x_2(t)) + x_3(t),\\ \dot{x}_3 &= -2x_3(t) + x_3(t-\tau) - x_2(t),\end{aligned} \quad (21)$$

where $\mathcal{F}_2$ is a set-valued mapping satisfying

$$\mathcal{F}_2(x_1, x_2) = \begin{cases} [-2x_1(t) - 1, -x_1(t) - 1]\,, x_1 > 0, x_2 > 0;\\ -x_1(t) - \mathrm{SGN}(x_2(t)), (x_1, x_2)\\ \in \{(x_1, x_2) : x_1 \leq 0, x_2 > 0\} \bigcup \{(x_1, x_2) : x_2 < 0\};\\ [-2x_1(t) - 1, -x_1(t) + 1]\,, x_2 = 0, x_1 > 0;\\ [-x_1(t) - 1, -x_1(t) + 1]\,, x_2 = 0, x_1 < 0;\\ [-1, 1]\,, x_1 = 0, x_2 = 0 \end{cases}$$

where $\mathrm{SGN}(y)$ is defined as follows

$$\mathrm{SGN}(y) = \begin{cases} 1, & y > 0\\ -1, & y < 0\\ [-1, 1], & y = 0. \end{cases} \quad (22)$$

It is known from [2] that $\mathcal{F}_2$ is an upper semicontinuous set-valued function and takes compact and convex values in $\mathbb{R}^2$. Hence, it is not hard to conclude that the set-valued functional at the right-hand side of system (21) satisfies the basic assumptions on C_τ and maps bounded sets of C_τ into bounded sets of $\mathbb{R}^n$. Choose a functional

$$\begin{aligned}V(\phi) = \frac{1}{2}\phi_1^2(0) + \frac{1}{2}\phi_2^2(0) + \frac{1}{2}\phi_3^2(0)\\ + \int_{-\tau}^{0} \phi_3^2(s)ds\end{aligned} \quad (23)$$

where $\phi(s) = \begin{bmatrix} \phi_1(s) & \phi_2(s) \end{bmatrix}^T \in \mathbb{R}^2, s \in [-\tau, 0]$. Obviously, the functional V satisfies the basic conditions and is locally Lipschitz continuous. Moreover, it is clear that $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(||\phi||_\tau)$ for some $\mathcal{K}_\infty$ functions

α_1 and α_2. Computing $D^+V(\phi, f)$ yields

$$D^+V(\phi,f) \in \begin{cases} [-1,0]\,\phi_1(0)\phi_2(0) - \phi_2(0) - \phi_3^2(0) \\ + \phi_3(0)\phi_3(-\tau) - \phi_3^2(-\tau), \phi_1(0) > 0, \phi_2(0) > 0; \\ -|\phi_2(0)| - \phi_3^2(0) + \phi_3(0)\phi_3(-\tau) - \phi_3^2(-\tau), \\ (\phi_1(0), \phi_2(0)) \in S; \\ -\phi_3^2(0) + \phi_3(0)\phi_3(-\tau) - \phi_3^2(-\tau), \phi_2(0) = 0, \end{cases}$$

where $S = \{(\phi_1(0), \phi_2(0)) : \phi_1(0) \le 0, \phi_2(0) > 0\} \bigcup \{(\phi_1(0), \phi_2(0)) : \phi_2(0) < 0\}$. Hence, we have

$$D^+V(\phi, f) \le 0 \; for \; any \; \phi \in C_r \; and \; f \in F(\phi). \quad (24)$$

As a consequence, any solution of system (21) is bounded. It is easy to confirm that

$$E_V = \{\phi : \phi_3(0) = \phi_3(-\tau) = 0, \phi_2(0) = 0\}. \quad (25)$$

Now our object is to determine the largest invariant subset in E_V. Note that, in the set E_V, one has

$$\mathcal{F}_2(\phi_1(0), \phi_2(0)) = \begin{cases} [-2\phi_1(0) - 1, -\phi_1(0) + 1]\,, \phi_2(0) = 0, \phi_1(0) > 0; \\ [-\phi_1(0) - 1, -\phi_1(0) + 1]\,, \phi_2(0) = 0, \phi_1(0) < 0; \\ [-1, 1]\,, \phi_1(0) = 0, \phi_2(0) = 0. \end{cases}$$

Any solution $x(t)$ of system (21) with an initial function $A(0)x \in E_V$ and $x_1(0) \subset \mathbb{R}\backslash[-1, 1]$ leaves E_V. Therefore, we have

$$M_V = \{\phi : \phi_3(s) = \phi_2(s) = 0, \\ \exists c \in [-1, 1] \; s.t. \; \phi_1(s) = c, \forall s \in [-\tau, 0]\}. \quad (26)$$

This implies that any solution of the system (21) is bounded. Moreover, $x_2(t)$ and $x_3(t)$ approach zero and $x_1(t)$ approaches a constant belonging to $[-1, 1]$.

6 Conclusions

The LaSalle's invariance principle was established for delay differential inclusions by locally Lipschitz functional. With the proposed invariance principle, some stability corollaries were given. Besides, the concept of limit delay differential inclusion has been proposed for the first time such that the invariance principle was generalized to time-varying delay differential inclusions. Some numerical examples were given to show the effectiveness of the newly proposed results. For future work, the stability theorems of delay differential inclusions will be explored by the newly developed results.

REFERENCES

[1] Z. Artstein, The limiting equations of nonautonomous ordinary differential equations, Journal of Differential Equations, 25(2):184–202, 1977.

[2] A. Bacciotti and F. Ceragioli, Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions, Control Optimisation and Calculus of Variations, 4:361–376, 1999.

[3] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, volume 178, Elsevier, 1985.

[4] A. F. Filippov, Differential Equations with Discontinuous Righthand Sides: Control Systems, volume 18, Springer, 1988.

[5] H. Gao, J. Lam, and C. Wang, Robust energy-to-peak filter design for stochastic time-delay systems, Systems and Control Letters, 55(2):101–111, 2006.

[6] K. Gu, V. Kharitonov, and J. Chen, Stability of Time-Delay Systems, Boston, Birkhauser, 2003.

[7] G. Haddad, Monotone viable trajectories for functional differential inclusions, Journal of Differential Equations, 42:1–24, 1981.

[8] J. R. Haddock, T. Krisztin, J. Terjéki, and J. H. Wu, An invariance principle of Lyapunov-Razumikhin type for neutral functional differential equations, Journal of differential equations, 107(2):395–417, 1994.

[9] J. K. Hale and S. V. Lunel, Introduction to functional differential equations, Springer-verlag, New York, 1993.

[10] J. K. Hale, A stability theorem for functional-differential equations, Proceedings of the National Academy of Sciences, 50(5):942–946, 1963.

[11] H. Khalil, Nonlinear Systems, Prentice Hall, 2002.

[12] J. P. LaSalle, The Stability of Dynamical Systems, 1976.

[13] T. C. Lee and Z. P. Jiang, A generalization of Krasovskii–LaSalle theorem for nonlinear time-varying systems: converse results and applications, IEEE Trans. Automat. Contr., 50(8):1147–1163, 2005.

[14] J. Liu, X. Z. Liu, and W. C. Xie, Invariance principles for impulsive switched systems, Dynamics of Continuous, Discrete & Impulsive Systems. Series B. Applications & Algorithms, 16:631–654, 2009.

[15] J. Liu, X. Z. Liu, and W. C. Xie, Generalized invariance principles for switched delay systems, IMA Journal of Mathematical Control and Information, 28:19-39, 2011.

[16] J. Liu and A. R. Teel, Hybrid systems with memory: modeling and stability analysis via generalized solutions, Proceedings of the 19th IFAC World Congress.

[17] K. Z. Liu, X. M. Sun, J. Liu, and A. R. Teel, Stability theorems for delay differential inclusions, Submitted to IEEE Trans. Automat. Contr.

[18] S. Niculescu, Delay effects on stability: A robust control approach, volume 269, Springer, 2001.

[19] P. Pepe, On Liapunov–Krasovskii functionals under carathéodory conditions, Automatica, 43(4):701–706, 2007.

[20] E. Ryan, An integral invariance principle for differential inclusions with applications in adaptive control, SIAM Journal on Control and Optimization, 36(3):960–980, 1998.

[21] D. Shevitz and B. Paden, Lyapunov stability theory of nonsmooth systems, IEEE Trans. Automat. contr., 39(9):1910–1914, 1994.

[22] X. M. Sun and W. Wang, Integral input-to-state stability for hybrid delayed systems with unstable continuous dynamics, Automatica, 48(9):2359–2364, 2012.

[23] X. M. Sun, J. Zhao, and D. J. Hill, Stability and L_2-gain analysis for switched delay systems: a delay-dependent method, Automatica, 42(10):1769–1774, 2006.

[24] A. V. Surkov, On the stability of functional-differential inclusions with the use of invariantly differentiable Lyapunov functionals, Differential Equations, 43(8):1079–1087, 2007.