Bisimulations for Input-Output Stability of Hybrid Systems

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Abstract— **In this paper, we investigate foundational questions related to the simplification based analysis of input-output stability of hybrid systems. Simulations and bisimulations are canonical notions that provide the relation between the original system and its abstraction that preserve several properties, including safety. However, recent investigations have shown that stability properties are not preserved by these notions. Hence, there have been efforts to strengthen these notions with additional requirements such as continuity on the relations, that enforce preservation of several stability notions including Lyapunov, asymptotic and input-to-state stability. Here we continue this line of work, and propose strengthenings of simulation/bisimulation relations for preservation of two variants of input-output stability for hybrid systems that are inspired by the incremental input-output stability and incremental state-independent input-output stability for continuous dynamical systems.**

I. INTRODUCTION

Hybrid systems [1] are complex control systems that exhibit mixed discrete-continuous behaviors, typically due to the interaction of the discrete logic implemented on an embedded processor with a continuously evolving physical system it controls. This is an extremely important class of control systems that has gained considerable interest in the recent past. Stability notions including Lyapunov and asymptotic stability, as well as input-to-state and input-to-output stability, capture the robustness of system behaviors with respect to perturbations in the initial state and/or input. Stability analysis for hybrid systems has been investigated extensively in the literature [2], [3], [4]. Methods based on common and multiple Lyapunov functions have been developed that provide proofs of stability [5], [6], [7], and automated methods that use certain semi-definite or sum-of-square programming problems [8], [9] to solve for parameters in candidate templates for these functions have been investigated. However, scalability of these methods remains a challenge owing to the complexity of these hybrid systems that arises due to non-trivial continuous dynamics and/or complex interactions between the continuous and discrete components. System simplification techniques that reduce the complexity of the system for the purpose of analysis are extremely important. In this work, we investigate the foundations of such simplification based analysis techniques for input-output stability of hybrid systems.

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Broadly, simplification or abstraction based analysis consists of constructing a simplified system (abstract system) from a given system (concrete system) such that the satisfaction of the desired property by the abstract system implies the satisfaction of the same by the concrete system. Such abstraction techniques have been investigate extensively in the context of safety analysis of hybrid systems using techniques such as predicate abstraction and hybridization [10], [11], [12] that construct finite state and hybrid system abstractions, respectively. Typically, the abstract system is conservative, and hence, violation of the property by the abstract system does not immediately imply the violation of the property by the concrete system. Hence, abstraction techniques are coupled with refinement techniques that iteratively reduce the conservativness in the abstraction [13], [14], [15].

Simulation and bisimultion relations [16] are canonical notions in process algebra that study the relation between the concrete and abstract systems for preservation of different properties. It is well known that if two systems are related by a bisimulation relation, then for several properties including safety and those expressible in certain temporal logics, either both systems satisfy the property or both do not [17]. These have been the basis of algorithmic techniques for safety analysis of several subclasses of hybrid systems including timed automata and o-minimal automata [18], [19]. Simulations are less restrictive and preserve property in only one direction and hence, allow a larger class of simplifications than bisimulations. The generalized notions of approximation simulation and bisimulation have also been investigated [20]. However, it was shown in [21] that even simulations and bisimulations do not preserve Lyapunov and asymptotic stability, and similar observations were made for input-to-state stability in [22]. Hence, these works develop new relations that strengthen the classical notions with additional constraints such as continuity and uniform continuity that enforce stability preservation. These foundational insights were used in [23], [24] to construct finite weighted graph abstractions for stability analysis. This is a novel approach in contrast to template based methods that that suffer from numerical issues as a results of numerical methods for solving the semi-definite and constraint solving problems. The broad goal of this work is to enable such new system simplification techniques for input-output stability by laying the foundations of abstractions.

First, we formalize two notions of input-output stability for hybrid systems with respect to a set of reference executions (which generalize equilibrium points), namely, *hIOS* and *hSIIOS*. Our definitions capture the essence of incremental input-ouput stability and incremental state-independent input-output stability whose non-incremental versions are defined in [25]. Next, we define two versions of uniformly continuous input-output simulations and bisimulations that preserve the two notions of stability formalized in the paper, respectively. We illustrate the notions using an example. Our notions generalize similar notions for input-to-state stability explored in [22]. In fact, the new definitions suggest a relaxed version of simulation/bisimulation for input-to-state stability than that explored in [22].

II. Hybrid Systems with Inputs and Outputs

Hybrid systems are systems exhibiting mixed discrete-continuous behaviors; hybrid automata [1] are a popular model that represent hybrid systems using a combination of finite state automata (for the discrete dynamics) and differential equations for the continuous dynamics. Here, we present *hybrid input/output transition systems*, that allow the modeling of the semantics of hybrid automata with inputs and outputs. In particular, we represent the continuous behavior using a triple consisting of input, state and output *trajectories* which capture the values of input, state and output over an interval of time; and represent the discrete behavior using *transitions* which capture instantaneous changes to the state and corresponding output due to impulse inputs. We take this simplistic model without concerning ourselves with continuity/differentiability of the solutions, since our theory can be developed in this general model. We will refer to these hybrid input/output transition systems as just *hybrid systems*.

A *trajectory* τ over a set A is a function $\tau : [0, T] \to A$ or $\tau : [0, \infty) \to A$, where $T \in \mathbb{R}^+$. If the domain, $dom(\tau)$, is a finite interval $[0, T]$, then $ltime(\tau)$ represents the last time T. Let $Traj(A)$ denote all trajectories over the set A.

Definition 1: A *hybrid I/O transition system* or *hybrid system* (HS) $\mathcal{H}$ is a tuple $(S, U, Y, \Sigma, \Delta)$, where S is a set of states, U is a set of inputs, Y is a set of outputs, $\Sigma \subseteq U \times S \times S \times Y \times Y$ is a set of transitions and $\Delta \subseteq Traj(U) \times Traj(S) \times Traj(Y)$ is a set of input-state-output trajectories such that for all $(\mathbf{u}, \mathbf{s}, \mathbf{y}) \in \Delta$, $dom(\mathbf{u}) = dom(\mathbf{s}) = dom(\mathbf{y})$.

Given an element $\sigma = (\sigma_u, \sigma_s, \sigma_y) \in \Delta$, we use $first_S(\sigma)$ to denote the first state of σ, namely, $\sigma_s(0)$ and if the domain of σ_s is a finite interval, then $last_S(\sigma)$ denotes the last state $\sigma_s(ltime(\sigma_s))$. Similarly, $first_Y(\sigma)$ and $last_Y(\sigma)$ will denote the first and last outputs. For $\sigma = (u, s, s', y, y') \in \Sigma$, $first_S(\sigma) = s$ and $last_S(\sigma) = s'$, $first_Y(\sigma) = y$ and $last_S(\sigma) = y'$.

Definition 2: An *execution* (finite or infinite) of a hybrid system $\mathcal{H}$ is a sequence $\{\sigma_i\}$, where $\sigma_i \in \Sigma \cup \Delta$, such that for each i, $last_S(\sigma_i) = first_S(\sigma_{i+1})$ and $last_Y(\sigma_i) = first_Y(\sigma_{i+1})$. Let $Exec(\mathcal{H})$ denote the set of all executions of $\mathcal{H}$.
Let $first_S(\{\sigma_i\}) = first_S(\sigma_0)$, and $first_Y(\{\sigma_i\}) = first_Y(\sigma_0)$.

Given an execution σ, we use σ^u, σ^s and σ^y to denote the projection of σ to the input, state and output components, also called the input, state and output signals.

In order to reason about stability of a system, one needs a notion of distance between the behaviors of the system. Hence, we extend the definition of the hybrid system with metrics on the state, input and output spaces which can then be extended to distance between executions.

Definition 3: A *metric hybrid system* (MHS) is a pair $(\mathcal{H}, d)$ where $\mathcal{H} = (S, U, Y, \Sigma, \Delta)$ is a hybrid system, $d = (d^s, d^u, d^y)$, and (S, d^s), (U, d^u) and (Y, d^y) are extended metric spaces (the range of the distance functions is $\mathbb{R}^+ \cup \{\infty\}$.

We will ignore the superscripts s, u, y in d^s, d^u, d^y, when it is clear from the context. We use graphical distance [26], [22] as the notion of distance between executions. Roughly, graphical distance between executions (or signals) $\sigma = \{\sigma_i\}$ and $\sigma' = \{\sigma'_i\}$, denoted $d(\sigma, \sigma')$, is less than or equal to ϵ if for every (i, t), there exists (j, t') such that the distance between $\sigma_i(t)$ and $\sigma_j(t')$ is within ϵ, and vice versa.

Example 1: We illustrate how to express a traditional representation of hybrid system in our semantic model. Consider a switched nonlinear system that switches arbitrarily (at any time) between two modes, say, q_1 and q_2, whose dynamics are given by the differential equations in $\dot{x}(t) = f_1(x(t), u(t))$ and $\dot{x}(t) = f_2(x(t), u(t))$, respectively, where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. Let $h : \mathbb{R}^n \to \mathbb{R}^l$ be an output map. The hybrid input/output transition $\mathcal{H} = (S, U, Y, \Sigma, \Delta)$ represents the above switched system, where the components are given by:

- $S = \{q_1, q_2\} \times \mathbb{R}^n$; $U = \mathbb{R}^m$; $Y = \mathbb{R}^l$;
- $\Sigma = \{(0, (q, x), (q', x'), y, y') \mid x' = x, y = h(x), y' = h(x')\}$;
- $\Delta = \{(\mathbf{u}, \mathbf{s}, \mathbf{y}) \mid \exists \mathbf{x} : \mathbf{s}(t) = (q, \mathbf{x}(t)) \forall t, (\mathbf{u}, \mathbf{x})$ satisfies $\dot{\mathbf{x}}(t) = f_1(\mathbf{x}(t), \mathbf{u}(t))$ if $q = q_1$ and satisfies $\dot{\mathbf{x}}(t) = f_2(\mathbf{x}(t), \mathbf{u}(t))$ if $q = q_2, \mathbf{y} = h(\mathbf{x})\}$.

The metric is defined by $d^u(u, u') = |u - u'|$; $d^y(y, y') = |y - y'|$; $d^s((q, x), (q', x')) = |x - x'|$ if $q = q'$ and ∞ otherwise.

III. Input-Output Stability

In this section, we define two notions of input to output stability of hybrid systems. We consider a set of executions as the reference behavior. Our definitions are inspired by the following two notions of incremental input-to-output stability for continuous dynamical systems whose non-incremental versions appear in [25]. Consider a continuous dynamical system with input u, state x and output y given by:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \mathbf{y} = h(\mathbf{x}) \tag{1}$$

Let $\mathbf{y}(\zeta, \mathbf{u})$ represent the output trajectory corresponding to the initial state ζ and input trajectory $\mathbf{u}$, and $\|\mathbf{u}\|$ correspond to the infinity norm on the signal $\mathbf{u}$. System 1 is *incrementally input-to-output stable* (δIOS) if there exists a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ such that for any time $t \geq 0$, any states ζ, ζ' and any pair of input trajectories $\mathbf{u}, \mathbf{u}'$, the following is true:

$$|\mathbf{y}(\zeta, \mathbf{u})(t) - \mathbf{y}(\zeta', \mathbf{u}')(t)| \leq \beta(|\zeta - \zeta'|, t) + \gamma(\|\mathbf{u} - \mathbf{u}'\|) \tag{2}$$

It states that the system is *δIOS* if small perturbations in the initial states or input signals lead to only small deviations in the output behaviors. Note that the initial state plays a role in this definition of input-output stability. An alternate notion that is independent of the initial state is the *state-independent incrementally input-to-output stability* (*δSIIOS*). *δSIIOS* is similar to *δIOS* except that the initial states on the right hand side are replaced by their corresponding outputs.

$$|\mathbf{y}(\zeta,\mathbf{u})(t)-\mathbf{y}(\zeta',\mathbf{u}')(t)| \leq \beta(|h(\zeta)-h(\zeta')|,t)+\gamma(\|\mathbf{u}-\mathbf{u}'\|) \quad (3)$$

Condition 3 states that small perturbations in the initial outputs and input signals lead to only small deviations in the overall output behaviors.

Next, we define a notion of (state-independent) input-to-output stability for hybrid systems with respect to a set of reference executions that captures the essence of conditions 2 and 3. Given a set of executions $\mathcal{T} \subseteq \mathit{Exec}(\mathcal{H})$, and an input signal $\mathbf{u}$, let $\mathcal{T}|_{\mathbf{u}}$ denote all executions for which $\mathbf{u}$ is the input, that is, the set $\{\sigma \in \mathcal{T} \mid \sigma^u = \mathbf{u}\}$.

Definition 4: A hybrid system $\mathcal{H}$ is *input-to-output stable* (*hIOS*) with respect to the set of executions $\mathcal{T} \subseteq \mathit{Exec}(\mathcal{H})$, if the following hold:

$(D1)$ for every $\epsilon > 0$, there exists a $\delta > 0$, such that for every input signal $\mathbf{u}$ and $\sigma \in \mathit{Exec}(\mathcal{H})|_{\mathbf{u}}$:

$$d(\mathit{first}_S(\sigma),\mathit{first}_S(\mathcal{T}|_{\mathbf{u}})) < \delta \Rightarrow \exists\hat{\sigma} \in \mathcal{T}|_{\mathbf{u}}, d(\sigma^y,\hat{\sigma}^y) < \epsilon$$

$(D2)$ there exists a $\eta > 0$ and a mapping $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every input signal $\mathbf{u}$ and for every $\sigma \in \mathit{Exec}(\mathcal{H})|_{\mathbf{u}}$:

$$d(\mathit{first}_S(\sigma),\mathit{first}_S(\mathcal{T}|_{\mathbf{u}})) < \eta \Rightarrow$$
$$\exists\hat{\sigma} \in \mathcal{T}|_{\mathbf{u}}, \forall\epsilon > 0, \mathit{within}(\sigma^y,\hat{\sigma}^y,\epsilon,g(\epsilon))$$

where $\mathit{within}(\sigma^y,\hat{\sigma}^y,\epsilon,T)$ states that the signals σ^y and $\hat{\sigma}^y$ are within ϵ at all time $t \geq T$.

$(D3)$ for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every input signal $\mathbf{u}$ and state ζ such that $\zeta \in \mathit{first}_S(\mathcal{T}|_{\mathbf{u}})$ and for all $\sigma \in \mathit{Exec}(\mathcal{H})$

$$(d(\sigma^u,\mathbf{u}) < \delta \wedge \mathit{first}_S(\sigma) = \zeta) \Rightarrow$$
$$(\exists\hat{\sigma} \in \mathcal{T}, \mathit{first}_S(\hat{\sigma}) = \zeta, d(\sigma^y,\hat{\sigma}^y) < \epsilon)$$

In [22], a characterization of input-to-state stability of continuous dynamical systems (Condition 2 with output replaced by state) is provided in terms of a decomposition into (a) Lyapunov stability of the system for a fixed input in which the $\epsilon-\delta$ dependence is independent of the input (b) convergence of the trajectories to the reference uniformly in time which is independent of the input (c) input-output stability for a fixed initial state in which the $\epsilon-\delta$ dependence is independent of the initial state. Conditions $(D1)-(D3)$ capture the conditions (a)-(c) corresponding to *δIOS*. More precisely, Condition $(D1)$ captures the fact that the β function in Equation 2 is a $\mathcal{K}$-function in the first argument, $(D2)$ that it is a $\mathcal{L}$ function in the second argument, and $(D3)$ that γ is a $\mathcal{K}_\infty$ function.

Similarly, we define *state-independent input-to-output stable* (*hSIIOS*) of $\mathcal{H}$ with respect to $\mathcal{T}$, by replacing all occurrences of $\mathit{first}_S(\cdot)$ by $\mathit{first}_Y(\cdot)$ in $(D1)$-$(D3)$ and considering an output o instead of state η in $(D3)$.

Remark 1: The two definitions, namely, *hIOS* and *hSIIOS*, generalize the notion of input-to-state stability of hybrid systems defined in [22]. However, when the output function is taken to be the identity function on the state, they are both equivalent to each other and to input-to-state stability. Further, note that the definition of *hSIIOS* is independent of the state component of the hybrid system. Hence, when analyzing *hSIIOS*, we can ignore the state component in the transitions and trajectories of a hybrid system.

IV. Preorders for I/O-stability preservation

Simplifications of models are extremely essential to deal with the analysis of complex systems such as hybrid systems. Foundations of such simplification based analyses lie in understanding the relation between the original (concrete) system and its simplification (abstraction) that preserve the property to be analysed. Simulation relations are certain binary relations between the state spaces of systems that induce a preorder on the class of systems such that given any two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ which are related, the satisfaction of the property by $\mathcal{H}_2$ implies the satisfaction of the property by $\mathcal{H}_1$. Several properties including safety and those expressible in temporal logics are known to be preserved by simulation and invariant under bisimulation (both systems satisfy the property or both do not) [17]. However, it was shown in [21], [22] that bisimulations do not preserve stability (Lyapunov, asymptotic as well as input-to-state stability). Hence, the notion was extended with continuity properties to ensure stability preservation. Here, we continue this line of work and examine such preorders for input-output stability notions. First, we review the classical notion of input-output (bi)simulation. Our definition is closely related to the definition of simulation defined in [27].

Definition 5: Given two hybrid systems $\mathcal{H}_1 = (S_1,U_1,Y_1,\Sigma_1,\Delta_1)$ and $\mathcal{H}_2 = (S_2,U_2,Y_2,\Sigma_2,\Delta_2)$, a triple of binary relations $R = (R_u,R_s,R_y)$, where $R_u \subseteq U_1 \times U_2$, $R_s \subseteq S_1 \times S_2$ and $R_y \subseteq Y_1 \times Y_2$ is called an *input-output (I/O) simulation relation* from $\mathcal{H}_1$ to $\mathcal{H}_2$, denoted $\mathcal{H}_1 \preceq_R \mathcal{H}_2$, if for every $(s_1,s_2) \in R_s$, the following hold:

- For every state s_1', input u_1 and outputs y_1,y_1' such that $(u_1,s_1,s_1',y_1,y_1') \in \Sigma_1$, there exist a state s_2', an input u_2 and outputs y_2,y_2' such that $R_s(s_1',s_2')$, $R_u(u_1,u_2)$, $R_y(y_1,y_2)$, $R_y(y_1',y_2')$ and $(u_2,s_2,s_2',y_2,y_2') \in \Sigma_2$.
- For every input-state-output trajectory $\sigma_1 = (\mathbf{u}_1,\mathbf{s}_1,\mathbf{y}_1) \in \Delta_1$ such that $\mathit{first}_S(\sigma_1) = s_1$, there exists an input-state-output trajectory $\sigma_2 = (\mathbf{u}_2,\mathbf{s}_2,\mathbf{y}_2) \in \Delta_2$ such that $\mathit{first}(\sigma_2) = s_2$, $\mathbf{s}_2 \in R_s(\mathbf{s}_1)$, $\mathbf{u}_2 \in R_u(\mathbf{u}_1)$ and $\mathbf{y}_2 \in R_y(\mathbf{y}_1)$.

Here $\tau' \in R(\tau)$ denotes that $R(\tau(t),\tau'(t))$ holds for all t.

Simulation guarantees that for every execution in the first system, there exists an execution in the second which is related by R pointwise in time. And, hence, the behaviors of $\mathcal{H}_2$ over-approximate the behaviors of $\mathcal{H}_1$.

Further, $R = (R_u,R_s,R_y)$ is an *I/O bisimulation relation* between $\mathcal{H}_1$ and $\mathcal{H}_2$ if both R and $R^{-1} = (R_u^{-1},R_s^{-1},R_y^{-1})$ are I/O simulation relations, that is, $\mathcal{H}_1 \preceq_R \mathcal{H}_2$ and

$\mathcal{H}_2 \preceq_{R^{-1}} \mathcal{H}_1$. Bisimulation relations define an equivalence relation on the class of systems. If two systems are related by a bisimulation relation, then they satisfy the same bisimulation invariant properties such as safety.

We also define *state-independent* versions of I/O (bi)simulation relations.

Definition 6: Given two hybrid systems $\mathcal{H}_1 = (S_1, U_1, Y_1, \Sigma_1, \Delta_1)$ and $\mathcal{H}_2 = (S_2, U_2, Y_2, \Sigma_2, \Delta_2)$, a triple of binary relations $R = (R_u, R'_y, R_y)$, where $R_u \subseteq U_1 \times U_2$, $R'_y \subseteq Y_1 \times Y_2$ and $R_y \subseteq Y_1 \times Y_2$ is called a *state-independent input-output simulation relation* from $\mathcal{H}_1$ to $\mathcal{H}_2$, denoted $\mathcal{H}_1 \preceq^I_R \mathcal{H}_2$, if for every $(y_1, y_2) \in R'_y$, the following hold:

- For all states s_1, s'_1, input u_1 and output y'_1 such that $(u_1, s_1, s'_1, y_1, y'_1) \in \Sigma_1$, there exist states s_2, s'_2, an input u_2 and output y'_2 such that $R_u(u_1, u_2)$, $R_y(y'_1, y'_2)$ and $(u_2, s_2, s'_2, y_2, y'_2) \in \Sigma_2$.
- For every input-state-output trajectory $\sigma_1 = (\mathbf{u}_1, \mathbf{s}_1, \mathbf{y}_1) \in \Delta_1$ such that *first*$_Y(\sigma_1) = y_1$, there exists an input-state-output trajectory $\sigma_2 = (\mathbf{u}_2, \mathbf{s}_2, \mathbf{y}_2) \in \Delta_2$ such that *first*$_Y(\sigma_2) = y_2$, $\mathbf{u}_2 \in R_u(\mathbf{u}_1)$ and $\mathbf{y}_2 \in R_y(\mathbf{y}_1)$.

Note that the above definition is independent of the state information; it does not have conditions that relate states in the transitions/trajectories in one system to those in the other. Also, when the output function is the identity function, the I/O simulaiton and its state-independent version coincide. We can similarly define state-independent I/O bisimulation.

A. Uniformly Continuous I/O (Bi)-Simulation

As shown in [21], [22], the classical notions of bisimulation and input bisimulation do not preserve stability notions including Lyapunov stability, asymptotic stability and input-to-state stability. Since, input-to-state stability is a specific instance of input-output stability (when the output function is identity), it follows that input-output bisimulations also do not preserve input-output stability. One can extend the examples in [22] to formally prove the insufficiency. Hence, our next goal is to extend the input-output bisimulation with additional requirements that enforce input-output stability preservation.

Definition 7: A relation $R \subseteq A \times B$ (on metric spaces A and B) is said to be *uniformly continuous* if

$$\forall \epsilon > 0, \exists \delta > 0 \,\text{s.t.}\, \forall a \in dom(A), R(B_\delta(a)) \subseteq B_\epsilon(R(a)).$$

Here $B_\gamma(a)$ denotes a ball of radius γ around a, and $R(S)$ is the set $\{s' \mid \exists s \in S, R(s, s')\}$. We denote by $uc(R, \epsilon, \delta)$ the fact that $\forall a \in dom(A), R(B_\delta(a)) \subseteq B_\epsilon(R(a))$. Given an $\epsilon > 0$, we call a $\delta > 0$ satisfying the above condition, a *uniformity constant of* R corresponding to ϵ.

Remark 2 (Uniform continuity): Note that if $d(a, b) < \delta$, then $R(b)$ is contained in $B_\epsilon(c)$ for *some* $c \in R(b)$. However, if $R(b)$ is a singleton, then $R(b) \subseteq B_\epsilon(c)$ for *any* $c \in R(b)$.

Definition 8: A triple $R = (R_u, R_s, R_y)$ is a *uniformly continuous I/O simulation* from $(\mathcal{H}_1, d_1)$ to $(\mathcal{H}_2, d_2)$, denoted $(\mathcal{H}_1, d_1) \preceq^C_R (\mathcal{H}_2, d_2)$, if R is an I/O simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and R_u, R_s and R_y^{-1} are uniformly continuous.

$\mathcal{H}_1 \preceq^C \mathcal{H}_2$ is used to denote the fact that there exists an R such that $\mathcal{H}_1 \preceq^C_R \mathcal{H}_2$.

Definition 9: A triple $R = (R_u, R'_y, R_y)$ is a *state-independent uniformly continuous I/O simulation* from $(\mathcal{H}_1, d_1)$ to $(\mathcal{H}_2, d_2)$, denoted $(\mathcal{H}_1, d_1) \preceq^{CI}_R (\mathcal{H}_2, d_2)$, if R is a state-independent I/O simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and R_u, R'_y and R_y^{-1} are uniformly continuous.

$\mathcal{H}_1 \preceq^{CI} \mathcal{H}_2$ is used to denote the fact that there exists an R such that $\mathcal{H}_1 \preceq^{CI}_R \mathcal{H}_2$.

Uniformly continuous input simulation relations that were defined in [22] require R_u, R_s and R_s^{-1} to be uniformly continuous. Note that our definition when restricted to identity output functions requires three uniformly continuous relations R_u, R_s and ${R'}_s^{-1}$. It is a relaxed notion, since it allows a potentially different uniformly continuous relation ${R'}_s^{-1}$ from the state space of $\mathcal{H}_2$ to the state space of $\mathcal{H}_1$ instead of requiring that R_s^{-1} be uniformly continuous.

The next theorem states that uniformly continuous I/O simulations define a preorder on hybrid systems. Recall that a preorder is a binary relation that is reflexive and transitive.

Theorem 1 (Preorder): Let $(\mathcal{H}_i, d_i)$, for $1 \le i \le 3$, where $\mathcal{H}_i = (S_i, U_i, Y_i, \Sigma_i, \Delta_i)$, be three metric hybrid systems. Then we have the following properties about $\preceq^C$:

- (Reflexivity) $\mathcal{H}_1 \preceq^C \mathcal{H}_1$.
- (Transitivity) If $\mathcal{H}_1 \preceq^C \mathcal{H}_2$ and $\mathcal{H}_2 \preceq^C \mathcal{H}_3$, then $\mathcal{H}_1 \preceq^C \mathcal{H}_3$,

The above theorem also holds when $\preceq^C$ is replaced by $\preceq^{CI}$.

Definition 10: R is a *(state-independent) uniformly continuous I/O bisimulation* from $\mathcal{H}_1$ to $\mathcal{H}_2$ if R is a (state-independent) uniformly continuous I/O simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and R^{-1} is a (state-independent) uniformly continuous I/O simulation from $\mathcal{H}_2$ to $\mathcal{H}_1$.

V. I/O Stability Preservation

In this section, we present the main result of the paper, namely, that uniformly continuous I/O simulations define a I/O stability preserving pre-order on the class of hybrid systems. However, we need the relations to satisfy certain additional conditions with respect to the reference executions. We summarize them below.

Definition 11: An I/O simulation $R = (R_u, R_s, R_y)$ from $\mathcal{H}_1 = (S_1, U_1, Y_1, \Sigma_1, \Delta_1)$ to $\mathcal{H}_2 = (S_2, U_2, Y_2, \Sigma_2, \Delta_2)$ is said to be *semi-consistent* with respect to the sets of executions $\mathcal{T}_1 \subseteq$ *Exec*$(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq$ *Exec*$(\mathcal{H}_2)$, if the following hold:

$(C1)$ $\exists \theta > 0$ such that for every $x_1 \in B_\theta($*first*$_S(\mathcal{T}_1))$, there exists a x_2 such that $R_s(x_1, x_2)$.

$(C2)$ For every $\mathbf{u}$, $R_s($*first*$_S(\mathcal{T}_1|_\mathbf{u})) \subseteq$ *first*$_S(\mathcal{T}_2|_{R_u(\mathbf{u})})$.

$(C3)$ For every $\sigma_2 \in \mathcal{T}_2$, such that there exists $\sigma \in \mathcal{T}_1$ with $R_s($*first*$_S(\sigma),$*first*$_S(\sigma_2))$ and $R_u(\sigma^u, \sigma_2^u)$, there exists $\sigma_1 \in \mathcal{T}_1$ such that $\sigma^u = \sigma_1^u$ and $R_y(\sigma_1^y, \sigma_2^y)$.

$(C4)$ For every y that appears in σ^y for some $\sigma \in \mathcal{T}_2$, $R_y^{-1}(y)$ is a singleton.

$(C5)$ For every u that appears in σ^u for some $\sigma \in \mathcal{T}_1$, $R_u(u)$ is a singleton.

$(C6)$ For every $\sigma_2 \in \mathcal{T}_2$, such that there exists $\sigma \in \mathcal{T}_1$ with $R_s($*first*$_S(\sigma),$*first*$_S(\sigma_2))$ and $R_u(\sigma^u, \sigma_2^u)$, there exists $\sigma_1 \in \mathcal{T}_1$ such that $\mathit{first}_S(\sigma) = \mathit{first}_S(\sigma_1)$ and $R_y(\sigma_1^y, \sigma_2^y)$.

These generalize the condition that in a purely continuous setting where the reference executions are unique equilibrium points, the equilibrium point of one system is uniquely mapped to the equilibrium point of the other system by the simulation relation. Note that semi-consistency only depends on $\mathcal{T}_1$ and $\mathcal{T}_2$ and not on other executions in $\mathcal{H}_1$ and $\mathcal{H}_2$. R is said to be *consistent* with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ if both R and R^{-1} are semi-consistent with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. The *state-independent* version of the (semi-)consistency will correspond to replacing the occurrences of $\mathit{first}_S(\cdot)$ by $\mathit{first}_Y(\cdot)$ and R_s by R'_y in the conditions $C1$-$C6$.

Theorem 2: Let $(\mathcal{H}_1, d_1)$ and $(\mathcal{H}_2, d_2)$ be two metric hybrid systems, and let $\mathcal{T}_1 \subseteq \mathit{Exec}(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq \mathit{Exec}(\mathcal{H}_2)$ be two sets of executions. Let R be a uniformly continuous I/O simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let R be semi-consistent with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then:

- If $\mathcal{H}_2$ is *hIOS* with respect to $\mathcal{T}_2$, then $\mathcal{H}_1$ is *hIOS* with respect to $\mathcal{T}_1$.

Proof: Let $\mathcal{H}_2$ be *hIOS* with respect to $\mathcal{T}_2$, that is, it satisfies conditions, $D1$-$D3$. We will show that $\mathcal{H}_1$ satisfies $D1$-$D3$ as well.

Let $\epsilon > 0$. We need to find a δ such that $D1$ is satisfied for $\mathcal{H}_1$. Let ϵ' be such that $uc(R_y^{-1}, \epsilon, \epsilon')$. Let δ' be a real number corresponding to ϵ' that witnesses $(D1)$ for $\mathcal{H}_2$. Finally, let δ be such that $uc(R_s, \delta', \delta)$. Let us assume that $\delta < \theta$ of Condition $(C1)$. Let us fix $\mathbf{u}$ and a $\sigma \in \mathit{Exec}(\mathcal{H}_1)|_{\mathbf{u}}$ such that $d(\mathit{first}_S(\sigma), \mathit{first}_S(\mathcal{T}_1|_{\mathbf{u}})) < \delta$. From condition $(C1)$ and R being a simulation, there exists $\sigma' \in \mathit{Exec}(\mathcal{H}_2)|_{R_u(\mathbf{u})}$ such that σ and σ' are related by R. Since $d(\mathit{first}_S(\sigma), \mathit{first}_S(\mathcal{T}_1|_{\mathbf{u}})) < \delta$, $d(\mathit{first}_S(\sigma'), R_s(\mathit{first}_S(\mathcal{T}_1|_{\mathbf{u}}))) < \delta'$ (from choice of δ'). From Condition $(C2)$, $d(\mathit{first}_S(\sigma'), \mathit{first}_S(\mathcal{T}_2|_{R_u(\mathbf{u})})) < \delta'$. Then there exists $\hat{\sigma}' \in \mathcal{T}_2|_{R_u(\mathbf{u})}$ such that $d(\sigma'^y, \hat{\sigma}'^y) < \epsilon'$ from satisfaction of $D1$ by $\mathcal{H}_2$. From Condition $(C3)$, there exists $\hat{\sigma} \in \mathcal{T}_1|_{\mathbf{u}}$ such that $R_y(\hat{\sigma}^y, \hat{\sigma'}^y)$. Finally, we show that $d(\sigma^y, \hat{\sigma}^y) < \epsilon$, thereby showing $\mathcal{H}_1$ satisfies $(D1)$. Note that $d(\sigma'^y, \hat{\sigma}'^y) < \epsilon$, $R_y(\sigma^y, \sigma'^y)$ and $R_y(\hat{\sigma^y}, \hat{\sigma}'^y)$. From the choice of ϵ and the fact that R_y^{-1} is a singleton on every y occuring in $\hat{\sigma}'^y$, we obtain that $d(\sigma^y, \hat{\sigma}^y) < \epsilon$.

Next, let us assume that Condition $(D2)$ holds for $\mathcal{H}_2$ and show that Condition $(D2)$ holds for $\mathcal{H}_1$. Let us fix $\eta_2 > 0$ and g_2 for $\mathcal{H}_2$. Consider η_1 such that $uc(R_s, \eta_2, \eta_1)$ holds. Define $h(\epsilon)$ to be some ϵ' such that $uc(R_y^{-1}, \epsilon, \epsilon')$ holds. Given $\mathbf{u}$ and $\sigma \in \mathit{Exec}(\mathcal{H}_1)|_{\mathbf{u}}$, define $\hat{\sigma}'$ and $\hat{\sigma}$ as in the proof for Condition $(D1)$. It is easy to see that $\mathit{within}(\sigma^y, \hat{\sigma}^y, \epsilon, g_2(h(\epsilon)))$ holds for every ϵ, since $\mathit{within}(\sigma'^y, \hat{\sigma}'^y, \epsilon', g_2(\epsilon'))$ holds.

Lastly, let us assume that Condition $(D3)$ holds for $\mathcal{H}_2$ and show that it holds for $\mathcal{H}_1$ as well. Given $\epsilon > 0$ for $\mathcal{H}_1$, choose $\delta > 0$ as in the proof of Condition $(D1)$. Let ζ_1, σ_1 and $\mathbf{u}_1$ be such that $\zeta_1 \in \mathit{first}_S(\mathcal{T}_1)\mathbf{u}_1$, $\mathit{first}_S(\sigma_1) = \zeta_1$ and $d(\sigma_1^u, \mathbf{u}_1) < \delta$. From Condition $(C2)$, we have a $\mathbf{u}_2$ and ζ_2 such that $\zeta_2 \in \mathit{first}_S(\mathcal{T}_2)\mathbf{u}_2$, $R_s(\zeta_1, \zeta_2)$ and $R_u(\mathbf{u}_1, \mathbf{u}_2)$. Since, R is a simulation, there exists an execution σ_2 in $\mathcal{H}_2$ that is related to σ_1 and $\mathit{first}_S(\sigma_2) = \zeta_2$. Using Condition $(C5)$, since, σ_1^u is within δ of $\mathbf{u}_1$, we obtain that σ_2^u is within δ' of $\mathbf{u}_2$. Hence, σ_2^y is within ϵ' of $\hat{\sigma}_2^y$ for some $\sigma_2 \in \mathcal{T}_2$ with $\mathit{first}_S(\sigma_2) = \eta_2$. Using Condition $(C6)$, there exists $\hat{\sigma}_1 \in \mathcal{T}_1$ such that $\mathit{first}_S(\hat{\sigma}_1) = \eta_1$ and $R(\hat{\sigma}_1^y, \hat{\sigma}_2^y)$. Since, σ_2^y is within ϵ' of $\hat{\sigma}_2^y$, σ_1^y is within ϵ of $\hat{\sigma}_1^y$ from Condition $(C4)$ and choice of ϵ. ■

Theorem 3: Let $(\mathcal{H}_1, d_1)$ and $(\mathcal{H}_2, d_2)$ be two metric hybrid systems, and let $\mathcal{T}_1 \subseteq \mathit{Exec}(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq \mathit{Exec}(\mathcal{H}_2)$ be two sets of executions. Let R be a state-independent uniformly continuous I/O simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let R be state-independent semi-consistent with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then:

- If $\mathcal{H}_2$ is *hSIIOS* with respect to $\mathcal{T}_2$, then $\mathcal{H}_1$ is *hSIIOS* with respect to $\mathcal{T}_1$.

Proof: The proof is similar to that of the proof of Theorem 2, except that $\mathit{first}_Y(\cdot)$ is used instead of $\mathit{first}_S(\cdot)$, R'_y is used instead of R_s and the state-independent versions of Conditions $C1$-$C6$ are used. ■

The next theorem is a corollary of Theorem 2 and Theorem 3 and states that hybrid (state-independent) I/O stability is a uniformly continuous (state-independent) I/O bisimulation invariant property.

Theorem 4: Let $(\mathcal{H}_1, d_1)$ and $(\mathcal{H}_2, d_2)$ be two metric hybrid systems, and $\mathcal{T}_1 \subseteq \mathit{Exec}(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq \mathit{Exec}(\mathcal{H}_2)$ be two sets of executions.

1) Let R be a uniformly continuous I/O bisimulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let R be consistent with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then the following holds: $\mathcal{H}_1$ is *hIOS* with respect to $\mathcal{T}_1 \Leftrightarrow \mathcal{H}_2$ is *hIOS* with respect to $\mathcal{T}_2$.
2) Let R be a state-independent uniformly continuous I/O bisimulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let R be state-independent consistent with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then the following holds: $\mathcal{H}_1$ is *hSIIOS* with respect to $\mathcal{T}_1 \Leftrightarrow \mathcal{H}_2$ is *hSIIOS* with respect to $\mathcal{T}_2$.

Theorem 2 and Theorem 3 state that uniformly continuous I/O simulations and their state-independent versions serve as a foundation for constructing abstractions for *hIOS* and *hSIIOS* analysis, respectively. Theorem 2 states that uniformly continuous I/O bisimulations and their state-independent versions define a notion of equivalence between systems with respect to *hIOS* and *hSIIOS*, respectively. Further, they also lay the foundations for refinements. More precisely, if $\mathcal{H}_2$ is an abstraction of $\mathcal{H}_1$, that is, it satisfies $\mathcal{H}_1 \preceq^\alpha \mathcal{H}_2$ (for $\alpha = C$ or CI), then a refinement would correspond to a system $\mathcal{H}_3$ such that $\mathcal{H}_1 \preceq^\alpha \mathcal{H}_3 \preceq^\alpha \mathcal{H}_2$. $\mathcal{H}_3$ is an abstraction that is "closer" to $\mathcal{H}_1$ than $\mathcal{H}_2$, in that it has "fewer" behaviors than $\mathcal{H}_2$.

VI. Illustration

In this section, we illustrate how Theorem 2 can be used to establish input-output stability of a hybrid system. Consider a switched nonlinear system that arbitrarily switches between two modes whose dynamics are as below:

$$\begin{cases} \dot{x}_1 &= -x_1 - 2x_1x_2^2 + u_1 \\ \dot{x}_2 &= -2x_2 - x_1^2x_2 + u_2 \\ \dot{x}_3 &= x_1^2 + x_1^2 + x_3 + u_3 \end{cases} \quad (4)$$

$$\begin{cases} \dot{x}_1 &= -2x_1 + x_1x_2^2 + u_1 \\ \dot{x}_2 &= -x_2 - 2x_1^2x_2 + u_2 \\ \dot{x}_3 &= -x_1^2 - x_1^2 + x_3 + u_3 \end{cases} \quad (5)$$

Let the input, state and output spaces be compact sets X_0, U and Y. Define an output map $h : \mathbb{R}^3 \to \mathbb{R}^2$ by $h(x_1, x_2, x_3) = (x_1, x_2)$, i.e., the projection of the state $x = (x_1, x_2, x_3)$ onto the (x_1, x_2)-plane. Let $\mathcal{H}_1$ denote the hybrid input/output transition system corresponding to this switched system as explained in the example in Section II.

We will show that $\mathcal{H}_1$ is input-to-output stable by constructing an abstract system $\mathcal{H}_2$ that uniformly continuously simulates $\mathcal{H}_1$ and which can be easily verified to be input-to-output stable. Define an abstract, 1-dimensional hybrid system $\mathcal{H}_2$ as follows:

- The state, input, and output spaces be $\mathbb{R}_{\geq 0}$;
- The set of transitions are identity transitions of the form $(0, x, x, \sqrt{x}, \sqrt{x})$
- The set of trajectories consist of triples $(\mathbf{u}, \mathbf{x}, \mathbf{y})$ satisfying $\dot{\mathbf{x}} \leq -\mathbf{x} + ||\mathbf{u}||^2$ and $\mathbf{y} = \sqrt{\mathbf{x}}$.

This abstract system is clearly input-to-output stable, because by a standard comparison argument $\mathbf{y}(t) \leq \sqrt{\mathbf{x}(0)}e^{-\frac{t}{2}} + ||\mathbf{u}||$ for all $t \geq 0$.

We now define the triple $R = (R_u, R_s, R_y)$ that relates the concrete switched system $\mathcal{H}_1$ with the abstract system $\mathcal{H}_2$ defined above. Let $V(x) = x_1^2 + x_2^2$.

- $R_s := \{(x, x') \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0} : \; x' = V(x)\}$.
- $R_u := \{(u, u') \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0} : \; u' = |u|^2\}$.
- $R_y := \{(y, y') \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} : \; y' = \sqrt{y_1^2 + y_2^2}\}$.

We observe that R is a I/O simulation. For every trajectory/execution of $\mathcal{H}_1$, its image under R is trajectory/execution of $\mathcal{H}_2$. This follows from the following inequalities: $\dot{V}\big|_{(4)} \leq -2x_1^2 - 4x_2^2 + 2x_1u_1 + 2x_2u_2 \leq -V(x) + ||u||^2$, $\dot{V}\big|_{(5)} \leq -4x_1^2 - 2x_2^2 + 2x_1u_1 + 2x_2u_2 \leq -V(x) + ||u||^2$. Moreover, $|h(x)|^2 = V(x) \leq |x|^2$ for all $x \in \mathbb{R}^3$. Further, the functions $V(\cdot)$ and $|\cdot|^2$ are continuous functions, and the relations R_s and R_u are over compact spaces. Therefore, R_s and R_u are uniformly continuous. R_y^{-1} maps every point $y \in \mathbb{R}^+$ to a circle of radius y in the two dimensional plane. Note that R_y^{-1} is continuous, since, small perturbations in y lead to small perturbations in the circles. Further, since, the domain and co-domain of R_y are compact, it is uniformly continuous.

By Theorem 2, we can conclude that the switched system $\mathcal{H}_1$ is also input-to-output stable. This agrees with the input-to-output stability results for continuous dynamical systems [25] and switched systems [28], since, $V(x)$ is a Lyapunov function for the system. An interesting research direction is to develop computational methods for constructing the simplified systems which we know uniformly continuously simulate the original system.

VII. Conclusion

In this paper, we defined uniformly continuous I/O simulations and bisimulations and their state-independent versions that serve as the foundations for abstraction and reduction of hybrid systems for reasoning about I/O stability and state-independent I/O stability. Main future work will consist of using the insights provided by these preorders to develop abstraction and model reduction techniques that construct simplified systems and hence, reduce the complexity of verifying I/O stability.

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