Hybrid Dynamical Systems with Finite Memory

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Abstract Hybrid systems with memory are dynamical systems that exhibit both hybrid and delay phenomena, as seen in many physical and engineered applications. A prominent example is the use of delayed hybrid feedback in control systems. This chapter outlines a framework that allows studying hybrid systems with delays through generalized solutions and summarizes some recent results on basic existence and well-posedness of solutions and stability analysis using Lyapunov-based methods.

1 Introduction

Hybrid systems with memory refer to dynamical systems exhibiting both hybrid and delay phenomena. Control systems with delayed hybrid feedback and interconnected hybrid systems with network delays are particular examples of such systems. In fact, delays are often inevitable in many control applications [17] and often cause instability and/or loss of robustness [2].

Motivated by robust stability issues in hybrid feedback control systems, generalized solutions of hybrid inclusions defined on hybrid time domains have been proposed for hybrid systems [4, 5, 15]. These generalized solutions have led to most of the stability analysis tools and results for classical nonlinear systems, including converse Lyapunov theorems, being successfully extended to hybrid systems (see [5, 16] and references therein).

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In this chapter, we summarize some recent developments [9–11] towards a framework that allows studying hybrid systems with memory through generalized solutions. The main results recalled here include basic existence and well-posedness of solutions and Lyapunov-based sufficient conditions for stability. While asymptotic stability for hybrid systems with delays have been addressed in the past in various settings (e.g., in [8, 13, 18, 19]), general results on robust asymptotic stability along the lines of [5] have not been available. This is partially owing to the fact that most current tools and results for such systems rely on standard concepts like uniform convergence, while this concept is not well-suited to handle discontinuities caused by jumps in hybrid systems, especially when structural properties of the solutions are concerned. It is from this perspective that we believe it is necessary to formulate hybrid systems with delays using generalized concepts of solutions. The main purpose of this chapter is to recapitulate this newly developed framework of hybrid systems with memory and demonstrate the feasibility of establishing some notion of robustness for asymptotic stability within this framework.

2 Preliminaries

Notation:
$\mathbb{R}^n$ denotes the n-dimensional Euclidean space with its norm denoted by $|\cdot|$; $\mathbb{Z}$ denotes the set of all integers; $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}_{\leq 0} = (-\infty, 0]$, $\mathbb{Z}_{\geq 0} = \{0,\ 1,\ 2,\ \cdots\}$, and $\mathbb{Z}_{\leq 0} = \{0,\ -1,\ -2,\ \cdots\}$; $C([a, b], \mathbb{R}^n)$ denotes the set of all continuous functions from $[a, b]$ to $\mathbb{R}^n$.

2.1 Hybrid Systems with Memory

We start with the definitions of hybrid time domains and hybrid arcs [3, 5] for hybrid systems with memory. All of the definitions in this section are recalled from [9–11].

Definition 1 Consider a subset $E \subseteq \mathbb{R} \times \mathbb{Z}$ with $E = E_{\geq 0} \cup E_{\leq 0}$, where $E_{\geq 0} := E \cap (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0})$ and $E_{\leq 0} := E \cap (\mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0})$. It is called a *compact hybrid time domain with memory* if

$$E_{\geq 0} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

and

$$E_{\leq 0} = \bigcup_{k=1}^{K} ([s_k, s_{k-1}], -k + 1)$$

for some finite sequence of times $s_K \le \cdots \le s_1 \le s_0 = 0 = t_0 \le t_1 \le \cdots \le t_J$. It is called a *hybrid time domain with memory* if, for all $(T, J) \in E_{\ge 0}$ and all $(S, K) \in \mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0}$, the set

$$(E_{\ge 0} \cap ([0, T] \times \{0, 1, \cdots, J\})) \cup (E_{\le 0} \cap ([-S, 0] \times \{-K, -K+1, \cdots, 0\}))$$

is a compact hybrid time domain with memory. The set $E_{\le 0}$ is called a *hybrid memory domain*.

Definition 2 A *hybrid arc with memory* is a pair consisting of a domain *dom* x, which is a hybrid time domain with memory, and a function $x : \operatorname{dom} x \to \mathbb{R}^n$ such that $x(\cdot, j)$ is locally absolutely continuous on $I_j = \{t : (t, j) \in \operatorname{dom} x\}$ for each $j \in \mathbb{Z}$ such that I_j has nonempty interior. In particular, a hybrid arc x with memory is called a *hybrid memory arc* if $\operatorname{dom} x \subseteq \mathbb{R}_{\le 0} \times \mathbb{Z}_{\le 0}$. We shall simply use the term *hybrid arc* if we do not have to distinguish between the above two. The collection of all hybrid memory arcs is denoted by $\mathscr{M}$.

Definition 3 Given a hybrid arc and any $(t, j) \in \operatorname{dom} x$, we define an operator $\mathscr{A}_{[t,j]}$ that maps x to $\mathscr{A}_{[t,j]}x \in \mathscr{M}$ given by

$$\mathscr{A}_{[t,j]}x(s, k) = x(t+s, j+k),$$

for all $(s, k) \in \operatorname{dom} \mathscr{A}_{[t,j]}x$, where $\operatorname{dom} \mathscr{A}_{[t,j]}x$ is defined by $(s, k) \in \operatorname{dom} \mathscr{A}_{[t,j]}x$ if and only if $(t+s, j+k) \in \operatorname{dom} x$.

Definition 4 Data of a *hybrid system with memory* in $\mathscr{M}$ consists of four elements:

- a set $\mathscr{C} \subseteq \mathscr{M}$, called the *flow set*;
- a set-valued functional $\mathscr{F} : \mathscr{M} \rightrightarrows \mathbb{R}^n$, called the *flow map*;
- a set $\mathscr{D} \subseteq \mathscr{M}$, called the *jump set*;
- a set-valued functional $\mathscr{G} : \mathscr{M} \rightrightarrows \mathbb{R}^n$, called the *jump map*.

The system is denoted by $\mathscr{H}_{\mathscr{M}} = (\mathscr{C}, \mathscr{F}, \mathscr{D}, \mathscr{G})$.

Definition 5 A hybrid arc x is a *solution to the hybrid system* $\mathscr{H}_{\mathscr{M}}$ if $\mathscr{A}_{[0,0]}x \in \mathscr{C} \cup \mathscr{D}$ and:

(S1) for all $j \in \mathbb{Z}_{\ge 0}$ and almost all $t \in I_j$,

$$\mathscr{A}_{[t,j]}x \in \mathscr{C}, \quad \dot{x}(t, j) \in \mathscr{F}(\mathscr{A}_{[t,j]}x), \tag{1}$$

(S2) for all $j \in \mathbb{Z}_{\ge 0}$ and $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$\mathscr{A}_{[t,j]}x \in \mathscr{D}, \quad x(t, j+1) \in \mathscr{G}(\mathscr{A}_{[t,j]}x). \tag{2}$$

The solution x is called *nontrivial* if its positive domain $\operatorname{dom}_{\ge 0}(x) = \operatorname{dom} x \cap (\mathbb{R}_{\ge 0} \times \mathbb{Z}_{\ge 0})$ has at least two points. It is called *complete* if $\operatorname{dom}_{\ge 0}(x)$ is unbounded. It is called *maximal* if there does not exist another solution y to $\mathscr{H}_{\mathscr{M}}$ such that dom x

is a proper subset of dom y and $x(t, j) = y(t, j)$ for all $(t, j) \in \text{dom}\, x$. The set of all maximal solutions to $\mathscr{H}_{\mathscr{M}}$ is denoted by $\mathscr{S}_{\mathscr{H}_{\mathscr{M}}}$.

2.2 Phase Space

The solutions of a hybrid system with memory will be considered in the space of hybrid memory arcs, namely $\mathscr{M}$. First, note that $\mathscr{M}$ is not a vector space, since different hybrid arcs may have different domains. We use the notion of *graphical distance* (or the set distance between graphs [14]) to quantify the distance between and convergence of hybrid memory arcs in $\mathscr{M}$.

Let cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$ denote the collection of all nonempty, closed subsets of $\mathbb{R}^n$. Given $\rho \geq 0$, for each pair $A, B \in$ cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$, define

$$\mathbf{d}_\rho(A, B) := \max_{|z|\leq\rho} \left|d(z, A) - d(z, B)\right|.$$

where $d(z, H)$ for $z \in \mathbb{R}^n$ and $H \subseteq \mathbb{R}^n$ is defined by $\inf_{w\in H} |w - z|$. Furthermore, define

$$\mathbf{d}(A, B) := \int_0^\infty \mathbf{d}_\rho(A, B)e^{-\rho} d\rho,$$

which is called the *(integrated) set distance* between A and B [14].

We adopt this distance on graphs of hybrid arcs as follows. Given a hybrid arc $\varphi :\ \text{dom}\, \varphi \to \mathbb{R}^n$, the graph of x is defined by $\text{gph}\, \varphi := \{(t, j, x) : x = \varphi(s, k)\}$. Given $\rho \geq 0$, for a pair of hybrid arcs φ and ψ, define $\mathbf{d}_\rho(\varphi, \psi) := \mathbf{d}_\rho(\text{gph}\, \varphi, \text{gph}\, \psi)$ and $\mathbf{d}(\varphi, \psi) := \mathbf{d}(\text{gph}\, \varphi, \text{gph}\, \psi)$, which is called the *graphical distance* between hybrid arcs. Note that the same notion of graphical distance applies to both hybrid arcs and hybrid memory arcs.

We now focus on hybrid memory arcs in $\mathscr{M}$. As a consequence of Theorem 4.42 in [14] and the fact that the graph of a hybrid memory arc is a nonempty, closed subset of $\mathbb{R}^{n+2}$, we know that the space $(\mathscr{M}, \mathbf{d})$ is a separable metric space. However, $(\mathscr{M}, \mathbf{d})$ is not complete, since the limit of a sequence of graphically convergent hybrid memory arcs may not be a hybrid memory arc. The following subspace of $(\mathscr{M}, \mathbf{d})$ is often used where such compactness are needed. Given $b, \lambda \in \mathbb{R}_{\geq 0}$, define

$$\mathscr{M}_b := \Big\{\varphi \in \mathscr{M} : \sup_{(s,k)\in\text{dom}\,\varphi} |\varphi(s, k)| \leq b\Big\},$$

$$\mathscr{M}_{b,\lambda} := \Big\{\varphi \in \mathscr{M}_b : \varphi \text{ is } \lambda\text{-Lipschitz}\Big\},$$

where $\varphi \in$ is said to be λ-*Lipschitz* if $\left|\varphi(s', k) - \psi(s'', k)\right| \leq \lambda \left|s - s'\right|$ holds for all $(s, k),\ (s', k) \in \text{dom}\, \varphi$.

Proposition 1 [10] *The space* $(\mathscr{M}_{b,\lambda}, \mathbf{d})$ *is a separable, locally compact, and complete metric space.*

In the above proposition, b and λ are fixed constants. Motivated by the need to consider constrained but possibly unbounded solutions of hybrid systems, we also consider the following set. Let $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a nondecreasing function and

$$\widehat{\mathscr{M}_\lambda} := \bigcup_{b \in \mathbb{R}_{\geq 0}} \mathscr{M}_{b,\lambda(b)}.$$

While $\widehat{\mathscr{M}_\lambda}$ is not closed under the graphical convergence topology, the set

$$\mathscr{M}_b \cap \widehat{\mathscr{M}_\lambda} = \mathscr{M}_{b,\lambda(b)}$$

is compact under the graphical convergence topology in $(\mathscr{M}, d)$ for each $b \geq 0$, according to Proposition 1.

2.3 *Hybrid Basic Conditions*

The regularity conditions on the data of a hybrid system with memory are formulated next, which are recalled from [10, 11].

Definition 6 (*Outer semicontinuous*) A set-valued functional $\mathscr{F} : \mathscr{M} \rightrightarrows \mathbb{R}^n$ is said to be *outer semicontinuous* at $\varphi \in \mathscr{M}$, if for all sequences of hybrid memory arcs in $\mathscr{M}_{b,\lambda}$ for some $b, \lambda \geq 0$ such that

$$\varphi_i \xrightarrow{\text{gph}} \varphi$$

and $y_i \to y$ with $y_i \in \mathscr{F}(\varphi_i)$, we have $y \in \mathscr{F}(\varphi)$.

Definition 7 (*Local boundedness*) A set-valued functional $\mathscr{F} : \mathscr{M} \rightrightarrows \mathbb{R}^n$ is said to be *locally bounded* at $\varphi \in \mathscr{M}$ if for each $b \geq 0$ there exists a neighborhood $\mathscr{U}_\varphi$ of φ (in graphical distance) such that the set $\mathscr{F}(\mathscr{U}_\varphi \cap \mathscr{M}_b) := \bigcup_{\psi \in \mathscr{U}_\varphi \cap \mathscr{M}_b} \mathscr{F}(\psi) \subseteq \mathbb{R}^n$ is bounded.

A stronger (but often easier to verify) condition for checking the local boundedness of $\mathscr{F}$ is that it maps every set $\mathscr{M}_b$, where $b \geq 0$, into a bounded set in $\mathbb{R}^n$.

In the above definitions, $\mathscr{F}$ is said to be outer semicontinuous (respectively, locally bounded) *relative* to a set $\mathscr{M}' \subseteq \mathscr{M}$ provided $\mathscr{F}|_{\mathscr{M}'}$ (defined by $\mathscr{F}|_{\mathscr{M}'}(\varphi) = \mathscr{F}(\varphi)$ if $\varphi \in \mathscr{M}'$ and $\mathscr{F}|_{\mathscr{M}'}(\varphi) = \emptyset$ elsewhere) is outer semicontinuous (respectively, locally bounded) at each $\varphi \in \mathscr{M}'$. Finally, the mapping $\mathscr{F}$ is said to be outer semicontinuous (respectively, locally bounded) if it is so relative to its domain.

The following are the basic conditions on the data of $\mathcal{H}_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$:

(A1) $\mathcal{C}$ and $\mathcal{D}$ are closed subsets of $\mathcal{M}$;
(A2) $\mathcal{F}$ is outer semicontinuous and locally bounded relative to the set $\mathcal{C}$ and $\mathcal{F}(\varphi)$ is nonempty and convex for each $\varphi \in \mathcal{C}$;
(A3) $\mathcal{G}$ is outer semicontinuous and locally bounded relative to $\mathcal{D}$, and $\mathcal{G}(\varphi)$ is nonempty for each $\varphi \in \mathcal{D}$.

The above definitions have been introduced and used in [10, 11] to establish basic existence and well-posedness for hybrid systems with memory.

2.4 Basic Existence and Continuation

This section recalls a result on the basic existence and continuation of solutions for hybrid systems with memory from [10, 11]. The result uses the notion of tangent cone, as in hybrid systems without memory [5]. The definition below is recalled from [10].

Definition 8 For any $\varphi \in \mathcal{K} \subseteq \mathcal{M}$, we define ${}^{\mathrm{T}}_{\mathcal{K}}(\varphi) \subseteq \mathbb{R}^n$ by $v \in{}^{\mathrm{T}}_{\mathcal{K}}(\varphi)$ if and only if, for any $\varepsilon > 0$, there exist $h \in (0, \varepsilon]$ and $x_h \in C([0, h], \mathbb{R}^n)$ such that
(1) $x_h(0) = \varphi(0, 0)$ and

$$\frac{x_h(h) - x_h(0)}{h} \in v + \varepsilon\mathbb{B};$$

(2) the hybrid memory arc ψ_{x_h} defined by

$$\psi_{x_h}(s, k) = \begin{cases} x_h(h + s), & \forall s \in [-h, 0],\ k = 0, \\ \varphi(h + s, k), & \forall (h + s, k) \in \operatorname{dom} \varphi, \end{cases} \tag{3}$$

lies in $\mathcal{K}$.

Theorem 1 [10, 11] *Let* $\mathcal{H}_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ *satisfy the conditions* (A1)–(A3). *If, for every* $\xi \in \mathcal{C} \backslash \mathcal{D}$,

$$\mathcal{F}(\xi) \cap^{T}_{\mathcal{C}}(\xi) \neq \emptyset, \tag{4}$$

then there exists a nontrivial solution to $\mathcal{H}_{\mathcal{M}}$ *from every initial condition* $\varphi \in \mathcal{C} \cup \mathcal{D}$ *such that* $\varphi \in \mathcal{M}_{b,\lambda}$ *for some* $b, \lambda \in \mathbb{R}_{\geq 0}$. *Moreover, every such maximal solution* x *satisfies exactly one of the following conditions:*

(a) x *is complete, i.e.,* $\operatorname{dom}_{\geq 0}(x)$ *is unbounded;*
(b) $\operatorname{dom}_{\geq 0}(x)$ *is bounded, the interval* I_J *has nonempty interior, and*

$$\limsup_{t \to T^-} |x(t, J)| = \infty,$$

where $T = \sup_t dom\, x := \sup\{t : (t, j) \in dom\, x\}$ *and*

$$J = \sup_j dom\, x := \sup\{j : (t, j) \in dom\, x\};$$

(c) $\varphi(T, J) \notin \mathscr{C} \cup \mathscr{D}$*, where* (T, J) *is as defined in (b).*

Furthermore, if $\mathscr{G}(\varphi) \subseteq \mathscr{C} \cup \mathscr{D}$ *for all* $\varphi \in \mathscr{D}$*, then only (a) or (b) above occurs.*

3 Hybrid Systems with Finite Memory

The formulation of hybrid systems with memory presented in Sect. 2 does not bear any information on the size of the memory. The focus of this chapter is on hybrid systems with *finite* memory. Given $\Delta \geq 0$, let $\mathscr{M}^{\Delta}$ denote the collection of hybrid memory arcs φ satisfying the following two conditions: (1) $s + k \geq -\Delta - 1$ for all $(s, k) \in \operatorname{dom}\varphi$; and (2) there exists $(s', k') \in \operatorname{dom}\varphi$ such that $s' + k' \leq -\Delta$. We can similarly define

$$\mathscr{M}_b^{\Delta} := \mathscr{M}^{\Delta} \cap \mathscr{M}_b,\ \mathscr{M}_{b,\lambda}^{\Delta} := \mathscr{M}^{\Delta} \cap \mathscr{M}_{b,\lambda}, \text{ and}$$
$$\widehat{\mathscr{M}_{\lambda}^{\Delta}} := \mathscr{M}^{\Delta} \cap \widehat{\mathscr{M}_{\lambda}}.$$

Definition 9 A hybrid system $\mathscr{H}_{\mathscr{M}} = (\mathscr{C}, \mathscr{F}, \mathscr{D}, \mathscr{G})$ is said to be with *finite memory* of size $\Delta \geq 0$ if

$$\mathscr{C} \cup \mathscr{D} \cup \operatorname{dom}\mathscr{F} \cup \operatorname{dom}\mathscr{G} \subseteq \mathscr{M}^{\Delta}.$$

The system is denoted by $\mathscr{H}_{\mathscr{M}}^{\Delta} = (\mathscr{C}, \mathscr{F}, \mathscr{D}, \mathscr{G})$.

The solution operator $\mathscr{A}_{[t,j]}$ also has to be adapted so that it maps hybrid arcs into hybrid memory arcs of finite memory size.

Definition 10 [11] Given a hybrid arc and any $(t, j) \in \operatorname{dom} x$, we define an operator $\mathscr{A}_{[t,j]}^{\Delta}$ that maps x to $\mathscr{A}_{[t,j]}^{\Delta} x \in \mathscr{M}^{\Delta}$ given by

$$\mathscr{A}_{[t,j]}^{\Delta} x(s, k) = x(t + s, j + k),$$

for all $(s, k) \in \operatorname{dom}(\mathscr{A}_{[t,j]}^{\Delta} x)$, where

$$\operatorname{dom}(\mathscr{A}_{[t,j]}^{\Delta} x) := \Big\{(s, k) \in \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0} : (t + s, j + k) \in \operatorname{dom} x,\ s + k \geq -\Delta_{\inf}\Big\},$$

where

$$\Delta_{\inf} := \inf\Big\{\delta \geq \Delta : \exists (t + s, j + k) \in \operatorname{dom} x \text{ s.t. } s + k = -\delta\Big\}.$$

4 Lyapunov Conditions for $\mathscr{K}\mathscr{L}$ Pre-asymptotic Stability

In this section, we discuss Lyapunov sufficient conditions for the asymptotic stability analysis of hybrid systems with memory. These results are adopted from [10, 12].

Definition 11 [10] Let $\mathscr{H}_{\mathscr{M}}^{\Delta}$ be a hybrid system in $\mathscr{M}^{\Delta}$ and $\mathscr{W} \subseteq \mathbb{R}^n$ be a closed set. The set $\mathscr{W}$ is said to be $\mathscr{K}\mathscr{L}$ *pre-asymptotically stable* for $\mathscr{H}_{\mathscr{M}}$ if there exists a $\mathscr{K}\mathscr{L}$ function β such that any solution x to $\mathscr{H}_{\mathscr{M}}^{\Delta}$ satisfies

$$|x(t, j)|_{\mathscr{W}} \leq \beta(\|\mathscr{A}_{[0,0]}x\|_{\mathscr{W}}^{\Delta}, t + j), \tag{5}$$

where $\|\varphi\|_{\mathscr{W}}^{\Delta} = \sup_{\substack{(t,j)\in \mathrm{dom}\,\varphi \\ -\Delta \leq t+j \leq 0}} \inf_{y\in\mathscr{W}} |y - \varphi(t, j)|$ for $\varphi \in \mathscr{M}^{\Delta}$ and

$$|x|_{\mathscr{W}} := \inf_{y\in\mathscr{W}} |y - x|$$

for $x \in \mathbb{R}^n$.

4.1 Lyapunov–Krasovskii Functionals

As in functional differential equations, Lyapunov functionals can be used to formulate sufficient conditions for analyzing stability of hybrid systems with memory. The following result provides a set of such conditions, which resemble that for hybrid systems without memory.

Given a functional $V : \mathscr{M}^{\Delta} \to \mathbb{R}_{\geq 0}$, the upper right-hand derivative of V at $\varphi \in \mathscr{M}^{\Delta}$ along the solutions of $\mathscr{H}_{\mathscr{M}}^{\Delta}$ can be defined as follows:

$$D^+V(\varphi) := \sup_{\substack{x\in\mathscr{S}_{\mathscr{H}_{\mathscr{M}}^{\mathscr{D}}} \\ \mathscr{A}_{[0,0]}^{\Delta}x=\varphi}} \limsup_{h\to 0} \frac{V(\mathscr{A}_{[h,0]}x) - V(\varphi)}{h}.$$

Theorem 2 [12] *Let $\mathscr{H}_{\mathscr{M}}^{\Delta}$ be a hybrid system in $\mathscr{M}^{\Delta}$ and let $\mathscr{W} \subseteq \mathbb{R}^n$ be a closed set. If there exists a functional $V : \mathscr{M}^{\Delta} \to \mathbb{R}_{\geq 0}$ and $\mathscr{K}_{\infty}$ functions α_i $(i = 1, 2, 3)$ such that the following hold:*

(i) $\alpha_1(|\varphi(0, 0)|_{\mathscr{W}}) \leq V(\varphi) \leq \alpha_2(\|\varphi\|_{W}^{\Delta})$ *for all* $\varphi \in \mathscr{C} \cup \mathscr{D} \cup \mathscr{G}(\mathscr{D})$*;*
(ii) $D^+V(\varphi) \leq -\alpha_3(|\varphi(0, 0)|_{\mathscr{W}})$ *for all* $\varphi \in \mathscr{C}$*;*
(iii) $V(\varphi_g^+) - V(\varphi) \leq -\alpha_3(|\varphi(0, 0)|_{\mathscr{W}})$ *for all* $\varphi \in \mathscr{D}$ *and* $g \in \mathscr{G}(\varphi)$*,*

where φ_g^+ is defined by

$$gph\,\varphi_v^+ = (gph\,\varphi + (0, -1, 0)) \cup \{(0, 0, g)\}$$

and D^+V is the upper right-hand derivative of V (defined below) along the solutions of $\mathscr{H}_{\mathscr{M}}^{\Delta}$, then $\mathscr{W}$ is $\mathscr{K}\mathscr{L}$ pre-asymptotically stable for $\mathscr{H}_{\mathscr{M}}$.

The proof of Theorem 2, given in [12], is similar to that of Theorem 3.18 in [5] and draws inspiration from that of [7, Theorem 2.1, Chap. 5]. The main difference lies in that the solutions now evolve in $\mathscr{M}^{\Delta}$. We also note that a special version of Lyapunov–Krasovskii theorem for hybrid systems with delays was proved in [1, Proposition1] in the context of reset control systems, where the reset map only acts on the controller state and the emphasis on the delay is in continuous time. Theorem 2 above covers the general case of hybrid systems with delays, where the delay can be in both continuous and discrete time.

4.2 Lyapunov–Razumikhin Functions

It is also possible to formulate conditions for $\mathscr{K}\mathscr{L}$ pre-asymptotical stability using Lyapunov functions defined on $\mathbb{R}^n$, as shown in the following result:

Theorem 3 [12] *Let $\mathscr{H}_{\mathscr{M}}^{\Delta} = (\mathscr{C}, \mathscr{F}, \mathscr{D}, \mathscr{G})$ be a hybrid system with finite memory (i.e., $\Delta < \infty$) and let $\mathscr{W} \subseteq \mathbb{R}^n$ be a closed set. If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\mathscr{K}_\infty$ functions α_i $(i = 1, 2, 3)$, and continuous functions $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $p(s) > s$ and $\rho(s) < s$ for all $s > 0$ such that the following hold:*

(i) $\alpha_1(|\varphi(0,0)|_{\mathscr{W}}) \leq V(\varphi(0,0)) \leq \alpha_2(|\varphi(0,0)|_{\mathscr{W}})$ *for all* $\varphi \in \mathscr{C} \cup \mathscr{D} \cup \mathscr{G}(\mathscr{D})$;
(ii) $\nabla V(\varphi(0,0)) \cdot f \leq -\alpha_3(V(\varphi(0,0)))$ *for all* $\varphi \in \mathscr{C}$ *such that* $p(V(\varphi(0,0))) \geq \overline{V}(\varphi)$ *and all* $f \in \mathscr{F}(\varphi)$;
(iii) $V(g) \leq \rho(\overline{V}(\varphi))$ *for all* $\varphi \in \mathscr{D}$ *and all* $g \in \mathscr{G}(\varphi)$,

where $\overline{V}(\varphi) = \max_{-\Delta-1 \leq s+k \leq 0} V(\varphi(s,k))$, then $\mathscr{W}$ is $\mathscr{K}\mathscr{L}$ pre-asymptotically stable for $\mathscr{H}_{\mathscr{M}}^{\Delta}$.

The following result, which first appeared in [10], can be seen as a corollary of the above theorem:

Corollary 1 [10] *Let $\mathscr{H}_{\mathscr{M}} = (\mathscr{C}, \mathscr{F}, \mathscr{D}, \mathscr{G})$ be a hybrid system with memory and let $\mathscr{W} \subseteq \mathbb{R}^n$ be a closed set. If there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\mathscr{K}_\infty$ functions α_i $(i = 1, 2)$, and positive constants $\mu > q$ and $\rho < 1$ such that*

(i) $\alpha_1(|\varphi(0,0)|_{\mathscr{W}}) \leq V(\varphi(0,0)) \leq \alpha_2(|\varphi(0,0)|_W)$ *for all* $\varphi \in \mathscr{C} \cup \mathscr{D} \cup \mathscr{G}(\mathscr{D})$;
(ii) $\nabla V(\varphi(0,0)) \cdot f \leq -\mu V(\varphi(0,0)) + q\overline{V}_{[0,0]}(\varphi)$ *for all* $\varphi \in \mathscr{C}$ *and* $f \in \mathscr{F}(\varphi)$;
(iii) $V(g) \leq \rho \overline{V}_{[0,0]}(\varphi)$ *for all* $\varphi \in \mathscr{D}$ *and* $g \in \mathscr{G}(\varphi)$,

all hold, where

$$\overline{V}_{[0,0]}(\varphi) = \max_{-\Delta \leq s+k \leq 0} V(\varphi(s,k)),$$

then $\mathscr{W}$ is $\mathscr{K}\mathscr{L}$ pre-asymptotically stable for $\mathscr{H}_{\mathscr{M}}$.

The conditions above in Corollary 1 are more in the spirit of Halanay-type inequalities [6, p. 378] than the general Razumikhin-type conditions used in Theorem 3.

5 Well-Posed Hybrid Systems with Memory and Robustness of $\mathcal{KL}$ Stability

In order to discuss robustness stability, we define perturbations of a hybrid system with memory as follows. It closely follows the notion of outer-perturbation of a hybrid system without memory [5], but formulated in a more restricted sense by making the following assumption on $\mathcal{H}^{\Delta}_{\mathcal{M}}$: there exists a nondecreasing function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\mathcal{C} \cup \mathcal{D} \subseteq \widehat{\mathcal{M}^{\Delta}_{\lambda}} := \bigcup_{b \in \mathbb{R}_{\geq 0}} \mathcal{M}^{\Delta}_{b,\lambda(b)},$$

and only defining perturbations within this set $\widehat{\mathcal{M}^{\Delta}_{\lambda}}$. The following definition is recalled from [11].

Definition 12 [11] Given a hybrid system with memory $\mathcal{H}^{\Delta}_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ and a functional $\rho : \mathcal{M}^{\Delta} \to \mathbb{R}_{\geq 0}$, the ρ-perturbation of $\mathcal{H}^{\Delta}_{\mathcal{M}}$, denoted by $(\mathcal{H}^{\Delta}_{\mathcal{M}})_{\rho}$, is the hybrid system with data:

- $\mathcal{C}_{\rho} = \left\{\varphi \in \widehat{\mathcal{M}^{\Delta}_{\lambda}} : \widehat{\mathbb{B}}(\varphi, \rho(\varphi)) \cap \mathcal{C} \neq \emptyset\right\}$;
- $\mathcal{F}_{\rho}(\varphi) = \overline{\text{con}}\mathcal{F}(\widehat{\mathbb{B}}(\varphi, \rho(\varphi)) \cap \mathcal{C}) + \rho(\varphi)\mathbb{B}$;
- $\mathcal{D}_{\rho} = \left\{\varphi \in \widehat{\mathcal{M}^{\Delta}_{\lambda}} : \widehat{\mathbb{B}}(\varphi, \rho(\varphi)) \cap \mathcal{D} \neq \emptyset\right\}$;
- $\mathcal{G}_{\rho}(\varphi) = \left\{y \in \mathbb{R}^n : y \in v + \rho(\varphi^+_v)\mathbb{B},\ v \in \mathcal{G}(\widehat{\mathbb{B}}(\varphi, \rho(\varphi)) \cap \mathcal{D})\right\}$, where φ^+_v is defined by $\text{gph}\,\varphi^+_v = (\text{gph}\,\varphi + (0, -1, 0)) \cup \{(0, 0, v)\}$.

where $\overline{\text{con}}(E)$ is the closed convex hull of a set $E \subseteq \mathbb{R}^n$ and

$$\widehat{\mathbb{B}}(\varphi, \rho(\varphi)) := \left\{\psi \in \widehat{\mathcal{M}^{\Delta}_{\lambda}} : \psi \text{ and } \varphi \text{ are } \rho(\varphi)\text{-close}\right\}$$

and $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$.

We can verify that continuous perturbations of a hybrid system still satisfy the regularity assumptions.

Proposition 2 [11] *Let ρ be a continuous functional $\rho : \mathcal{M}^{\Delta} \to \mathbb{R}_{\geq 0}$ and $\mathcal{H}^{\Delta}_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ be a hybrid system satisfying assumptions* (A1)–(A3). *Then $(\mathcal{H}^{\Delta}_{\mathcal{M}})_{\rho} = (\mathcal{C}_{\rho}, \mathcal{F}_{\rho}, \mathcal{D}_{\rho}, \mathcal{G}_{\rho})$ satisfies assumptions* (A1)–(A3).

Definition 13 (*Well-posedness of hybrid systems with finite memory* [11]) A hybrid system $\mathcal{H}^{\Delta}_{\mathcal{M}}$ is said to be *well-posed* if the following properties hold: for any given continuous function $\rho : \mathcal{M}^{\Delta} \to \mathbb{R}_{\geq 0}$, a decreasing sequence $\{\delta_i\}_{i=1}^{\infty}$ in $(0, 1)$

with $\delta_i \to 0$ as $i \to \infty$, and for every graphically convergent sequence $\{x_i\}_{i=1}^{\infty}$ of solutions to $(\mathscr{H}_{\mathscr{M}}^{\Delta})_{\delta_i \rho}$ with $\mathscr{A}_{[0,0]} x_i \in \mathscr{M}_b^{\Delta}$ for some $b \geq 0$ and

$$\mathscr{A}_{[0,0]} x_i \xrightarrow{\text{gph}} \varphi \in \mathscr{M}^{\Delta},$$

we have

(a) if the sequence $\{x_i\}_{i=1}^{\infty}$ is locally eventually bounded (that is, for any $m > 0$, there exists $N > 0$ and $k > 0$ such that, for all $i > N$ and all $(t, j) \in \operatorname{dom} x_i$ with $t + j < m$, $|x_i(t, j)| < k$), then its graphical limit x is a solution to $\mathscr{H}_{\mathscr{M}}^{\Delta}$ with $\mathscr{A}_{[0,0]} x = \varphi$ and $\operatorname{dom} x = \lim_{i \to \infty} \operatorname{dom} x_i$;
(b) if the sequence $\{x_i\}_{i=1}^{\infty}$ is not locally eventually bounded, then there exist some T and J in $(0, \infty)$ and a sequence $\{t_i\}_{i=1}^{\infty}$ with $(t_i, J) \in \operatorname{dom} x_i$ for sufficiently large i such that $\lim_{i \to \infty} t_i = T$, $\lim_{i \to \infty} |x_i(t_i, J)| = \infty$, and the limit $x = \lim \operatorname{gph}_{i \to \infty} x_i$ restricted to the domain $\{(t, j) \in \operatorname{dom} x : t + j < T + J\}$ is a maximal solution to $\mathscr{H}_{\mathscr{M}}^{\Delta}$ with $\mathscr{A}_{[0,0]} x = \varphi$ and $\lim_{t \to T} |x(t, J)| = \infty$.

Theorem 4 [11] *If a hybrid system with finite memory* $\mathscr{H}_{\mathscr{M}}^{\Delta} = (\mathscr{C}, \mathscr{F}, \mathscr{D}, \mathscr{G})$ *satisfies Assumptions* (A1)–(A3)*, then it is well-posed.*

One of the main consequences of well-posedness is that $\mathscr{KL}$ pre-asymptotic stability of hybrid systems with memory is robust in the following sense:

Definition 14 (*Robust $\mathscr{KL}$ pre-asymptotic stability* [11]) Let $\mathscr{W} \subseteq \mathbb{R}^n$ be a compact set and $\mathscr{H}_{\mathscr{M}}^{\Delta}$ be a hybrid system with memory.

(a) The set $\mathscr{W}$ is *robustly $\mathscr{KL}$ pre-asymptotically stable* for $\mathscr{H}_{\mathscr{M}}^{\Delta}$ if there exists a continuous function $\rho : \mathscr{M}^{\Delta} \to \mathbb{R}_{\geq 0}$ that is positive on

$$\mathscr{M}^{\Delta} \backslash \mathscr{W} := \left\{\varphi \in \mathscr{M}^{\Delta} : \exists (s, k) \in \operatorname{dom} \varphi \text{ s.t. } \varphi(s, k) \notin \mathscr{W}\right\}$$

such that

$$\omega_2(x(t, j)) \leq \beta(\omega_1(\mathscr{A}_{[0,0]} x), t + j), \quad \forall (t, j) \in \operatorname{dom} x,$$

where

$$w_2(z) = |z|_{\mathscr{W}} := \inf_{y \in \mathscr{W}} |y - z|$$

for $z \in \mathbb{R}^n$ and

$$w_1(\varphi) = \|\varphi\|_{\mathscr{W}} = \sup_{(s,k) \in \operatorname{dom} \varphi} \inf_{y \in \mathscr{W}} |y - \varphi(s, k)|$$

for all $\varphi \in \mathscr{M}^{\Delta}$, holds for all solutions x of $(\mathscr{H}_{\mathscr{M}}^{\Delta})_{\rho}$.
(b) The set $\mathscr{W}$ is *semi-globally practically robustly $\mathscr{KL}$ pre-asymptotically stable* for $\mathscr{H}_{\mathscr{M}}^{\Delta}$ if there exists $\beta \in \mathscr{KL}$ such that

$$\omega_2(x(t,j)) \le \beta(\omega_1(\mathscr{A}_{[0,0]}x), t+j), \quad \forall (t,j) \in \operatorname{dom} x,$$

holds for all solutions x of $\mathscr{H}^{\Delta}_{\mathscr{M}}$, and such that for every continuous function $\rho : \mathscr{M}^{\Delta} \to \mathbb{R}_{\ge 0}$ that is positive on $\mathscr{M}^{\Delta}\backslash\mathscr{W}$, the following holds: for every $\mathscr{M}^{\Delta}_{b}$ with $b \ge 0$ and every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that every solution x to $(\mathscr{H}^{\Delta}_{\mathscr{M}})_{\delta\rho}$ satisfies

$$\omega_2(x(t,j)) \le \beta(\omega_1(\mathscr{A}_{[0,0]}x), t+j) + \varepsilon, \quad \forall (t,j) \in \operatorname{dom} x.$$

Theorem 5 [11](*Robustness of pre-asymptotic stability*) *Let $\mathscr{H}^{\Delta}_{\mathscr{M}}$ be a well-posed hybrid system with memory and $\mathscr{W} \subseteq \mathbb{R}^n$ be a compact set. If $\mathscr{W}$ is $\mathscr{K}\mathscr{L}$ pre-asymptotically stable, then it is also robustly $\mathscr{K}\mathscr{L}$ pre-asymptotically stable.*

6 Conclusions

We have summarized a framework for studying hybrid dynamical systems with memory using generalized concepts of solutions. Such solutions are defined in the phase space of hybrid memory arcs equipped with graphical distance topology. Regularity conditions on the hybrid data are formulated in terms of the graphical topology, which allows us to establish basic existence and well-posedness of hybrid systems with memory, using ideas from functional viability theory. We then discussed Lyapunov-based sufficient conditions for the stability of hybrid systems with delays via generalized solutions. The main motivation for using generalized solutions for studying hybrid systems with memory is to establish well-posedness and its consequence in robust stability. We recalled some recent results in this direction, which show that pre-asymptotic stability of well-posed hybrid systems with memory is robust and, therefore, provide justification for using this generalized framework to study hybrid systems with memory.

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