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# Stability theorems for delay differential inclusions

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***Abstract*—This paper addresses stability problems of delay differential inclusions. A chain rule is proposed for a wide class of Lyapunov-Krasovskii functionals which are not necessarily invariantly differentiable. We also obtain an invariance-like theorem, where the derivative of the candidate functional is bounded above by a continuous negative semidefinite function. Several examples are given to show the effectiveness of the results.**

***Index Terms*—Delay differential inclusions, Krasovskii functional, invariance-like theorems, adaptive control**

## I. INTRODUCTION

Stability analysis for differential inclusions has important applications such as differential equations with discontinuous righthand side. Various kinds of stability theorems for ordinary differential equations have been extended to differential inclusions. There are three main stability arguments. One is the most well-known Lyapunov theorem that concludes asymptotic stability by requiring strict decrease of a Lyapunov function along the solutions [9]. Many papers have extended this idea to differential inclusions ( [33], [6], [27], [1]). The second is the LaSalle's invariance principle, which is generally applicable for autonomous systems [12]. It asserts that a solution approaches the largest invariant subset of a set in which the derivative of the candidate Lyapunov function is zero. When the largest invariant set is the origin, uniform convergence of solutions to the origin can be concluded. There are multiple kinds of important variations and extensions of the invariance principle, for example, [13], [24], [2], [25], [34], [4]. Integral invariance principle has been extended to differential inclusions in [24]. LaSalle-Yoshizawa theorem for differential inclusions has been developed in [4]. All these extensions to differential inclusions show their great importance in practical applications notably in adaptive control [11]. The third one is the Matrosov theorem, which proves asymptotic stability by a positive definite function and multiple auxiliary functions [17]. Different from the LaSalle's invariance principle, this approach does not need to identify an invariant set, while having weaker conclusions than those of LaSalle's invariance principle. Along this research line, Matrosov's theorem has also various extensions and applications (e.g., [16], [26], [31], [18]).

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The increasing interest in control system with delays stems from the importance and inevitability of delays in practical applications (see, for example, [5], [28], [29], [23] and references therein). Stability analysis for delayed systems depends mainly on Krasovskii functional and Razumikhin function approaches [8]. Both of the two tools have been extended to various systems such as switched delayed systems [29], impulsive delayed systems [15], hybrid systems with memory [14].

The stability theorems of delay differential inclusions have received some attention. Boundedness of the solutions of delay differential inclusions are investigated in [19] by a class of specific functionals, which exclude multiple integrals and time-varying delays. Lyapunov function is used in [32] to analyze the stability of delay differential inclusions. Some stability conditions are given in [35] for delay differential inclusions, where the derivative of the Krasovskii functional is not formulated in the Driver's form, but is formulated involving formally the solutions of the delay differential inclusion. Invariance principles are developed for autonomous delay differential inclusions in [30] and some asymptotic stability conditions are also presented in [30] by using a class of rather general invariantly differentiable functionals. However, the functionals in [30] are often constrained for some systems. In the paper [22], the input-to-state stability and integral input-to-state stability redesign are addressed for nonsmooth delayed systems in the sense of Krasovskii solutions by invariantly differentiable functionals. A more general locally Lipschitz functional is discussed in [21]. It is asserted in [21] that a locally Lipschitz functional can be used as far as the input-to-state stability is concerned for delay differential equations, since the problem of the absolute continuity of the composite function of the Krasovskii functional and the solution is overcome.

The main contributions of this paper consist of two parts. The first one is that the invariantly differentiable functional in [30] is relaxed and a chain rule for more general locally Lipschitz functional is proposed, which is not addressed in the related literature [19], [30], [35]. Two examples are given to illustrate the effectiveness of this result. The second one is that a new invariance-like theorem is proposed for nonautonomous delay differential inclusions. The derivative of the candidate functional is bounded above by a continuous negative semidefinite function. This result can be used to analyze the convergence of solutions for nonautonomous delay differential inclusions. An example is also presented to show the validity of this result.

## II. PRELIMINARIES

The following notation will be adopted in this paper:

$\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space and $\mathbb{R}_{\geq 0} = [0, \infty)$. $|x|$ and $||x||_1$ are respectively the Euclidean norm and

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1-norm for all $x \in \mathbb{R}^n$. Given $x \in \mathbb{R}^n$, $\text{sgn}(x)$ is defined as $\text{sgn}(x) = \left[ \text{sgn}(x_1) \quad \text{sgn}(x_2) \quad \cdots \quad \text{sgn}(x_n) \right]^T$ where $\text{sgn}(x_j), j \in \{1, 2, \cdots, n\}$ is a sign function. Denote the collection of all continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ by $C_r$. For a continuous function $x$ defined at least on $[t-r, t]$, $x_t$ is a function defined on $[-r, 0]$ with $x_t(\theta) = x(t+\theta), \theta \in [-r, 0]$. For $\phi \in C_r$, $||\phi||_r = \max_{-r \le s \le 0} |\phi(s)|$. A metric space $\mathbb{R}_{\ge 0} \times C_r$ is endowed with the metric $||y - z|| = \max\{|t - s|, ||\phi - \varphi||_r\}$, where $y = (t, \phi)$ and $z = (s, \varphi)$ belong to $\mathbb{R}_{\ge 0} \times C_r$. A set-valued functional $\mathcal{F} : \mathbb{R}_{\ge 0} \times C_r \rightrightarrows \mathbb{R}^n$ is upper semicontinuous on $\mathbb{R}_{\ge 0} \times D \subseteq \mathbb{R}_{\ge 0} \times C_r$ if, given $\bar{y} \in \mathbb{R}_{\ge 0} \times D$, for each $\epsilon > 0$ there exists $\delta > 0$ such that, for all $\bar{z} \in \mathbb{R}_{\ge 0} \times D$ satisfying $||\bar{z} - \bar{y}|| < \delta$ we have $\mathcal{F}(\bar{z}) \subseteq \mathcal{F}(\bar{y}) + \epsilon B$ where $B$ is the unit open ball in $\mathbb{R}^n$.

Let $D \subseteq C_r$ be an open set containing $\phi = 0 \in C_r$. Consider the following delay differential inclusion

$$\dot{x}(t) \in \mathcal{F}(t, x_t) \tag{1}$$

where $\mathcal{F} : \mathbb{R}_{\ge 0} \times D \rightrightarrows \mathbb{R}^n$ is a set-valued functional that is bounded on closed bounded subsets of $D$. The delay differential inclusion (1) will be denoted by $\mathcal{DI}$ for brevity.

The set-valued functional $\mathcal{F}$ satisfies the following **basic assumptions**: it is upper semicontinuous and takes nonempty, convex and compact values on $\mathbb{R}_{\ge 0} \times D$. Under the basic assumptions, there exists at least a solution for $\mathcal{DI}$ on interval $[t_0 - r, T)$ for some $T > 0$ (Theorem 2.2 in [7]). A function $x : [t_0 - r, T) \to \mathbb{R}^n$ is said to be a solution of $\mathcal{DI}$ with initial time $t_0$ and initial value $\phi \in C_r$ if it is a solution of $\mathcal{DI}$ and verifies $x_{t_0} = \phi$. For simplicity, this solution will be denoted by $x(t, t_0, \phi)$.

We now give a lemma related to the properties of solutions of $\mathcal{DI}$. This result is similar to Theorem 3.1 in Chapter 2 of [8].

*Lemma 1:* Suppose that $\mathcal{F} : \mathbb{R}_{\ge 0} \times D \rightrightarrows \mathbb{R}^n$ satisfies the basic assumptions on $\mathbb{R}_{\ge 0} \times D$. Then for any initial value $(t_0, \phi) \in \mathbb{R}_{\ge 0} \times D$, there exist solutions of $\mathcal{DI}$ satisfying $x_{t_0} = \phi$. If a solution $x(t, t_0, \phi)$ is defined on a finite interval $[t_0 - r, T)$ and cannot be continued, then for any compact set $W$ in $D$, there is a $t_W$ such that $x_t \notin W$ for $t_W \le t < T$.

The proof is similar to Theorem 3.1 in Chapter 2 of [8].

Similar to Theorem 3.2 in Chapter 2 of [8], we will relax the compact set in Lemma 1 to a bounded closed set .

*Theorem 1:* Suppose that $\mathcal{F} : \mathbb{R}_{\ge 0} \times D \rightrightarrows \mathbb{R}^n$ is bounded on closed bounded set of $D$ and satisfies the basic assumptions on $\mathbb{R}_{\ge 0} \times D$. Then for any initial values $(t_0, \phi) \in \mathbb{R}_{\ge 0} \times D$, there exists a solution of $\mathcal{DI}$ satisfying $x_{t_0} = \phi$. If a solution $x(t, t_0, \phi)$ is defined on a finite interval $[t_0 - r, T)$ and cannot be continued, then for any bounded closed set $U$ in $C_r$, $U$ in $D$, there is a $t_U$ such that $x_t \notin U$ for $t_U \le t < T$.

*Proof:* Let $(t_0, \phi) \in \mathbb{R}_{\ge 0} \times D$. Existence of solutions of $\mathcal{DI}$ with initial values $(t_0, \phi)$ under **basic assumptions** follows from Theorem 2.2 in [7] immediately. Choose any solution $x(t, t_0, \phi)$ of $\mathcal{DI}$ with initial values $(t_0, \phi)$. This solution is defined on $[t_0 - r, T)$ and can not be continued. Suppose that the conclusion of this lemma is not true. Then there exist a bounded closed set $U$ and a sequence of real numbers $t_k \to T^-$ such that $x_{t_k} \in U$ for all $k$. Since $r > 0$, this implies that $x(t), t_0 - r \le t < T$ is bounded. Consequently, there is a constant $M$ such that $|\mathcal{F}(t, \varphi)| \le M$ in the closure of $\{x_t : t_0 \le t < T\}$ for all $t \ge t_0$. Based on the definition of solutions of $\mathcal{DI}$, it follows that $\dot{x}(t) \in \mathcal{F}(t, x_t)$ for almost all $t \in [t_0, T)$. This results in $|x(t+\tau) - x(t)| \le M\tau$ for all $t$, $t + \tau < T$. Thus, $x$ is uniformly continuous on $[t_0 - r, T)$. This implies that $\{x_t : t_0 \le t < T\}$ belongs to a compact set in $D$. This contradicts Lemma 1 and proves our assertion. ■

*Remark 1:* If $D = C_r$, any solution of $\mathcal{DI}$ diverges to $\infty$ in finite time or is defined on $[t_0 - r, \infty)$.

For a function $\varphi \in C_r$ and any $\Delta > 0$, denote the set of all continuous extensions of $\varphi$ to the interval $(-\Delta - r, \Delta)$ by $E_\Delta(\varphi)$.

*Definition 1:* A functional $W : C_r \to \mathbb{R}$ is said to have an invariant derivative $\partial_\varphi W$ at a point $\varphi \in C_r$ if, for each $\Psi \in E_\Delta(\varphi)$, the function $Y_\Psi(\xi) = W(\Psi_\xi)$, where $\xi \in (-\Delta, \Delta)$ and $\Psi_\xi(\theta) = \Psi(\xi + \theta), -r \le \theta \le 0$, has a finite derivative $\frac{\partial Y_\Psi}{\partial \xi}|_{\xi=0}$ at zero and this derivative $\partial_\varphi W = \frac{\partial Y_\Psi}{\partial \xi}|_{\xi=0}$ is independent of the choice of $\Psi \in E_\Delta(\varphi)$; i.e., the value of the derivative at zero is the same for all functions $\Psi \in E_\Delta(\varphi)$.

*Remark 2:* The concept of the invariant derivative is taken from [30], [10] with minor changes to accommodate the formulations of this paper. The difference is that the invariant derivative in [30], [10] is defined by the right derivative.

The following definition can be found in [3].

*Definition 2:* For a functional $f : \mathbb{R}^m \times C_r \to \mathbb{R}$, the one-sided directional derivative and general derivative at a point $(x, \varphi) \in \mathbb{R}^m \times C_r$ with respect to the first variable are defined respectively as follows

$$f'(x, \varphi, z) = \lim_{h \downarrow 0} \frac{f(x + hz, \varphi) - f(x, \varphi)}{h},$$
$$f^\circ(x, \varphi, z) = \limsup_{y \to x, h \downarrow 0} \frac{f(y + hz, \varphi) - f(y, \varphi)}{h}. \tag{2}$$

*Definition 3:* A functional $f : \mathbb{R}^m \times C_r \to \mathbb{R}$ is said to have invariant directional derivative at a point $(x, \varphi) \in \mathbb{R}^m \times C_r$ for $z \in \mathbb{R}^m$ if, at that point, there exists a finite directional derivative $f'(x, \varphi, z)$ and an invariant derivative $\partial_\varphi f(x, \varphi)$ such that

$$\lim_{h \downarrow 0} \frac{f(x + hz, \Psi_h) - f(x, \varphi)}{h} = f'(x, \varphi, z) + \partial_\varphi f(x, \varphi)$$
$$\lim_{h \downarrow 0} \frac{f(x + hz, \Psi_{-h}) - f(x, \varphi)}{h} = f'(x, \varphi, z) - \partial_\varphi f(x, \varphi). \tag{3}$$

*Definition 4:* A scalar functional $f : \mathbb{R}^m \times C_r \to \mathbb{R}$ is said to be locally Lipschitz if, for any $(x, \varphi) \in \mathbb{R}^m \times C_r$, there exists $l_{(x,\varphi)} > 0$ such that the following inequality holds in some neighborhood $N_{(x,\varphi)}$ of $(x, \varphi)$

$$|f(x_1, \varphi_1) - f(x_2, \varphi_2)| \le l_{(x,\varphi)} \max\{|x_2 - x_1|, ||\varphi_1 - \varphi_2||_r\}, \forall (x_1, \varphi_1), (x_2, \varphi_2) \in N_{(x,\varphi)}.$$

## III. MAIN RESULTS

In this section, we will present some sufficient conditions to guarantee the stability of $\mathcal{DI}$.

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### A. Sufficient conditions for asymptotic stability by locally Lipschitz functional

The following stability concepts are trivial extensions of that for delay differential equations.

*Definition 5:* The trivial solution of the delay differential inclusion $\mathcal{DI}$ is said to be:

(S1) *uniformly stable* if, for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ independent of $t_0$ such that any solution $x(t, t_0, \phi)$ of $\mathcal{DI}$ with $||\phi||_r \leq \delta$ satisfies $|x(t, t_0, \phi)| \leq \epsilon$ for all $t \geq t_0$;

(S2) *uniformly attractive* if there exists $\delta > 0$ such that, for any $\epsilon > 0$, there exists $T(\delta, \epsilon) > 0$ such that any solution $x(t, t_0, \phi)$ of $\mathcal{DI}$ with $||\phi||_r \leq \delta$ satisfies $|x(t, t_0, \phi)| \leq \epsilon$ for all $t \geq T(\delta, \epsilon)$;

(S3) *uniformly asymptotically stable* (UAS) if it is uniformly stable and uniformly attractive;

(S4) *uniformly globally asymptotically stable* if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon\to\infty} \delta(\epsilon) = \infty$, and, for each pair of positive numbers $\eta$ and $c$, there is $T = T(\eta, c) > 0$ such that any solution $x(t, t_0, \phi)$ of $\mathcal{DI}$ with $||\phi||_r \leq c$ satisfies $|x(t, t_0, \phi)| \leq \eta$ for all $t \geq T(\eta, c)$.

*Definition 6:* A continuous functional $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times D \to \mathbb{R}$ is said to be composite locally absolutely continuous if, for any continuous function $x : [t_0 - r, T) \to \mathbb{R}^n$ that is locally absolutely continuous in $[t_0, T)$ and satisfies $x_t \in D$ for all $t \in [t_0, T)$, the composite function $V(t, x(t), x_t)$ is locally absolutely continuous in $[t_0, T)$.

*Remark 3:* A sufficient condition guaranteeing that $V : C_r \to \mathbb{R}$ is composite locally absolutely continuous is that $V$ is locally Lipschitz and the initial function $x(s), s \in [t_0-r, t_0]$ is continuously differentiable (Theorem 5 in [21]). For more relaxed conditions such that $V$ is composite locally absolutely continuous, the readers can refer to [21] for detailed discussions about this problem.

*Definition 7:* Let $f : \mathbb{R}^m \times C_r \to \mathbb{R}$ be a locally Lipschitz functional that has invariant derivative $\partial_\varphi f(x, \varphi)$ with respect to $\varphi$ at $(x, \varphi) \in \mathbb{R}^m \times C_r$. Define the general invariant gradient of $f$ at the point $(x, \varphi)$ by

$$\partial f(x, \varphi) = \overline{\text{co}}\{(\lim \nabla_{x_i} f(x_i, \varphi), \partial_\varphi f(x, \varphi))|x_i \to x, \\ x_i \notin \Omega_f\}, \quad (4)$$

where $\Omega_f$ denotes the set of measure zero in which $\nabla_{x_i} f(x_i, \varphi)$ does not exist.

*Example 1:* Consider the following functional $V : \mathbb{R} \times \mathbb{R}^n \times C_r \to \mathbb{R}$:

$$\begin{aligned} V(t, x, \varphi) = {} & Q(t, x) + \int_{-d(t)}^{0} P(t, \varphi(s))ds \\ & + \int_{-d(t)}^{0} \int_{s}^{0} L(t, \varphi(w))dwds \\ & + \int_{-d(t)}^{0} \int_{s}^{0} \int_{\theta}^{0} S(t, \varphi(w))dwd\theta ds \end{aligned} \quad (5)$$

where $P(t, z)$, $L(t, z)$ and $S(t, z)$ are continuous functions from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$ and have continuous first-order partial derivatives with respect to $t$, $Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, $d(t) : \mathbb{R}_{\geq 0} \to [0, r]$ is a continuously differentiable function. Moreover, suppose that $Q$ is a regular function and the one-sided directional derivative exists for each $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ ( [3], [27]). Then it holds by computation that for each $w = (t, x) \in \mathbb{R} \times \mathbb{R}^n$ and $z = (\xi, y) \in \mathbb{R} \times \mathbb{R}^n$

$$\lim_{h\downarrow 0} \frac{V(w + hz, \Psi_h) - V(w, \varphi)}{h} = V'(w, \varphi, z) + \partial_\varphi V(w, \varphi),$$

$$\lim_{h\downarrow 0} \frac{V(w + hz, \Psi_{-h}) - V(w, \varphi)}{h} = V'(w, \varphi, z) - \partial_\varphi V(w, \varphi),$$

where

$$\begin{aligned} \partial_\varphi V(w, \varphi) = {} & P(t, \varphi(0)) - P(t, \varphi(-d(t))) \\ & + d(t)L(t, \varphi(0)) - \int_{-d(t)}^{0} L(t, \varphi(s))ds \\ & + \frac{d^2(t)}{2} S(t, \varphi(0)) - \int_{-d(t)}^{0} \int_{\theta}^{0} S(t, \varphi(s))dsd\theta \end{aligned} \quad (6)$$

and

$$\begin{aligned} & V'(w, \varphi, z) \\ & = Q'(w, z) - \xi\dot{d}(t)P(t, \varphi(-d(t))) + \xi \int_{-d(t)}^{0} \frac{\partial P(t, \varphi(s))}{\partial t} ds \\ & - \xi\dot{d}(t) \int_{-d(t)}^{0} L(t, \varphi(s))ds + \xi \int_{-d(t)}^{0} \int_{s}^{0} \frac{\partial L(t, \varphi(\theta))}{\partial t} d\theta ds \\ & - \xi\dot{d}(t) \int_{-d(t)}^{0} \int_{s}^{0} S(t, \varphi(\theta))d\theta ds \\ & + \xi \int_{-d(t)}^{0} \int_{s}^{0} \int_{\theta}^{0} \frac{\partial S(t, \varphi(w))}{\partial t} dwd\theta ds. \end{aligned} \quad (7)$$

Moreover, the general invariant gradient of $V$ at the point $(w, \varphi)$ can be given as follows

$$\partial V(w, \varphi) = \overline{\text{co}} \left\{ \begin{bmatrix} \lim(\nabla_{w_i} V(w_i, \varphi))^T \\ \partial_\varphi V(w, \varphi) \end{bmatrix}^T \middle| w_i \to w, \; w_i \notin \Omega_V \right\} \quad (8)$$

where $\nabla_{w_i} V(w_i, \varphi)$ is the gradient of $V$ with respect to $w_i$ at the point $w_i \in \mathbb{R} \times \mathbb{R}^n$, $\Omega$ is the set of measure zero in which $\nabla_{w_i} V(w_i, \varphi)$ does not exist.

Next, a chain rule will be given to compute the derivative of a Lyapunov functional for system (1). This result is similar to the chain rule of a Lyapunov function for differential inclusions [27] and captures almost all the functionals in [30] and [19].

*Proposition 1:* Consider the delay differential inclusion $\mathcal{DI}$. Suppose that there exists a locally Lipschitz functional $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times C_r \to \mathbb{R}_{\geq 0}$ that satisfies the following conditions:

(c1) it is composite locally absolutely continuous;

(c2) the invariant directional derivative exists at each point $(w, \varphi) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}^n) \times C_r$;

(c3) $V'(w, \varphi, z) = V^\circ(w, \varphi, z)$ for each $(w, \varphi) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}^n) \times C_r$ and $z \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$.

Then for any solution $x : [t_0 - r, T)$ of (1), it holds that for almost all $t \in [t_0, T)$,

$$\dot{V}(t, x(t), x_t) \in \dot{\tilde{V}}(t, x(t), x_t) \quad (9)$$

where $\dot{\bar{V}}(t,x(t),x_t)=\bigcap_{\xi\in\partial V(w,x_t)}\xi\begin{bmatrix}1\\ \mathcal{F}(t,x_t)\\ 1\end{bmatrix}$ and $w=(t,x(t))$.

*Proof:* Since $x(t)$ is locally absolutely continuous and $V(t,x,\varphi)$ is composite locally absolutely continuous, $V(t,x(t),x_t)$ is locally absolutely continuous. Let $E_0$ be the set of measure zero where $x(t)$ and $V(t,x(t),x_t)$ are not differentiable. Then by combining Lemma 2.1 of [27] and the conditions of Proposition 1, it holds that for all $t\in[t_0,T)-E_0$,

$$\begin{aligned}&\dot{V}(t,x(t),x_t)\\ &=\lim_{h\downarrow 0}\frac{V(t+h,x(t+h),x_{t+h})-V(t,x(t),x_t)}{h}\\ &=\lim_{h\downarrow 0}\frac{V(t+h,x(t)+h\dot{x}(t),x_{t+h})+o(h)-V(t,x(t),x_t)}{h}\\ &=V'(w,x_t,z)+\partial_\varphi V(w,x_t)\\ &=V^\circ(w,x_t,z)+\partial_\varphi V(w,x_t)\\ &=\max\left\{\xi\begin{bmatrix}1\\ \dot{x}(t)\\ 1\end{bmatrix}\middle|\xi\in\partial V(w,x_t)\right\}\end{aligned}\tag{10}$$

where $w=(t,x(t))$, $\varphi=x_t$ and $z=(1,\dot{x}(t))$. Similarly, by taking the left derivative of $V(t,x(t),x_t)$, the following relation holds for all $t\in[t_0,T)-E_0$

$$\begin{aligned}&\dot{V}(t,x(t),x_t)\\ &=\lim_{h\downarrow 0}\frac{V(t-h,x(t-h),x_{t-h})-V(t,x(t),x_t)}{-h}\\ &=\lim_{h\downarrow 0}\frac{V(t-h,x(t)-h\dot{x}(t),x_{t-h})+o(-h)-V(t,x(t),x_t)}{-h}\\ &=-V'(w,x_t,z)+\partial_\varphi V(w,x_t)\\ &=-V^\circ(w,x_t,z)+\partial_\varphi V(w,x_t)\\ &=-\max\left\{\xi\begin{bmatrix}-1\\ -\dot{x}(t)\\ -1\end{bmatrix}\middle|\xi\in\partial V(w,x_t)\right\}\\ &=\min\left\{\xi\begin{bmatrix}1\\ \dot{x}(t)\\ 1\end{bmatrix}\middle|\xi\in\partial V(w,x_t)\right\}\end{aligned}\tag{11}$$

where $w=(t,x(t))$, $\varphi=x_t$ and $z=(-1,-\dot{x}(t))$. Therefore, (9) holds by combining (10) and (11). ■

*Remark 4:* Proposition 1 gives a chain rule for a wide class of functionals which are not necessary to be invariantly differentiable compared with the papers [20], [30]. Compared with [19], a chain rule is given for general Lyapunov functionals.

*Theorem 2:* Consider the delay differential inclusion $\mathcal{DI}$. Suppose that there exists a locally Lipschitz functional $V:\mathbb{R}_{\geq 0}\times\mathbb{R}^n\times C_r\rightarrow\mathbb{R}_{\geq 0}$ satisfying all the conditions of Proposition 1, $\alpha_i\in\mathcal{K}(i=1,2)$, and a continuous positive definite function $\alpha_3$ such that

(1) $\alpha_1(|\varphi(0)|)\leq V(t,\varphi(0),\varphi)\leq\alpha_2(||\varphi||_r)$ for all $(t,\varphi)\in\mathbb{R}_{\geq 0}\times D$;

(2) $\dot{\bar{V}}(t,\varphi(0),\varphi)\leq-\alpha_3(\varphi(0))$ for all $(t,\varphi)\in\mathbb{R}_{\geq 0}\times D$.

Then the trivial solution is uniformly asymptotically stable. Moreover, if $D=C_r$ and $\alpha_1$ is a class-$\mathcal{K}_\infty$ function, the trivial solution of $\mathcal{DI}$ is uniformly globally asymptotically stable.

The proof is similar to that of delay differential equations [8].

*Proof:* Let $\epsilon>0$ be such that $\{\phi\in C_r\big|||\phi||_r\leq\epsilon\}\subset D$. Choose $\epsilon_1$ with $0<\epsilon_1<\epsilon$ and $\delta>0$ such that $\alpha_2(\delta)\leq\alpha_1(\epsilon_1)$. For any solution $x=x(t,t_0,\phi)$ of system (1) with $||\phi||_r\leq\delta$, it holds by combining Proposition 1 and the conditions of Theorem 2,

$$\alpha_1(|x(t)|)\leq V(t,x(t),x_t)\leq\alpha_2(\delta)\leq\alpha_1(\epsilon_1).\tag{12}$$

Thus the solution $x(t)$ is defined for all $t\geq t_0$ by Theorem 1 and $|x(t)|\leq\epsilon$ for all $t\geq t_0$.

We now prove uniform attractivity. Let $\delta$ and $\epsilon_1$ be constants from above uniform stability. We just need to show that for any $\bar{\epsilon}>0$, there exists $T(\delta,\bar{\epsilon})>0$ such that any solution $x=x(t,t_0,\phi)$ with $||\phi||_r\leq\delta$ satisfies $|x(t)|\leq\bar{\epsilon}$ for all $t\geq T(\delta,\bar{\epsilon})$. Let $\delta(\bar{\epsilon})>0$ and $\bar{\epsilon}$ be the above constants for uniform stability with $\delta(\bar{\epsilon})\leq\delta$. Suppose that such $T(\delta,\bar{\epsilon})$ does not exist, then $||x_t||_r>\delta(\bar{\epsilon})$ for all $t\geq t_0$. Since each interval of length $r$ contains $s$ such that $|x(s)|>\delta(\bar{\epsilon})$, there exists a sequence $t_k$ such that

$$t_0+(2k-1)r\leq t_k\leq t_0+2kr,k=1,2,\cdots\tag{13}$$

and $|x(t_k)|>\delta(\bar{\epsilon})$. By the assumption of $\mathcal{F}$, there exists $L>0$ such that $|\mathcal{F}(t,\phi)|\leq L$ for all $||\phi||_r\leq\epsilon_1$. Then $|x(t)|>\frac{\delta(\bar{\epsilon})}{2}$ for all $t_k-\frac{\delta(\bar{\epsilon})}{2L}\leq t\leq t_k+\frac{\delta(\bar{\epsilon})}{2L}$ (there intervals do not overlap if $L$ is chosen large). Denote $\rho=\min_{\frac{\delta(\bar{\epsilon})}{2}\leq|s|\leq\epsilon_1}\alpha_3(s)$. Combining condition (2) and Proposition 1 leads to

$$\begin{aligned}V(t_k,x(t_k),x_{t_k})&\leq\alpha_2(\delta)-\int_{t_0}^{t_k}\alpha_3(x(s))ds\\ &\leq\alpha_2(\delta)-\frac{\rho\delta(\bar{\epsilon})}{L}(k-1).\end{aligned}\tag{14}$$

If $k>\bar{k}=1+\lfloor\frac{\alpha_2(\delta)L}{\rho\delta(\bar{\epsilon})}\rfloor$ with $\lfloor\frac{\alpha_2(\delta)L}{\rho\delta(\bar{\epsilon})}\rfloor$ denoting the smallest integer larger than or equal to $\frac{\alpha_2(\delta)L}{\rho\delta(\bar{\epsilon})}$, then $V(t_k,x(t_k),x_{t_k})<0$, which is a contradiction. Therefore, $||x_t||_r\leq\delta(\bar{\epsilon})$ for some $t_0<t\leq t_0+2r\bar{k}$ and $|x(t,t_0,\phi)|\leq\bar{\epsilon}$ for all $t\geq t_0+2r\bar{k}$. ■

*Example 2:* Consider the following system

$$\dot{x}\ \in\ -A(\partial||x(t)||_1)^T-Nx(t)+\sum_{j=1}^{N}x(t-d_j)\tag{15}$$

where $A=\begin{bmatrix}0&-1\\ 1&0\end{bmatrix}$, $d_j(j=1,2,\cdots,N)$ are constant delays with $0<d_j\leq r$ and $\partial||x||_1$ is the general gradient. Choose functional $V(x,\varphi)=||x||_1+\sum_{j=1}^{N}\int_{-d_j}^{0}||\varphi(s)||_1ds$ that satisfies the conditions of Proposition 1.
If $\varphi_1(0)\neq 0,\varphi_2(0)\neq 0$, then

$$\begin{aligned}\dot{\bar{V}}(\varphi(0),\varphi))&=\bigcap_{\zeta\in\partial V(\varphi(0),\varphi)}\zeta(\mathcal{F}^T(\varphi),1)^T\\ &=\sum_{j=1}^{N}(\mathrm{sgn}^T(\varphi(0))\varphi(-d_j)-||\varphi(-d_j)||_1)\leq 0.\end{aligned}\tag{16}$$

If $\varphi_1(0) > 0, \varphi_2(0) = 0$, then

$$\begin{aligned}
&\dot{\tilde{V}}(\varphi(0), \varphi) = \bigcap_{\zeta \in \partial V(\varphi(0),\varphi)} \zeta(\mathcal{F}^T(\varphi), 1)^T \\
&= \bigcap_{\zeta \in [-1,1]} \Bigg( [\ 1 \quad \zeta\ ] \Bigg( \left( \begin{array}{c} [-1,1] \\ -1 \end{array} \right) - \\
&\qquad \sum_{j=1}^{N} (\varphi(0) - \varphi(-d_j)) \Bigg) \\
&+ \sum_{j=1}^{N} (||\varphi(0)||_1 - ||\varphi(-d_j)||_1) \Bigg) \\
&= \bigcap_{\zeta \in [-1,1]} [-1 + c_1 + (c_2 - 1)\zeta, 1 + c_1 + (c_2 - 1)\zeta]. \quad (17)
\end{aligned}$$

where $c_1 = \sum_{j=1}^N (-\varphi_1(0) + \varphi_1(-d_j)) + \sum_{j=1}^N (||\varphi(0)||_1 - ||\varphi(-d_j)||_1)$, $c_2 = \sum_{j=1}^N (-\varphi_2(0) + \varphi_2(-d_j))$ for brief. Note that

$$\begin{aligned}
&\bigcap_{\zeta \in [-1,1]} [-1 + c_1 + (c_2 - 1)\zeta, 1 + c_1 + (c_2 - 1)\zeta] \\
&\subset [c_1 + c_2 - 2, c_1 + c_2] \bigcap [c_1 - c_2, c_1 - c_2 + 2]. \quad (18)
\end{aligned}$$

From (17) and (18), $\dot{\tilde{V}}(\varphi(0), \varphi) \le 0$ or $\dot{\tilde{V}}(\varphi(0), \varphi) = \emptyset$.

Similarly, we can derive $\dot{\tilde{V}}(\varphi(0), \varphi) \le 0$ for all $\varphi \in C_r \backslash E_0$ where $E_0$ denotes the set in which $\dot{\tilde{V}}(\varphi(0), \varphi) = \emptyset$. Therefore, the trivial solution is stable.

*Remark 5:* Note that the stability of system (15) cannot be analyzed by the commonly used invariantly differentiable functional $V(x, \varphi) = ax^T x + b \sum_{j=1}^N \int_{-d_j}^0 \varphi^T(s)\varphi(s)ds$ in [30]. The specific functionals given in [19] preclude the multiple delays and thus are not applicable for system (15).

### *B. Invariance-like theorem for delay differential inclusions*

Theorem 2 requires that the composite function $t \mapsto V(t, x(t), x_t)$ is strictly decreasing along the solutions of $\mathcal{DI}$. We next give a result that can relax this condition.

*Theorem 3:* Consider the delay differential inclusion $\mathcal{DI}$. Suppose that there exists a locally Lipschitz functional $V : \mathbb{R}_{\ge 0} \times \mathbb{R}^n \times C_r \to \mathbb{R}_{\ge 0}$ satisfying all the conditions of Proposition 1, $\alpha_i \in \mathcal{K}_\infty (i = 1, 2)$ and a continuous positive semidefinite function $\alpha_3$ such that

(1) $\alpha_1(|\varphi(0)|) \le V(t, \varphi(0), \varphi) \le \alpha_2(||\varphi||_r)$ for all $(t, \varphi) \in \mathbb{R}_{\ge 0} \times D$,

(2) $\dot{\tilde{V}}(t, \varphi(0), \varphi) \le -\alpha_3(\varphi(0))$ for all $(t, \varphi) \in \mathbb{R}_{\ge 0} \times D$.

Choose a ball $\mathcal{B}_\delta \subseteq D$ and let $\bar{\delta}$ be any positive constant with $\bar{\delta} < \delta$. Then any solution $x(t, t_0, \phi)$ of $\mathcal{DI}$ with initial value $\phi \in \mathcal{B}_\eta$ where $\eta = \alpha_2^{-1} \circ \alpha_1(\bar{\delta})$, is bounded and satisfies $x(t) \to \{z \in \mathbb{R}^n | \alpha_3(z) = 0\}$ as $t \to \infty$.

*Proof:* Similar to the proof of Theorem 2, it is shown that

$$\phi \in \mathcal{B}_\eta \Rightarrow x_t \in \mathcal{B}_{\bar{\delta}} \Rightarrow x_t \in \mathcal{B}_\delta. \quad (19)$$

Thus any solution $x = x(t, t_0, \phi)$ of $\mathcal{DI}$ with $\phi \in \mathcal{B}_\eta$ is bounded and defined on $[t_0, \infty)$.

Since $t \mapsto V(t, x(t), x_t)$ is monotonically nonincreasing and bounded below by zero, it converges as $t \to \infty$. Condition (2) and Proposition 1 imply

$$\int_{t_0}^{t} \alpha_3(x(s, t_0, \phi))ds \le V(t_0, x(t_0), \phi) - V(t, x(t), x_t) \quad (20)$$

for all $t \ge t_0$. Now it follows that $\lim_{t\to\infty} \int_{t_0}^t \alpha_3(x(s, t_0, \phi))ds$ exists and is finite from the fact that $\int_{t_0}^t \alpha_3(x(s, t_0, \phi))ds$ is monotonically increasing and bounded above. $x(t, t_0, \phi)$ is bounded uniformly in $t$ and then $\mathcal{F}(t, x_t)$ is uniformly bounded in $t$. Thus $t \mapsto x(t, t_0, \phi)$ is uniformly continuous and then $t \mapsto \alpha_3(x(t, t_0, \phi))$ is uniformly continuous because $\alpha_3(x)$ is uniformly continuous in $x$ with $x \in \{x \in \mathbb{R}^n : |x| \le \bar{\delta}\}$. The Barbalat's Lemma results in $\lim_{t\to\infty} \alpha_3(x(t, t_0, \phi)) = 0$. Therefore, $x(t, t_0, \phi) \to \{z \in \mathbb{R}^n | \alpha_3(z) = 0\}$. ■

*Remark 6:* Theorem 3 is a version of LaSalle-Yoshizawa theorem for nonautonomous delay differential inclusions parallel to LaSalle-Yoshizawa theorem for ordinary differential equations (refer to Theorem 2.1 in [11]) and its corollary to differential inclusions [4]. Theorem 3 can be used to analyze the convergence of the solutions for nonautonomous delay differential inclusions. This is different from the LaSalle's invariance principle that applies the invariant properties of the solutions for autonomous delay differential inclusions [30].

## IV. AN APPLICATION TO ADAPTIVE CONTROL

Consider the following nonlinear delayed system

$$\dot{x} = u(t) + F(x(t))\theta + H(x(t - \tau)) + d(x) \quad (21)$$

where $x \in \mathbb{R}^n$ is the state, $\theta \in \mathbb{R}^p$ is a vector of unknown constant parameters, $F : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ is a known continuous function, $H : \mathbb{R}^n \to \mathbb{R}^n$ is a known continuous function satisfying $|H(\bar{x})| \le m|\bar{x}|$ for all $\bar{x} \in \mathbb{R}^n$, the delay constant $\tau$ is an unknown positive number, $d(x)$ is an unknown continuous function satisfying $|d(x)| \le \bar{d}$ with some known bound $\bar{d} > 0$.

An adaptive controller is given *a priori*

$$\begin{aligned}
u(t) &\in -F(x(t))\hat{\theta}(t) - mx(t) - k\text{SGN}(x(t)) \\
\dot{\hat{\theta}}(t) &= \Gamma^{-1} F^T(x(t))x(t) \quad (22)
\end{aligned}$$

where $\Gamma$ is a positive gain matrix, $k > \bar{d}$ is a constant, the $j$-th element of $\text{SGN}(x)$ is such that $(\text{SGN}(x))_j = 1$ if $x_j > 0$, $[-1, 1]$ if $x_j = 0$, and $-1$ if $x_j < 0$ $\forall j = 1, 2, \cdots, n$. $u$ takes the value $-F(0)\hat{\theta}$ when $x = 0$. The evolution of the resulting feedback system is governed by the following delay differential inclusion

$$\begin{aligned}
\dot{x} &\in -mx(t) - k\text{SGN}(x(t)) + F(x(t))\theta - F(x(t))\hat{\theta} \\
&\quad + H(x(t - \tau)) + d(x(t)), \\
\dot{\hat{\theta}} &= \Gamma^{-1} F^T(x(t))x(t). \quad (23)
\end{aligned}$$

From the properties of the function $F$ and $H$, the set-valued functional at the righthand side of system (23) is bounded on bounded closed subset of $C_\tau$ and satisfies **basic assumptions**.

Define $\tilde{\theta} = \theta - \hat{\theta}$. Then system (23) can be rewritten as

$$\begin{aligned} \dot{x} &\in -mx(t) - k\mathrm{SGN}(x(t)) + F(x(t))\tilde{\theta} \\ &\quad +H(x(t-\tau)) + d(x(t)) \\ \dot{\tilde{\theta}} &= -\Gamma^{-1}F^T(x(t))x(t). \end{aligned} \tag{24}$$

Choose the following functional

$$\begin{aligned} V(\phi) =& \frac{1}{2}\varphi^T(0)\varphi(0) + \frac{m}{2}\int_{-\tau}^{0}\varphi^T(s)\varphi(s)ds \\ &+ \frac{1}{2}\phi_{n+1}^T(0)\Gamma\phi_{n+1}(0) \end{aligned}$$

where $\phi(s) = \begin{bmatrix} \phi_1(s) & \phi_2(s) & \cdots & \phi_{n+1}^T(s) \end{bmatrix}^T \in \mathbb{R}^{n+p}$, $\varphi(s) = \begin{bmatrix} \phi_1(s) & \phi_2(s) & \cdots & \phi_n(s) \end{bmatrix}^T \in \mathbb{R}^n, s \in [-\tau, 0]$. Computing the derivative of $V$ for system (24) yields

$$\begin{aligned} \dot{\tilde{V}}(\phi(0), \phi) =& -m\varphi^T(0)\varphi(0) - k\varphi^T(0)\mathrm{SGN}(\varphi(0)) \\ &+ \varphi^T(0)H(\varphi(-\tau)) + \varphi^T(0)d(\varphi(0)) \\ &+ \frac{m}{2}\varphi^T(0)\varphi(0) - \frac{m}{2}\varphi^T(-\tau)\varphi(-\tau). \end{aligned}$$

The above relation implies

$$\dot{\tilde{V}}(\phi(0), \phi) \leq (-k + \bar{d})\varphi^T(0)\mathrm{sgn}(\varphi(0)). \tag{25}$$

Note that we have applied the inequality: $\varphi^T(0)H(\varphi(-\tau)) \leq \frac{m}{2}\varphi^T(0)\varphi(0) + \frac{m}{2}\varphi^T(-\tau)\varphi(-\tau)$. Hence the system (23) is stable and the estimate error $\tilde{\theta}$ is bounded. Further, Theorem 3 shows that the state $x$ converges to zero.

## V. Conclusion

In this paper, stability of general delay differential inclusions is investigated. A chain rule for a wide class of Lyapunov-Krasovskii functionals is proposed and such functionals are not necessarily invariantly differentiable. Moreover, an invariance-like theorem is proposed for the first time for general delay differential inclusions here, where the derivative of the candidate functional is bounded above by a continuous negative semidefinite function. Several examples are given to show the effectiveness of these results, including an application in adaptive control.

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