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Simulations and bisimulations for analysis of stability with respect to inputs of hybrid systems

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Abstract Simulation and bisimulation relations define pre-orders on processes which serve as the basis for approximation based verification techniques, and have been extended towards the design of continuous and hybrid systems with complex logic specifications. We study pre-orders between hybrid systems which preserve stability properties with respect to input. We show that these properties are not bisimulation invariant, and hence propose stronger notions which strengthen simulation and bisimulation relations with uniform continuity constraints. We show that uniform continuity is necessary on the relations corresponding to both the state-space and the input-space, and continuity itself does not suffice. Finally, we demonstrate the satisfiability of our definitions by casting the well-known Lyapunov function based techniques for stability analysis as constructing a simple one-dimensional system which is stable and uniformly continuously simulates the original system.

Keywords Bisimulations · Stability · Hybrid systems · Abstractions · Input-to-state stability · Incremental input-to-state stability

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1 Introduction

Embedded systems manifest extensively in safety-critical application domains such as aeronautics, automotive and medical devices, and hence, reliability is an extremely important issue. Correctness of embedded systems depends crucially on their interaction with the physical world; hence, formal reasoning entails modeling both the discrete and the continuous behaviors. Such systems with mixed discrete-continuous behaviors are popularly referred to as hybrid systems. This paper investigates pre-orders on hybrid systems as a foundational basis for developing abstraction mechanisms for reasoning about stability properties with respect to input.

Stability is a fundamental property in control system design. Intuitively, it captures the notion that small perturbations in the initial state or input lead to only small variations in the behavior of the system. In this paper, we focus explicitly on stability properties of hybrid systems with respect to inputs. Stability with respect to inputs is a robustness property wherein the small perturbations in input capture inconsistencies such as quantization errors in actuators and sensors, which are often ignored during control design, but, are nevertheless, present in any digital implementation of the control law.

Our first step is to define a notion of stability with respect to inputs for hybrid systems. We present a definition which generalizes the existing definitions in the literature for purely continuous systems. The classical notion of stability with respect to inputs for purely continuous systems is the input-to-state stability (Sontag and Wang 1995; Sontag 2006). It refers to the stability of the system with respect to an equilibrium point - a state of the system which does not change with time as long as no inputs are applied. Incremental input-to-state stability (Angeli 2002) generalizes input-to-state stability where the reference behavior is a trajectory and not necessarily an equilibrium point. The notion of input-to-state stability with respect to equilibrium points has been defined for hybrid systems and has been well-investigated (Cai and Teel 2009; Müller and Liberzon 2012; Heemels and Weiland 2008). More recently, incremental stability for nonlinear switched systems is proposed and characterized in (Girard et al. 2010) and input-to-output stability notions have been explored for discrete systems (Tabuada et al. 2012). Since the publication of the preliminary version of this paper (Prabhakar et al. 2013), several works on the notions of incremental stability for hybrid systems including their characterization using Lyapunov functions have appeared (Postoyan et al. 2015; Li and Sanfelice 2015). In this paper, we generalize the definition of incremental input-to-state stability of continuous systems to a notion of input-to-state stability with respect to trajectories for hybrid systems. For this, we provide an alternate characterization of the classical definition of incremental input-to-state stability using comparison functions in terms of super-position theorem "separating" the stability with respect to states and the stability with respect to the inputs. In particular, it provides an $\epsilon - \delta$ definition from which the generalization is natural and straightforward.

Simulations and bisimulations (Milner 1989) are the canonical notions of pre-order and equivalence on processes studied in process algebra. In the context of formal verification, they form the basis for state-space reduction and minimization techniques (Lee and Yannakakis 1992; Clarke et al. 1999). Several discrete-time properties including those expressible in LTL, CTL and μ-calculus, are known to be invariant under bisimulation; that is, if two systems are bisimilar then either both satisfy the property or both do not. We investigate pre-orders on hybrid systems which preserve stability properties with respect to input.

The broad motivation for this work is to develop an automated abstraction refinement framework for verification of stability properties with respect to inputs. This is to overcome some serious shortcoming of the current techniques for stability verification based on Lyapunov functions. These methods provide a certificate of stability in the form of a Lyapunov function (see, for example, Khalil 1996)- a continuously differentiable function which is positive definite and whose value decreases along any trajectory of the system. Automated methods for stability verification based on Lyapunov functions essentially indulge in search for parameters in a template which serves as a candidate Lyapunov function. One such method encodes the conditions of the Lyapunov function as a sum-of-square programming problem and solves it for the parameters (Papachristodoulou and Prajna 2002; Parrilo 2000). A major drawback of template based methods is the lack of efficient methods to iterate over the templates when a particular parameter search for a template fails. An abstraction-refinement framework on the other hand provides qualitatively better abstractions through refinement. For some recent work on abstraction refinement techniques for stability properties without input, see Prabhakar and Soto (2013).

The foundations of an abstraction refinement framework lie in understanding the pre-orders on hybrid systems which preserve the properties of interest. In the case without input, it has been shown in Cuijpers (2007) and Prabhakar et al. (2012) that simulations and bisimulations do not preserve the two classical notions of stability in control theory, namely, Lyapunov and asymptotic stability. Hence, additional constraints such as continuity and uniform continuity are imposed on the simulation and bisimulation relations to force stability preservation. In this paper, we investigate pre-orders for input-to-state stability of hybrid systems. We show that the classical definitions of input simulations and bisimulations do not preserve incremental input-to-state stability, even with continuity constraints imposed on the relations over the input and state spaces. Hence, uniform continuity constraints are added to the relations corresponding to both the state space and the input space. Finally, we demonstrate that the definitions are reasonable, by exhibiting concrete abstraction functions which satisfy the definition of the pre-order. To this end, we cast the well-known Lyapunov function based analysis as a concrete abstraction function which reduces a system to a simple one-dimensional system, for which incremental input-to-state stability can be easily inferred. The future work will focus on developing new abstraction-refinement mechanisms.

Organization of the paper In Section 2, we define the preliminaries and in Section 3, we define hybrid input transition systems and related concepts. Next, in Section 4, we show how to model the system description in terms of the traditional differential equations in our framework. In Section 5, we present a superposition type characterization of the notion of incremental input-to-state stability of continuous dynamical systems, and use it to present a definition of hybrid input-to-state stability. In Section 6, we define the hybrid input-to-state stability preserving pre-orders on hybrid systems. In Section 7, we present the stability preservation theorem and inadequacies of weaker notions of pre-orders. Finally, in Section 8, we present concrete abstraction techniques based on Lyapunov function based input-to-state stability analysis, and conclude with Section 9.

A preliminary version of the paper appeared in the Proceedings of the International Conference on Embedded Software (EMSOFT), 2013 (Prabhakar et al. 2013). The current draft consists of substantial reorganization and detailed explanations to improve the readability of the paper. In particular, Section 7 has been substantially extended to provide all the details of the counter-examples and their correctness. In addition, Section 8 has been added with a new result, namely, Theorem 8.

2 Preliminaries

Notation Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the set of reals and non-negative reals, respectively. Let $\mathbb{R}_\infty$ denote the set $\mathbb{R}_{\geq 0} \cup \{\infty\}$, where ∞ denotes the largest element of $\mathbb{R}_\infty$, that is, $x < \infty$ for all $x \in \mathbb{R}_{\geq 0}$. Also, for all $x \in \mathbb{R}_\infty$, $x + \infty = \infty$. Let $\mathbb{N}$ denote the set of all natural numbers $\{0, 1, 2, \cdots\}$, and let $[n]$ denote the first n natural numbers, that is, $[n] = \{0, 1, 2, \cdots, n-1\}$. Let $PreInt$ denote the set consisting of all closed intervals of the form $[0, T]$, where $T \in \mathbb{R}_{\geq 0}$, and the infinite interval $[0, \infty)$. Given an $x \in \mathbb{R}^n$, we use $|x|$ to denote the Euclidean norm of x. And, given a function $f : A \to \mathbb{R}^m$, we use $\|f\|_\infty$ to denote $\sup_{a \in A} |f(a)|$.

Functions and relations A binary relation R between A and B is a subset of $A \times B$. We denote the domain of R, namely, A, by $dom(R)$. The inverse of R, R^{-1} denotes the set $\{(b, a) \mid (a, b) \in R\}$. Given $a \in A$ and $A' \subseteq A$, $R(a) = \{b \mid (a, b) \in R\}$, $R(A') = \{b \mid \exists a \in A', (a, b) \in R\}$ and $R[A']$ is a relation between A' and B given by $R \cap (A' \times B)$. The composition of two relations $R \subseteq A \times B$ and $R' \subseteq B \times C$ is the relation between A and C, denoted $R \circ R'$, given by $\{(a, c) \mid \exists b \in B, (a, b) \in R, (b, c) \in R'\}$. A function F from A to B, denoted $F : A \to B$, is a relation between A and B such that $F(a)$ is a singleton for every $a \in A$.

Sequences A sequence σ is a function whose domain is either $[n]$ for some $n \in \mathbb{N}$ or the set of natural numbers $\mathbb{N}$. We denote the set of all domains of sequences as *SeqDom*. Length of a sequence σ, denoted $|\sigma|$, is n if $dom(\sigma) = [n]$ or ∞ otherwise. Given a sequence $\sigma : \mathbb{N} \to \mathbb{R}$ and an element r of $\mathbb{R}_\infty$ we use $\sum_{i=0}^{\infty} \sigma(i) = r$ to denote the standard limit condition $\lim_{N \to \infty} \sum_{i=0}^{N} \sigma(i) = r$.

Extended metric space An *extended metric space* is a pair (M, d) where M is a set and $d : M \times M \to \mathbb{R}_\infty$ is a distance function such that for all m_1, m_2 and m_3, the following hold: (Identity of indiscernibles) $d(m_1, m_2) = 0$ if and only if $m_1 = m_2$, (Symmetry) $d(m_1, m_2) = d(m_2, m_1)$, and (Triangle inequality) $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$. When the metric on M is clear we will simply refer to M as a metric space.

We define an open ball of radius ϵ around a point x to be the set of all points which are within a distance ϵ from x. Formally, an *open ball* is a set of the form $B_\epsilon(x) = \{y \in M \mid d(x, y) < \epsilon\}$. An *open set* is a subset of M which is a union of open balls. Given a set $X \subseteq M$, a *neighborhood* of X is an open set in M which contains X. Given a subset X of M, an ϵ-neighborhood of X is the set $B_\epsilon(X) = \bigcup_{x \in X} B_\epsilon(x)$. A subset X of M is *compact* if for every collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that $X \subseteq \bigcup_{\alpha \in A} U_\alpha$, there is a finite subset J of A such that $X \subseteq \bigcup_{i \in J} U_i$.

Set valued functions We consider set valued functions and define continuity of these functions. We choose not to treat set valued functions as single valued functions whose co-domain is a power set, since as argued in Aubin and Frankowska (1990), it leads to strong notions of continuity, which are not satisfied by many functions. A *set valued function* $F : A \rightsquigarrow B$ is a function which maps every element of A to a set of elements in B. A set valued function is another representation of a relation from A to B.

Continuity of set valued functions Let $F : A \rightsquigarrow B$ be a set valued function, where A and B are extended metric spaces. We define upper semi-continuity of F which is a generalization of the "ϵ, δ - definition" of continuity for single valued functions (Aubin and Frankowska 1990). The function $F : A \rightsquigarrow B$ is said to be *upper semi-continuous* at $a \in dom(F)$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } F(B_\delta(a)) \subseteq B_\epsilon(F(a)).$$

If F is upper semi-continuous at every $a \in dom(F)$ we simply say that F is upper semi-continuous. Next we define a "uniform" version of the above definition, where, analogous to the case of single valued functions, corresponding to an ϵ, there exists a δ which works for every point in the domain.

Definition 1 A function $F : A \rightsquigarrow B$ is said to be *uniformly continuous* if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall a \in dom(A),\ F(B_\delta(a)) \subseteq B_\epsilon(F(a)).$$

Given an $\epsilon > 0$, we call a $\delta > 0$ satisfying the above condition, a *uniformity constant of* F corresponding to ϵ. We refer to uniform upper semi-continuity as just uniform continuity, because uniform upper semi-continuity implies lower semi-continuity.

Proposition 1 *Given uniformly continuous functions $F : A \rightsquigarrow B$ and $G : B \rightsquigarrow C$, the composition $F \circ G$ is uniformly continuous.*

Proof Let F and G be uniformly continuous. Given $\epsilon > 0$, let $\gamma > 0$ be a uniformity constant of G with respect to ϵ, and let $\delta > 0$ be uniformity constant of F with respect to γ. It is straightforward to verify that δ is a uniformity constant of $F \circ G$ with respect to ϵ. Therefore, $F \circ G$ is uniformly continuous. □

Comparison functions: class $\mathcal{K}$, L, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions Next, we define some comparison functions which are used to define the notions of stability. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to *class* $\mathcal{K}_\infty$ if α is a class $\mathcal{K}$ function and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\varphi : [0, \infty) \to [0, \infty)$ is said to be of *class* L if it is monotonically decreasing and $\lim_{s\to\infty} \varphi(s) = 0$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is a *class* $\mathcal{KL}$ function if it is a class $\mathcal{K}$ function with respect to the first argument and class L with respect to the second argument, that is, for a fixed s, $\beta(r, s)$ is a class $\mathcal{K}$ function and for a fixed non-zero r, $\beta(r, s)$ is a class L function.

Remark 1 Given a class $\mathcal{K}$ function α and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, if f satisfies $|f(x_1) - f(x_2)| \leq \alpha(|x_1 - x_2|)$, then it implies that f is uniformly continuous. On the other hand, if β is a class L function and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a function satisfying $|g(t)| \leq \beta(t)$, then $|g(t)|$ converges to 0 as $t \to \infty$.

3 Hybrid systems with input

In this section, we present a general formalism for representing hybrid systems with inputs, called *hybrid input transition systems*. Hybrid systems are systems exhibiting mixed discrete-continuous behaviors. We represent the continuous behavior using a pair of input and state *trajectories* which capture the values of input and state over an interval of time;

and represent the discrete behavior using *transitions* which capture instantaneous changes to the state due to impulse inputs. We will not concern ourselves with the exact representation of the models, see, for example, the hybrid automaton model (Henzinger 1996). However, our abstract model captures the behaviors arising from a hybrid automaton model.

3.1 Trajectories

A *trajectory* τ over a set A is a function $\tau : I \to A$, where $I \in PreInt$. We denote the set of all trajectories over A as $Traj(A)$. Let us define a function $Size : Traj(A) \to \mathbb{R}_\infty$ which assigns a size to the trajectories. For $\tau \in Traj(A)$, $Size(\tau) = T$ if $dom(\tau) = [0, T]$ and $Size(\tau) = \infty$ if $dom(\tau) = [0, \infty)$.

Relating trajectories Given a relation $R \subseteq A_1 \times A_2$ and trajectories $\mathbf{a}_1 \in Traj(A_1)$ and $\mathbf{a}_2 \in Traj(A_2)$, we say that $\mathbf{a}_1$ and $\mathbf{a}_2$ are related by R, denoted $R(\mathbf{a}_1, \mathbf{a}_2)$ if $dom(\mathbf{a}_1) = dom(\mathbf{a}_2)$ and for every $t \in dom(\mathbf{a}_1)$, $R(\mathbf{a}_1(t), \mathbf{a}_2(t))$. We use $R(\mathbf{a}_1)$ to denote the set $\{\mathbf{a}_2 \mid R(\mathbf{a}_1, \mathbf{a}_2)\}$.

Input-state trajectories An input-state trajectory specifies the state evolution on an input signal. Let us fix a set of inputs U and a set of states S. An *input-state trajectory* over a pair (U, S) is a pair of trajectories $(\mathbf{u}, \mathbf{s})$ from $Traj(U) \times Traj(S)$ such that $dom(\mathbf{u}) = dom(\mathbf{s})$. We call $\mathbf{u}$ an *input trajectory* and $\mathbf{s}$ a *state trajectory*. We will use $ISTraj(U, S)$ to denote the set of all input-state trajectories over (U, S).

Size, first, last, states, inputs of input-state trajectories We extend *Size* to input-state trajectories in the natural way, namely, $Size(\mathbf{u}, \mathbf{s}) = Size(\mathbf{u}) = Size(\mathbf{s})$. We use $First((\mathbf{u}, \mathbf{s}))$ to denote the initial state, that is, $\mathbf{s}(0)$, and $Last((\mathbf{u}, \mathbf{s}))$ to denote the last state, that is, $\mathbf{s}(Size(\mathbf{s}))$, if $Size(\mathbf{s})$ is not ∞, and is not defined otherwise. Given a state trajectory $\mathbf{s}$, we use $States(\mathbf{s})$ to denote the set of states occurring in $\mathbf{s}$, namely, $\{\mathbf{s}(t) \mid t \in dom(\mathbf{s})\}$. Also, for an input-state trajectory we use $States((\mathbf{u}, \mathbf{s}))$ to denote $States(\mathbf{s})$. Similarly, for an input trajectory $\mathbf{u}$, we use $Inputs(\mathbf{u})$ to denote the set of inputs occurring in $\mathbf{u}$, namely, $\{\mathbf{u}(t) \mid t \in dom(\mathbf{u})\}$.

3.2 Transitions

A transition specifies the instantaneous change in a state resulting from an impulse input. A *transition* over a pair (U, S) is an element of $U \times (S \times S)$. A transition $(u, (s_1, s_2))$ denotes the fact that if an input impulse u is applied to the system in state s_1, then the system state changes to s_2. We will represent a transition $(u, (s_1, s_2))$ as $s_1 \xrightarrow{u} s_2$. We denote the set of all transition over a pair (U, S) as $Trans(U, S)$.

Size, first, last, states, inputs of transitions We define *Size* of a transition $(u, (s_1, s_2))$ to be 0. As before, given $\tau = (u, (s_1, s_2))$, we use $First(\tau)$ and $Last(\tau)$ to denote the state of the system before and after the transition, namely, $First(\tau) = s_1$ and $Last(\tau) = s_2$. Also, $First((s_1, s_2)) = s_1$ and $Last((s_1, s_2)) = s_2$. Similarly, $States((s_1, s_2)) = States((u, (s_1, s_2))) = \{s_1, s_2\}$. And, $Inputs(u) = \{u\}$, for an input u.

3.3 Hybrid input transition systems

We can now define a hybrid input transition system as consisting of sets of input-state trajectories and transitions.

Definition 2 A *hybrid input transition system* ($HITS$) $\mathcal{H}$ is a tuple (S, U, Σ, Δ), where S is a set of states, U is a set of inputs, $\Sigma \subseteq Trans(U, S)$ is a set of transitions and $\Delta \subseteq ISTraj(U, S)$ is a set of input-state trajectories.

We will just use hybrid system or hybrid transition system to refer to the above entity. Next, we define an execution of a hybrid transition system, which is a behavior of the system. An execution is a finite or infinite sequence of trajectories and transitions which have matching end-points.

Definition 3 An *execution* of a hybrid input transition system $\mathcal{H}$ is a sequence $\sigma : M \to \Sigma \cup \Delta$, where $M \in SeqDom$, such that for each $0 \leq i < |\sigma| - 1$, $Last(\sigma(i)) = First(\sigma(i+1))$. Let $Exec(\mathcal{H})$ denote the set of all executions of $\mathcal{H}$.

We can view an execution as a pair consisting of an input signal and state signal. Let $\sigma \in Exec(\mathcal{H})$. Then we also use (σ^u, σ^s) to denote the execution σ, where the domains of σ^u and σ^s are the same as the domain of σ, and if $\sigma(i) = (\mathbf{u}_i, \mathbf{s}_i)$ is a trajectory, then $\sigma^u(i) = \mathbf{u}_i$ and $\sigma^s(i) = \mathbf{s}_i$, and if $\sigma(i) = (u, (s_1, s_2))$ is a transition, then $\sigma^u(i) = u$ and $\sigma^s(i) = (s_1, s_2)$.

Given a set of executions $\mathcal{T}$ and an input signal σ^u, we use $\mathcal{T}|_{\sigma^u}$ to denote the set of all state signals of executions in $\mathcal{T}$ which result from application of the input signal σ^u. Formally, $\mathcal{T}|_{\sigma^u} = \{\sigma^s \mid (\sigma^u, \sigma^s) \in \mathcal{T}\}$.

First, last, states, inputs of executions We extend *first* and *last* to executions and state signals in the natural way, that is, the *first* of the first element in the sequence and the *last* of the last element if the sequence is finite. Formally, for an execution or a state signal σ, $First(\sigma) = First(\sigma(0))$ and $Last(\sigma)$ is defined only if $dom(\sigma) = [n]$ for some $n \in \mathbb{N}$ and is equal to $Last(\sigma(n))$. Similarly, $States(\sigma) = \bigcup_{i \in dom(\sigma)} States(\sigma(i))$. Also, for an input signal σ^u, $Inputs(\sigma^u) = \bigcup_{i \in dom(\sigma^u)} Inputs(\sigma^u(i))$. The functions are extended to sets of trajectories, state signals and executions in a natural manner. Let $States(\mathcal{H})$ denote $States(\Sigma) \cup States(\Delta)$ and $Inputs(\mathcal{H})$ denote $Inputs(\Sigma) \cup Inputs(\Delta)$.

Graph of an execution In order to define distance between executions, we interpret the input and state signals as sets called the graphs which have information about the linear ordering between the states and inputs at various times. The set corresponding to a state signal σ^s consists of triples (t, i, s) such that s is a state that is reached after time t has elapsed along the execution, and i is the number of discrete transitions that have taken place before time t. Similarly, the set corresponding to an input signal σ^u consists of triples (t, i, u) such that the input u was applied at time t, and the number of impulse inputs applied before time t is i.

Definition 4 For an input or state signal σ^a and $j \in dom(\sigma^a)$, let $T_j = \sum_{k=0}^{j-1} Size(\sigma^a\ (k))$ and $K_j = |\{k \mid k < j,\ \sigma^a(k) \text{ is not a trajectory}\}|$. The *graph* of the signal σ^a, denoted $gr(\sigma^a)$, is the set of all triples (t, i, x) such that there exists $j \in dom(\sigma^a)$ satisfying the following:

- $t \in [T_j, T_j + Size(\sigma^a(j))]]$.
- If $\sigma^a(j)$ is a trajectory, then $i = K_j$ and $x = \sigma^a(j)(t - T_j)$.
- If $\sigma^a(j)$ is not a trajectory, then
  - if σ^a is a state signal and $\sigma^a(j) = (s_1, s_2)$, then either $i = K_j$ and $x = s_1$, or $i = K_j + 1$ and $x = s_2$.
  - if σ^a is an input signal and $\sigma^a(i) = u$, then $i = K_j$ and $x = u$.

3.4 Metric hybrid input transition system

In order to reason about stability of a system, one needs a notion of distance between behaviors of the system. Hence, we extend the definition of the hybrid system with a metric on the states and inputs which can then be extended to distance between signals and executions.

A metric hybrid input transition system is a hybrid input transition system whose state and input spaces are equipped with a metric. A *metric hybrid input transition system* (*MHS*) is a pair $(\mathcal{H}, d^s, d^u)$ where $\mathcal{H} = (S, U, \Sigma, \Delta)$ is a hybrid input transition system, and (S, d^s) and (U, d^u) are extended metric spaces. The metric d^s on the state space can be lifted to state signals and d^u to input signals, which will then be used to define input-to-state stability notions. Before defining this extension, recall that given an extended metric space (M, d), the *Hausdorff distance* between $A, B \subseteq M$, also denoted $d(A, B)$, is given by the maximum of

$$\{\sup_{p\in A} \inf_{q\in B} d(p, q), \sup_{p\in B} \inf_{q\in A} d(p, q)\}.$$

We extend d to triples used in the definition of graphs.

Definition 5 For $(t_1, i_1, x_1), (t_2, i_2, x_2) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times M$, let

$$d((t_1, i_1, x_1), (t_2, i_2, x_2)) = \max\{|t_1 - t_2|, |i_1 - i_2|, d(x_1, x_2)\}.$$

Hence, given two sets $A, B \subseteq \mathbb{R}_{\geq 0} \times \mathbb{N} \times M$, $d(A, B)$, denotes the Hausdorff distance between the sets A and B, where $d(a, b)$ for any triples $a \in A$ and $b \in B$ is interpreted as in Definition 5. Now we can define the distance between state signals and input signals.

Definition 6 Let $(\mathcal{H}, d^s, d^u)$ be a metric hybrid input transition system with $\mathcal{H} = (S, U, \Sigma, \Delta)$. The *distance between state signals* ${\sigma^s}_1, {\sigma^s}_2$, denoted as $d^s({\sigma^s}_1, {\sigma^s}_2)$, is defined to be the Hausdorff distance between their graphs, that is, $d^s(gr({\sigma^s}_1), gr({\sigma^s}_2))$, and the *distance between input signals* ${\sigma^u}_1, {\sigma^u}_2$, denoted $d^u({\sigma^u}_1, {\sigma^u}_2)$, is defined as $d^u(gr({\sigma^u}_1), gr({\sigma^u}_2))$.

Distance between executions as defined above, called *graphical distance*, captures the notion that two executions are close if their states are close at approximately same times. The notion of graphical distance is similar to the notion of ϵ-closeness from Goebel et al. (2009), where it has been argued that allowing a wiggle time is necessary when one considers hybrid executions. Graphical distance between two executions is illustrated in Fig. 1. Note that the two executions σ and σ' are not close at all times t, for example, at a time $t \in (t_1, t_2)$, the states are very far. However, for every time t and corresponding state s of σ, there exists a time $t' \in [t - \epsilon, t + \epsilon]$ such that s is close to the state of σ' at time t'. For example, s_2 is close to s_2' and times t_1 and t_2 are close.

In order to define convergence, we need the distance between suffixes of signals starting from some time T. Given a subset G of $\mathbb{R}_{\geq 0} \times \mathbb{N} \times A$ and a $T \in \mathbb{R}_{\geq 0}$, let us denote by $G|_T$ the set $\{(t, i, x) \in G \,|\, t \geq T\}$. Given two signals σ_1, σ_2 and a $T \in \mathbb{R}_{\geq 0}$, we define $d(\sigma_1|_T, \sigma_2|_T)$ to be $d(gr(\sigma_1)|_T, gr(\sigma_2)|_T)$.

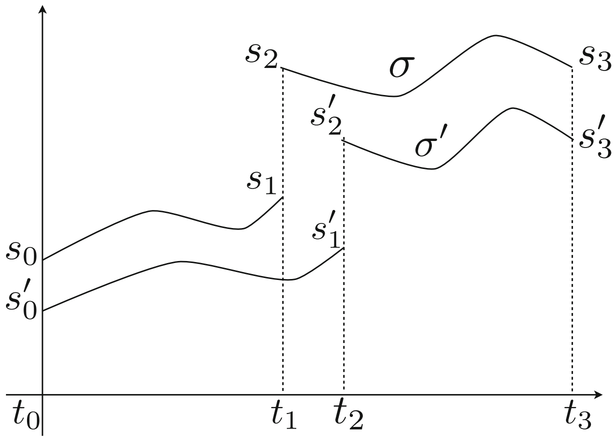

Fig. 1 Graphical distance between executions

4 Continuous dynamical systems and switched systems with inputs

In this section, we model the classical representations for continuous systems and switched system in terms of differential equations as the hybrid input transition systems they represent.

4.1 Continuous dynamical systems with input

Consider a continuous dynamical system with input:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in X = \mathbb{R}^n, x(0) \in X_0 \subseteq X, u(t) \in U \subseteq \mathbb{R}^m, \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and u, X_0 is the set of initial conditions, and U is the set of inputs. Let $\bar{0}$ be an equilibrium point, that is, $\bar{0} \in X_0$ and $\bar{0} \in U$ and $f(\bar{0}, \bar{0}) = \bar{0}$. We will assume that the input signal space $\mathcal{U}$ consists of piecewise continuous trajectories over U. The above conditions guarantee that for every $x_0 \in X_0$ and $\mathbf{u} \in \mathcal{U}$, there exists a unique trajectory $\mathbf{x}$ over $\mathbb{R}^n$ with $dom(\mathbf{x}) = dom(\mathbf{u})$ and $\mathbf{x}(0) = x_0$ which satisfies the Eq. 1, that is, for all $t \in dom(\mathbf{u})$, $\frac{d}{dt}\mathbf{x}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$. We will denote the unique solution corresponding to an x_0 and a $\mathbf{u}$ as $\Phi(x_0, \mathbf{u})$.

We define the hybrid system corresponding to the System (1) to be the following: $\mathcal{H}_{f,X,X_0,U} = (S, U, \Sigma, \Delta)$, where:

- $S = \mathbb{R}^n$,
- $\Sigma = \emptyset$, and
- Δ is the set of pairs $(\mathbf{u}, \mathbf{x})$, where $\mathbf{u}$ is in $\mathcal{U}$, $\mathbf{x}(0) \in X_0$ and $\mathbf{x} = \Phi(\mathbf{x}(0), \mathbf{u})$.

Let d^s and d^u be the standard Euclidean norms on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The set of reference trajectories $\mathcal{T}_{X_0,U}$ is the singleton $\{(\mathbf{u}, \mathbf{x}) | \mathbf{u}(t) = \bar{0} \text{ and } \mathbf{x}(t) = \bar{0} \text{ for all } t\}$.

4.2 Switched systems with inputs and state jumps

A switched system with state jumps consists of a set of dynamical systems, a switching signal which specifies the times at which the system switches its dynamics between the specified set of dynamical systems and jump functions with each switching of the dynamics.

A *switching signal* $\alpha = (\{t_i\}_{i \in \mathbb{N}}, \{\omega_i\}_{i \in \mathbb{N}})$ is a monotonically increasing divergent sequence, that is, it satisfies $t_0 = 0$, $t_i < t_j$ for $j > i$ and $\lim_{i \to \infty} t_i = \infty$.

A *switched system with state jumps* (also referred as impulsive switched systems (Hespanha et al. 2008; Liu et al. 2012)) is given by:

$$\begin{aligned} \dot{x} &= f_p(x, u), \;\; p \in [N], \;\; x \in X = \mathbb{R}^n, x_0 \in X_0 \subseteq X, u \in U \subseteq \mathbb{R}^m, \\ x^+ &= g_p(x, u), \;\; g_p : X \to X, \;\; p \in [N], \\ \alpha &= (\{t_i\}_{i \in \mathbb{N}}, \{\omega_i\}_{i \in \mathbb{N}}), t_i \in \mathbb{R}_{\geq 0}, \omega_i \in [N] \text{ is a switching signal.} \end{aligned} \tag{2}$$

We assume that each $f_p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and u, X_0 is the set of initial conditions, U is the set of inputs, and the input signal space $\mathcal{U}$ consists of piecewise continuous trajectories over U. The above conditions guarantee that given a $\mathbf{u} \in \mathcal{U}$ and an $x_0 \in \mathbb{R}^n$, there exists a unique solution for each system $\dot{x} = f_p(x, u)$ starting from x_0; we denote the unique solution by $\Phi_p(x_0, \mathbf{u})$. Then, there also exists a unique solution of System 2 starting from $x_0 \in X_0$ and $\mathbf{u} : [0, \infty) \to U$ in $\mathcal{U}$, denoted $\Phi(x_0, \mathbf{u})$, which is a sequence of trajectories $\Phi_{\omega_0}(x_0, \mathbf{u}[t_0, t_1])\Phi_{\omega_1}(x_1, \mathbf{u}[t_1, t_2])\Phi_{\omega_2}(x_2, \mathbf{u}[t_2, t_3]) \cdots$, where $x_i = g_{\omega_{i-1}}(\Phi_{\omega_{i-1}}(x_{i-1}, \mathbf{u}[t_{i-1}, t_i])(t_i - t_{i-1}), \mathbf{u}(t_i))$, for $i > 0$.

We associate a hybrid input transition system $\mathcal{H}_{f_1, \cdots, f_N, g_1, \cdots, g_N, X, X_0, U, \alpha}$ with the switched system in (2) given by (S, U, Σ, Δ), where:

- $S = \mathbb{R}_{\geq 0} \times \mathbb{R}^n$,
- $\Sigma = \{(\emptyset, (s, t), (s', t')) \mid \exists i \in \mathbb{N}, t = t' = t_i, s' = g_{\omega_{i-1}}(s)\}$, and
- Δ consists of input-state trajectories $(\mathbf{u}, \mathbf{s})$ such that there exists an $i \in \mathbb{N}$ and an $x_i \in \mathbb{R}^n$ with $dom(\mathbf{s}) = [0, t_{i+1} - t_i]$ and $\mathbf{s}(t) = (t_i + t, \Phi_{\omega_i}(x_i, \mathbf{u}))$.

In an execution, the first component time is used to identify the mode of the system, that is, the dynamics associated with the trajectory. Hence, we define the metric d_t over a space $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$ as, $d_t((t_1, x_1), (t_2, x_2))$ is Euclidean distance between x_1 and x_2 if $t_1 = t_2$, and ∞ otherwise. The metric space associated with S will be d_t. Let $\mathcal{T}_{X_0, U, \alpha}$ be the singleton set with pair (σ_u, σ_x) such that $dom(\sigma_u) = dom(\sigma_x) = \mathbb{N}$, for each $i \in \mathbb{N}$, $dom(\sigma_u(i)) = dom(\sigma_x(i)) = [0, t_{i+1} - t_i]$, and for each $t \in dom(\sigma_u(i))$, $\sigma_u(i)(t) = \bar{0}$ and $\sigma_x(i)(t) = (t, \bar{0})$.

5 Input-to-state stability of hybrid input transition systems

In this section, we present a definition of input-to-state stability for hybrid systems with respect to a set of reference executions. Our definition is motivated by the definition of incremental input-to-state stability for purely continuous dynamical systems presented in Angeli (2002). The definition refers to properties which state conditions on the distance between trajectories as opposed to definitions of Lyapunov and asymptotic stability which refer to the distance of a trajectory to an equilibrium point. Hence, it is an appropriate definition to generalize to the hybrid setting. However, the definition is provided in terms of comparison functions. It is not straightforward to accomodate different notions of distances between executions, and also to generalize the proofs of the stability preservation theorems for Lyapunov and asymptotic stability. Hence, we first provide an alternate characterization of incremental input-to-state stability in terms of an ϵ-δ definition, and use this to provide a definition of input-to-state stability for hybrid systems with respect to a set of reference executions.

5.1 An alternate characterization of incremental input-to-state stability of continuous dynamical systems

We present the definition of incremental input-to-state stability of Angeli (2002) and prove a super-position theorem which presents a characterization of the same in terms of an ϵ-δ definition.

We model the solutions of a continuous dynamical systems as a set of trajectories. Let $\mathcal{T}$ be a set of input-state trajectories over $(\mathbb{R}^m, \mathbb{R}^n)$ such that for each $\zeta \in \mathbb{R}^n$ and input trajectory $\mathbf{u} : [0, \infty) \to \mathbb{R}^m$, there exists a unique element $(\mathbf{u}, \mathbf{s}) \in \mathcal{T}$, where $\mathbf{s} : [0, \infty) \to \mathbb{R}^n$ and $First(\mathbf{s}) = \zeta$. Given ζ and $\mathbf{u}$, let us denote the unique trajectory $\mathbf{s}$ by $\Phi(\zeta, \mathbf{u})$. The definition of incremental input-to-state stability (δISS) from Angeli (2002) is as follows:

Definition 7 (δISS for input-state trajectories) The set of input-state trajectories $\mathcal{T}$ is said to be *incrementally input-to-state stable* if there exists a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ such that for any $t \geq 0$, any ζ_1, ζ_2 and any pair of input trajectories $\mathbf{u}_1, \mathbf{u}_2$, the following is true:

$$|\Phi(\zeta_1, \mathbf{u}_1)(t) - \Phi(\zeta_2, \mathbf{u}_2)(t)| \leq \beta(|\zeta_1 - \zeta_2|, t) + \gamma(\|\mathbf{u}_1 - \mathbf{u}_2\|_\infty). \tag{3}$$

Note that when there is no input, then the above definition reduces to asymptotic stability for a fixed equilibrium point $\zeta_2 = 0$. Next, we present certain properties which are implied by the definition of incremental input-to-state stability.

($C1$) *The system is Lyapunov stable "uniformly" in the input.*
For every $\epsilon > 0$, there exists a $\delta > 0$, such that for every input trajectory $\mathbf{u}$, and for all initial states ζ_1, ζ_2, the following holds for every $t \geq 0$.

$$|\zeta_1 - \zeta_2| < \delta \Rightarrow |\Phi(\zeta_1, \mathbf{u})(t) - \Phi(\zeta_2, \mathbf{u})(t)| < \epsilon.$$

(Note that δ depends only on ϵ, in particular, it is independent of the input trajectory $\mathbf{u}$.)

($C2$) *The system converges "uniformly" in the input.*
There exists $\delta > 0$, for any $\epsilon > 0$ and initial states ζ_1, ζ_2 with $|\zeta_1 - \zeta_2| < \delta$, there exists a $T \geq 0$, such that for every input signal $\mathbf{u}$,

$$|\Phi(\zeta_1, \mathbf{u})(t) - \Phi(\zeta_2, \mathbf{u})(t)| < \epsilon, \forall t > T.$$

(Note that T depends only on ϵ and is independent of $\mathbf{u}$.)

($C3$) *The system is input-to-state stable "uniformly" in the initial state.*
For any $\epsilon > 0$, there exists a $\delta > 0$ such that for all input signals $\mathbf{u}_1, \mathbf{u}_2$, initial state ζ and time $t \geq 0$, the following holds:

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_\infty < \delta \Rightarrow |\Phi(\zeta, \mathbf{u}_1)(t) - \Phi(\zeta, \mathbf{u}_2)(t)| < \epsilon.$$

(Note the independence of δ with respect to ζ.)

Conditions $C1$ and $C2$ state that the system is asymptotically stable in a uniform sense. The last condition $C3$ states the small perturbations in the infinity norm of the input signals leads to only small variations in the infinity norm of the state signals resulting from the input signals for a fixed initial state. Next, we show that the conditions $(C1) - (C3)$ in fact characterize incremental input-to-state stability as given in Definition 7. This is summarized in the following theorem:

Theorem 1 (Super-position theorem) *A set of input-state trajectories $\mathcal{T}$ is δISS iff it satisfies Conditions* $(C1) - (C3)$.

5.2 Input-to-state stability of hybrid input transition systems with respect to a set of reference executions

In this section, we define a notion of input-to-state stability of hybrid input transition systems with respect to a set of executions (*hISS*) by generalizing the characterization of incremental input-to-state stability for continuous dynamical systems given by Theorem 1. However, we specialize the definition to refer to a set of reference executions. One central notion we need is to define distances between executions, for which we borrow the notion of graphical distance introduced in Goebel et al. (2009) for hybrid trajectories. However, the results in the paper are not sensitive to the particular definition of distance.

Definition 8 (*hISS*: Hybrid Input-to-State Stability) Given a hybrid input transition system $\mathcal{H}$ and a set of executions $\mathcal{T} \subseteq Exec(\mathcal{H})$, we say that $\mathcal{H}$ is *input-to-state stable* (*hISS*) with respect to the set of executions $\mathcal{T}$, if the following hold:

$(D1)$ for every $\epsilon > 0$, there exists a $\delta > 0$, such that the following holds for every input signal σ^u and for every $\eta_1 \in First(Exec(\mathcal{H})|_{\sigma^u})$ and $\eta_2 \in First(\mathcal{T}|_{\sigma^u})$:

$$d^s(\eta_1, \eta_2) < \delta \Rightarrow [\forall(\sigma^u, \sigma^s) \in Exec(\mathcal{H}), First(\sigma^s) = \eta_1 \\ \Rightarrow \exists(\sigma^u, \hat{\sigma}^s) \in \mathcal{T}, First(\hat{\sigma}^s) = \eta_2, d^s(\sigma^s, \hat{\sigma}^s) < \epsilon.]$$

$(D2)$ for every $\delta > 0$, for every $\epsilon > 0$, there exists $T \geq 0$ such that for every $\eta_1 \in First(Exec(\mathcal{H})|_{\sigma^u})$ and $\eta_2 \in First(\mathcal{T}|_{\sigma^u})$:

$$d^s(\eta_1, \eta_2) < \delta \Rightarrow [\forall(\sigma^u, \sigma^s) \in Exec(\mathcal{H}), First(\sigma^s) = \eta_1 \\ \Rightarrow \exists(\sigma^u, \hat{\sigma}^s) \in \mathcal{T}, First(\hat{\sigma}^s) = \eta_2, d^s(\sigma^s|_T, \hat{\sigma}^s|_T) < \epsilon.]$$

$(D3)$ for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every input signals $\sigma^u, \hat{\sigma}^u$ and state $\eta \in First(Exec(\mathcal{T})|_{\hat{\sigma}^u})$, the following holds:

$$d^u(\sigma^u, \hat{\sigma}^u) < \delta \Rightarrow [\forall(\sigma^u, \sigma^s) \in Exec(\mathcal{H}), First(\sigma^s) = \eta \\ \Rightarrow \exists(\hat{\sigma}^u, \hat{\sigma}^s) \in \mathcal{T}, First(\hat{\sigma}^s) = \eta, d^s(\sigma^s, \hat{\sigma}^s) < \epsilon.]$$

In the above definition, we choose to use "there exists a trajectory in $\mathcal{T}$" to capture the scenario, for example, where we consider ISS of a system with respect to multiple equilibrium points, that is, $\mathcal{T}$ consists of a finite set of trajectories corresponding to the equilibrium points. In this case we want every trajectory that starts close to $\mathcal{T}$ (that is, to one of the equilibrium points) to converge only to one of the equilibrium points (hence the use of "there exists") and not all of them.

Remark 2 If we consider a purely continuous dynamical systems whose (infinite time) solutions are the trajectories of $\mathcal{H}$ and take the references trajectories $\mathcal{T}$ to also be $Exec(\mathcal{H})$, then the definition of *hISS* coincides with that of incremental input-to-state stability. More precisely, Eq. 1 is incrementally input-to-state stable according to Eq. 3 if and only if $\mathcal{H}_{f,X,X_0,U}$ is *hISS* with respect to $Exec(\mathcal{H}_{f,X,X_0,U})$.

Remark 3 If Eq. 1 is input-to-state stable according to the definition in (Sontag and Wang 1995), then $\mathcal{H}_{f,X,X_0,U}$ is *hISS* with respect to $\mathcal{T}_{X_0,U}$. Our hybrid input-to-state stability notion on $\mathcal{H}_{f_1,\cdots,f_N,g_1,\cdots,g_N,X,X_0,U,\alpha}$ coincides with the input-to-state stability notion with respect to 0 initial state and 0 constant input (as defined in Vu et al. 2007; Liu et al. 2012) for System (2).

Remark 4 There have been several proposals for defining metrics on the set of executions, including the Skorokhod metric (see Davoren 2009; Caspi and Benveniste 2002 for more details), wherein, two executions are considered close if there exists a bijective, strictly order-preserving function between the time domains of the executions, such that the distance between a time point and its image under the function is small and the values of the corresponding states are small. However, the results in the paper are not sensitive to the particular choice of the distance metric.

6 Pre-orders

Pre-orders on a class of systems which preserve properties of interest are the basis for constructing abstraction refinement frameworks. A pre-order $\preceq$ is a binary relation which is reflexive and transitive. A pre-order $\preceq$ on a class of systems preserves a property if given any two element $\mathcal{H}_1$ and $\mathcal{H}_2$ which are related, $\mathcal{H}_1 \preceq \mathcal{H}_2$, the satisfaction of the property by $\mathcal{H}_2$ implies the satisfaction of the property by $\mathcal{H}_1$. Pre-orders preserving a property essentially specify the relation between a concrete system and its abstraction with respect to the property, and the ordering specifies qualitatively better abstractions. An abstraction refinement framework can be obtained by defining an abstraction function which takes as input a concrete system and constructs an abstract system which is related to the concrete system according to the pre-order.

We first define a classical pre-order from concurrency theory, namely, simulations. Simulations preserve several properties studied in the context of verification such as those expressible in the safe fragments of modal logics. We will show later that these pre-orders do not preserve the stability properties of interest in the paper. Hence, we strengthen the definition of simulation to force the preservation of stability properties.

6.1 Input (Bi)-simulations

The notion of input simulation is an extension of the classical notion of simulation with inputs for hybrid input transition systems. Our definition is closely related to the definition of simulation defined in Kaynar et al. (2003).

Definition 9 Given two hybrid input transition systems $\mathcal{H}_1 = (S_1, U_1, \Sigma_1, \Delta_1)$ and $\mathcal{H}_2 = (S_2, U_2, \Sigma_2, \Delta_2)$, a pair of binary relations (R_1, R_2), where $R_1 \subseteq S_1 \times S_2$ and $R_2 \subseteq U_1 \times U_2$, is called an *input simulation relation* from $\mathcal{H}_1$ to $\mathcal{H}_2$, denoted $\mathcal{H}_1 \preceq_{(R_1,R_2)} \mathcal{H}_2$, if for every $(s_1, s_2) \in R_1$, the following hold:

– For every state s_1' and input u_1 such that $(u_1, (s_1, s_1')) \in \Sigma_1$, there exist a state s_2' and an input u_2 such that $R_1(s_1', s_2')$, $R_2(u_1, u_2)$ and $(u_2, (s_2, s_2')) \in \Sigma_2$.

- For every input-state trajectory $(\mathbf{u}_1, \mathbf{s}_1) \in \Delta_1$ such that $First(\mathbf{s}_1) = s_1$, there exists an input-state trajectory $(\mathbf{u}_2, \mathbf{s}_2) \in \Delta_2$ such that $First(\mathbf{s}_2) = s_2$, $\mathbf{s}_2 \in R_1(\mathbf{s}_1)$ and $\mathbf{u}_2 \in R_2(\mathbf{u}_1)$.

Simulation guarantees that for every execution in the first system, there exists an execution in the second which is related by R pointwise in time. And, the behaviors of $\mathcal{H}_2$ over-approximate the behaviors of $\mathcal{H}_1$.

Further, (R_1, R_2) is an *input bisimulation relation* between $\mathcal{H}_1$ and $\mathcal{H}_2$ if both (R_1, R_2) and (R_1^{-1}, R_2^{-1}) are input simulation relations, that is, $\mathcal{H}_1 \preceq_{(R_1,R_2)} \mathcal{H}_2$ and $\mathcal{H}_2 \preceq_{(R_1^{-1},R_2^{-1})} \mathcal{H}_1$. Bisimulations define an equivalence relation on the class of systems; hence, if two systems are bisimilar, then they satisfy the same bisimulation invariant properties.

We will refer to abstract systems which are obtained by applying "input simulation relations" on the concrete system. Hence we define the image of a hybrid system under a pair of binary relations (R_1, R_2).

Definition 10 Given a hybrid input transition system $\mathcal{H} = (S, U, \Sigma, \Delta)$, and a pair (R_1, R_2), where $R_1 \subseteq S \times S'$ and $R_2 \subseteq U \times U'$, for some S' and U', we define the *image of* $\mathcal{H}$ *under* (R_1, R_2), denoted $(R_1, R_2)(\mathcal{H})$, to be the hybrid input transition system $(S', U', \Sigma', \Delta')$, where:

- $\Sigma' = \{(u', (s_1', s_2')) \mid \exists (u, (s_1, s_2)) \in \Sigma, u' \in R_2(u), s_1' \in R_1(s_1), s_2' \in R_1(s_2)\}$.
- $\Delta' = \{(\mathbf{u}', \mathbf{s}') \mid \exists (\mathbf{u}, \mathbf{s}) \in \Delta, \mathbf{u}' \in R_2(\mathbf{u}), \mathbf{s}' \in R_1(\mathbf{s})\}$.

6.2 Uniformly continuous input (Bi)-simulation

The notion of input bisimulation does not preserve hybrid input-to-state stability. Hence, we strengthen the binary relations on the state-spaces and input-spaces with uniform continuity condition.

Let $(\mathcal{H}_1, {d^s}_1, {d^u}_1)$ and $(\mathcal{H}_2, {d^s}_2, {d^u}_2)$ be two metric input hybrid transition systems.

Definition 11 A pair (R_1, R_2) is a *uniformly continuous input simulation* from $\mathcal{H}_1$ to $\mathcal{H}_2$, denoted $\mathcal{H}_1 \preceq^C_{(R_1,R_2)} \mathcal{H}_2$, if (R_1, R_2) is an input simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and R_1, R_1^{-1} and R_2 are uniformly continuous.

$\mathcal{H}_1 \preceq^C \mathcal{H}_2$ is used to denote the fact that there exists a pair (R_1, R_2) such that $\mathcal{H}_1 \preceq^C_{(R_1,R_2)} \mathcal{H}_2$.

Definition 12 A pair (R_1, R_2) is a *uniformly continuous input bisimulation* from $\mathcal{H}_1$ to $\mathcal{H}_2$ if (R_1, R_2) is a uniformly continuous input simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and (R_1^{-1}, R_2^{-1}) is a uniformly continuous input simulation from $\mathcal{H}_2$ to $\mathcal{H}_1$.

Uniformly continuous input simulations define a pre-order on hybrid input transitions systems.

Theorem 2 (Pre-order) *Let* $(\mathcal{H}_i, {d^s}_i, {d^u}_i)$*, for* $1 \leq i \leq 3$*, where* $\mathcal{H}_i = (S_i, U_i, \Sigma_i, \Delta_i)$*, be three metric hybrid transition systems. Then we have the following properties about* $\preceq^C$*:*

- *(Reflexivity)* $\mathcal{H}_1 \preceq^C \mathcal{H}_1$.
- *(Transitivity) If* $\mathcal{H}_1 \preceq^C \mathcal{H}_2$ *and* $\mathcal{H}_2 \preceq^C \mathcal{H}_3$*, then* $\mathcal{H}_1 \preceq^C \mathcal{H}_3$,

Proof (Reflexivity) Given $\mathcal{H}_1$, consider the pair (Id_1, Id_2), where $Id_1 = \{(s, s) \mid s \in S_1\}$ and $Id_2 = \{(u, u) \mid u \in U_1\}$. Note that (Id_1, Id_2) is an input simulation from $\mathcal{H}_1$ and $\mathcal{H}_1$. Further, Id_1, Id_2 and their inverses are all identity functions, and hence, uniformly continous.

(Transitivity) Suppose $\mathcal{H}_1 \preceq^C_{(R_1,R_2)} \mathcal{H}_2$ and $\mathcal{H}_2 \preceq^C_{(R'_1,R'_2)} \mathcal{H}_3$. It is well known that composition of input simulations is an input simulation, therefore, $(R_1 \circ R'_1, R_2 \circ R'_2)$ is an input simulation from $\mathcal{H}_1$ to $\mathcal{H}_3$. Further, since R_1, R_2, R'_1 and R'_2 and their inverses are all uniformly continuous, from Proposition 1, $R_1 \circ R'_1$, $R_2 \circ R'_2$ and their inverses are uniformly continuous as well. Therefore, $\mathcal{H}_1 \preceq^C_{(R_1 \circ R'_1, R_2 \circ R'_2)} \mathcal{H}_3$. □

6.3 Consistency with respect to the reference executions

In addition to a uniformly continuous simulation between two systems, we require certain additional conditions between the input simulation relations and the reference executions of the systems, for property preservation.

Definition 13 A pair of relations (R_1, R_2), where $R_1 \subseteq S_1 \times S_2$ and $R_2 \subseteq U_1 \times U_2$, is said to be *semi-consistent* with respect to the sets of executions $\mathcal{T}_1$ and $\mathcal{T}_2$ over (S_1, U_1) and (S_2, U_2), respectively, if the following hold:

$(A1)$ For every $\sigma^u{}_1$ and ζ_1 such that $\zeta_1 \in First(\mathcal{T}_1|_{\sigma^u{}_1})$, there exist $\sigma^u{}_2$ and ζ_2 such that $\zeta_2 \in First(\mathcal{T}_2|_{\sigma^u{}_2})$, $R_2(\sigma^u{}_1, \sigma^u{}_2)$ and $R_1(First(\sigma^s{}_1), First(\sigma^s{}_2))$.

$(A2)$ For every $(\sigma^u{}_2, \sigma^s{}_2) \in \mathcal{T}_2$, for every $\sigma^u{}_1 \in R_2^{-1}(\sigma^u{}_2)$ and $\zeta_1 \in R_2^{-1}(First(\sigma^s{}_2))$ with $\zeta_1 \in First(\mathcal{T}_1|_{\sigma^u{}_1})$ there exists $\sigma^s{}_1$ with $First(\sigma^s{}_1) = \zeta_1$, $R_1(\sigma^s{}_1, \sigma^s{}_2)$ and $(\sigma^u{}_1, \sigma^s{}_1) \in \mathcal{T}_1$.

$(A3)$ $R_2(u)$ is a singleton for every $u \in Inputs(\mathcal{T}_1)$.

$(A4)$ $R_1^{-1}(s)$ is singleton for every $s \in States(\mathcal{T}_2)$.

Remark 5 Later we show that Lyapunov function based analysis can be interpreted as a uniformly continuous simulation based analysis where the simulation relation is given by the Lyapunov function. The condition that $V(0) = 0$ and $V(x) \neq 0$ for any $x \neq 0$ can be interpreted as the equilibrium point of a given system being uniquely mapped to the equilibrium point of the abstract one dimensional system obtained by applying V. The semi-consistency conditions similarly provide constraints between the set of reference trajectories of the two systems and generalize the condition required by the Lyapunov function.

(R_1, R_2) is said to be *consistent* with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ if both (R_1, R_2) and (R_1^{-1}, R_2^{-2}) are semi-consistent with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$.

7 Incremental Input-to-State stability preservation

In this section, we present the main result of the paper, namely, that uniformly continuous input simulations define a hybrid input-to-state stability preserving pre-orders on the class of hybrid input transition systems. The detailed proofs are provided in the Appendix.

Theorem 3 *Let* $(\mathcal{H}_1, d^s{}_1, d^u{}_1)$ *and* $(\mathcal{H}_2, d^s{}_2, d^u{}_2)$, *where* $\mathcal{H}_1 = (S_1, U_1, \Sigma_1, \Delta_1)$ *and* $\mathcal{H}_2 = (S_2, U_2, \Sigma_2, \Delta_2)$, *be two metric hybrid input transition systems, and let* $\mathcal{T}_1 \subseteq Exec(\mathcal{H}_1)$ *and* $\mathcal{T}_2 \subseteq Exec(\mathcal{H}_2)$ *be two sets of executions. Let* (R_1, R_2) *be a uniformly* *continuous input simulation from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$*, and let* (R_1, R_2) *be semi-consistent with respect to* $\mathcal{T}_1$ *and* $\mathcal{T}_2$*. Then the following holds:*

$$\mathcal{H}_2 \text{ is } hISS \text{ with respect to } \mathcal{T}_2 \Rightarrow \mathcal{H}_1 \text{ is } hISS \text{ with respect to } \mathcal{T}_1.$$

The next theorem states that hybrid input-to-state stability is a uniformly continuous input bisimulation invariant property.

Theorem 4 *Let* $(\mathcal{H}_1, {d^s}_1, {d^u}_2)$ *and* $(\mathcal{H}_2, {d^s}_2, {d^u}_2)$*, where* $\mathcal{H}_1 = (S_1, U_1, \Sigma_1, \Delta_1)$ *and* $\mathcal{H}_2 = (S_2, U_2, \Sigma_2, \Delta_2)$*, be two metric hybrid input transition systems, and* $\mathcal{T}_1 \subseteq Exec(\mathcal{H}_1)$ *and* $\mathcal{T}_2 \subseteq Exec(\mathcal{H}_2)$ *be two sets of executions. Let* (R_1, R_2) *be a uniformly continuous input bisimulation from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$*, and let* (R_1, R_2) *be consistent with respect to* $\mathcal{T}_1$ *and* $\mathcal{T}_2$*. Then the following holds:*

$$\mathcal{H}_2 \text{ is } hISS \text{ with respect to } \mathcal{T}_2 \Leftrightarrow \mathcal{H}_1 \text{ is } hISS \text{ with respect to } \mathcal{T}_1.$$

Theorem 3 states that uniformly continuous input simulations serve as a notion of abstraction with respect to *hISS* and Theorem 4 states that uniformly continuous input bisimulations are a notion of equivalence between systems with respect to hybrid input-to-state stability.

7.1 Illustration of uniformly continuous input simulations for input-to-state stability analysis of linear systems

We illustrate through a simple example of a linear system, how the input-to-state stability can be analysed by reducing it to a simpler system which uniformly continuously simulates the original system, and for which input-to-state stability can be easily determined.

Consider a linear system with inputs:

$$\dot{x} = f(x, u) = Ax + Bu,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \qquad (4)$$
$$x \in X = \mathbb{R}^n,\ u \in U \subseteq \mathbb{R}^m,\ x_0 \in X_0 \subseteq X,$$

where X_0 and U are compact sets. It is a well-known fact from linear systems theory that System 4 is an input to state stable, when A is a Hurwitz matrix, that is, all the eigen values have negative real parts. We will reprove this theorem by constructing a simple one-dimensional system which is input-to-state stable and uniformly continuously simulates System 4.

When A is Hurwitz, we can find a positive definite symmetric matrix P satisfying $A^T P + PA = -I$, where I is the identity matrix. Consider a function $R_1 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by $R_1(x) = x^T Px$ and a function $R_2 : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ given by $R_2(u) = |u|$. Then,

$$\begin{aligned}
\dot{R}_1(x) &= \dot{x}^T Px + x^T P\dot{x} \\
&= x^T(A^T P + PA)x + u^T B^T Px + x^T PBu \\
&\leq -\|x\|^2 + \frac{1}{2}\|x\|^2 + 2\|B\|^2\|P\|^2\|u\|^2 \\
&\leq -\frac{1}{2\lambda_{\max}(P)} x^T Px + 2\|B\|^2\|P\|^2 u_m \|u\|_\infty \\
&\leq -\lambda R_1(x) + \mu\|u\|_\infty,
\end{aligned}$$

where $\lambda = \frac{1}{2\lambda_{\max}(P)}$, $\lambda_{\max}(P)$ is the maximum eigenvalue of P, and $\mu = 2\|B\|^2\|P\|^2 u_m$, and $u_m = \max_{u\in U}\|u\|$. We have used an elementary inequality $ab \leq \varepsilon a^2 + (1/\varepsilon)b^2$ with $\varepsilon = 1/2$.

Consider the one-dimensional system:

$$\dot{y} \leq -\lambda y + \mu\|v\|_\infty,\ y \geq 0. \tag{5}$$

Note that the solutions to the system satisfy $y(t) \leq e^{-\lambda t}y(0) + \mu/\lambda\|v\|_\infty$. It is easy to check from the solutions of this system, that it is trivially input-to-state stable.

We will show that (R_1, R_2) is a uniformly continuous input simulation from System (4) to System (5). Input simulation follows from the fact that if $(\mathbf{x}, \mathbf{u})$ satisfies $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$ for all $t \geq 0$, then by construction, $(R_1(\mathbf{x}), R_2(\mathbf{u}))$ satisfies $\dot{R}_1(\mathbf{x}) \leq -\lambda R_1(\mathbf{x}) + \mu\|R_2(\mathbf{u})\|_\infty$. Also, when R_1 and R_2 are interpreted as relations or set valued functions, then R_1, R_1^{-1}, R_2 and R_2^{-1} are continuous. Furthermore, these functions are uniformly continuous over $States(\mathcal{H}_{f,X_0,X,U})$ and $Inputs(\mathcal{H}_{f,X_0,X,U})$, where, since X_0 and U are compact, both $States(\mathcal{H}_{f,X_0,X,U})$ and $Inputs(\mathcal{H}_{f,X_0,X,U})$ are bounded in view of Eq. 5. Note that the set of reference executions in both the systems is $\{(\mathbf{0}, \mathbf{0})\}$, where $\mathbf{0}$ is of appropriate dimension. It is easy to see that semi-consistency is trivially satisfied. Hence, from Theorem 3 System (4) is incremental input-to-state stable.

7.2 Inadequacy of weaker notions of input bisimulations

We show that weaker extensions to the definition of input bisimulation which require only continuity instead of uniform continuity on either the input space or the state space, do not suffice to preserve input-to-state stability.

7.2.1 Necessity of uniform continuity on the input space

We define two hybrid systems $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_1$ is *hISS*, where as $\mathcal{H}_2$ is not. We show that there exists an input bisimulation between the two which is uniformly continuous on the state-space but is only continuous on the input space.

Let $\mathcal{H}_1 = (S_1, U_1, \Sigma_1, \Delta_1)$ and $\mathcal{H}_2 = (S_2, U_2, \Sigma_2, \Delta_2)$. The state spaces $S_1 = S_2 = \mathbb{R}^2$, input spaces $U_1 = U_2 = \mathbb{R}$ and $\Sigma_1 = \Sigma_2 = \emptyset$. Δ_1 consists of trajectories $(\mathbf{u}, \Phi((x_0, y_0), \mathbf{u}))$, where $\mathbf{u}$ is a constant trajectory with $dom(\mathbf{u}) = [0, \infty)$, $x_0 = 0$, $y_0 \in [-1, 1]$, and

$$\Phi((x_0, y_0), \mathbf{u})(t) = (t, e^{-t}y_0 + \mathbf{u}(0)\bar{t}),$$

where $\bar{t} = t$ if $t \leq 1$, and 1 otherwise. The reference executions $\mathcal{T}_1$ and $\mathcal{T}_2$ contain the singleton set $\{(\mathbf{u}, \mathbf{x})\}$, where $\mathbf{u}$ and $\mathbf{x}$ are constant $\bar{0}$ trajectories. Δ_2 is same as Δ_1 except that Φ is defined as:

$$\Phi((x_0, y_0), \mathbf{u})(t) = (t, e^{-t}y_0 + e^{\mathbf{u}(0)}\bar{t}),$$

where $\bar{t} = t$ if $t \leq 1$, and 1 otherwise.

Proposition 2 *$\mathcal{H}_1$ is hISS and $\mathcal{H}_2$ is not hISS.*

Proof Both the systems $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy Conditions D_1 and D_2. Given any $(\sigma_u, \sigma^s) \in Exec(\mathcal{H}_1)$, if $d^s(First(\sigma^s), First(\mathcal{T}|_{\sigma^u})) < \delta$, then σ^s is a trajectory which starts at $(0, y)$ for some $y \in (-\delta, \delta)$, then $d^s(\sigma^s, \hat{\sigma}^s)$, where $\hat{\sigma}^s$ is the constant $\bar{0}$ trajectory, is bounded by the difference between $(t, e^{-t}y + \mathbf{u}(0)\bar{t})$ and $(t, e^{-1}0 + \mathbf{u}(0)\bar{t})$, which is bounded by $e^{-t}\delta$. Note that the bound is independent of $\mathbf{u}$. Therefore, since e^{-t} is bounded for all $t \geq 0$, $\mathcal{H}_1$ satisfies D_1 and since e^{-t} converges to 0 as $t \to \infty$, $\mathcal{H}_1$ satisfies D_2. The argument for $\mathcal{H}_2$ satisfying D_1 and D_2 is similar.

$\mathcal{H}_1$ satisfies D_3, but $\mathcal{H}_2$ does not. Given $\mathbf{u}_1$ and $\mathbf{u}_2$ with $d^u(\mathbf{u}_1, \mathbf{u}_2) < \delta$, the distance in $\mathcal{H}_1$ between $\Phi((x_0, y_0), \mathbf{u}_1)$ and $\Phi((x_0, y_0), \mathbf{u}_2)$ is bounded by $\mathbf{u}_1(0)\bar{t} - \mathbf{u}_2(0)\bar{t} \leq (\mathbf{u}_1(0) - \mathbf{u}_2(0))\bar{t} \leq \delta \times 1 \leq \delta$. Hence, it satisfies D_3. On the other hand, in $\mathcal{H}_2$, $\Phi((x_0, y_0), \mathbf{u}_1)$ and $\Phi((x_0, y_0), \mathbf{u}_2)$ is bounded by $e^{\mathbf{u}_1(0)}\bar{t} - e^{\mathbf{u}_2(0)}\bar{t} \leq e^{\mathbf{u}_1(0)} - e^{\mathbf{u}_2(0)}$. Even if $\mathbf{u}_1(0) - \mathbf{u}_2(0) < \delta$, the difference $e^{\mathbf{u}_1(0)} - e^{\mathbf{u}_2(0)}$ diverges with larger values of $\mathbf{u}_1$ and $\mathbf{u}_2$. Therefore $\mathcal{H}_2$ does not satisfy D_3. □

Proposition 3 *Let* $R_1 = \{((x, y), (x, y))\}$, $R_2 = \{(e^u, u)\}$. *(R_1, R_2) is an input bisimulation from $\mathcal{H}_1$ to $\mathcal{H}_2$ which is consistent with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$, and R_1, R_1^{-1} are uniformly continuous, where as, R_2^{-1} is continous, but not uniformly continuous.*

Proof This is an input bisimulation, since the trajectories starting from the initial state (x_0, y_0) and input signal e^u in $\mathcal{H}_1$ is related by R_1 to the trajectory starting from the initial state (x_0, y_0) and the input signal u in $\mathcal{H}_2$. Further, R_1 and R_1^{-1} are uniformly continuous, where as, R_2^{-1} is only continuous. It is trivial to check that the consistence conditions hold, since 0 is mapped only to 0 by R_1, R_2, R_1^{-1} and R_2^{-1}. □

Propositions 2 and 3 imply that Theorems 3 and 4 will not hold if the uniformity condition on the continuity of the relation between the input-spaces is relaxed.

7.2.2 Necessity of uniform continuity on the state space

Next, we show the necessity of uniform continuity on the relation between the state-spaces. We extend the counter-example in Prabhakar et al. (2012), which shows the necessity of uniform continuity on the state space for asymptotic stability preservation (without inputs).

The first system $\mathcal{H}_3$ is the same as $\mathcal{H}_1$ above, except that the input space is restricted to be $\{0\}$. We define $\mathcal{H}_4$ which is the same as $\mathcal{H}_3$ except that the trajectories do not converge to the reference trajectory. The dynamics of the system $\mathcal{H}_4$ is as follows:

$$\Phi((x_0, y_0), \mathbf{u}) = (t, y_0).$$

Proposition 4 *$\mathcal{H}_3$ is hISS and $\mathcal{H}_4$ is not hISS.*

Proof Since $\mathcal{H}_3$ is the same as $\mathcal{H}_1$ except for a smaller input space, $\mathcal{H}_3$ is *hISS*.

The proof is along the same lines as Proposition 2. $\mathcal{H}_4$ satisfies Condition $D1$, since when the difference $d^s(\sigma^s, \hat{\sigma}^s) < \delta$, the difference between (t, y) and $(t, 0)$ is $d^s(y, 0) \leq \delta$. However, since $d^s(y, 0)$ does not converge to 0 as $t \to \infty$, $\mathcal{H}_4$ does not satisfy Condition $D2$. Given $\mathbf{u}_1$ and $\mathbf{u}_2$ with $d^u(\mathbf{u}_1, \mathbf{u}_2) < \delta$, the distance in $\mathcal{H}_4$ between $\Phi((x_0, y_0), \mathbf{u}_1)$ and $\Phi((x_0, y_0), \mathbf{u}_2)$ is bounded by 0, therefore $\mathcal{H}_4$ trivially satisfies Condition $D3$. □

Proposition 5 *Let* $R_1 = \{((x_1, y_1), (x_2, y_2)) \mid x_1 = x_2, y_1 = e^{-x_1} y_2\}$ *and* $R_2 = \{(u, u)\}$. *(R_1, R_2) is an input bisimulation from $\mathcal{H}_3$ to $\mathcal{H}_4$ which is consistent with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$, and R_2, R_1^{-2} are uniformly continuous, where as, R_1 is continuous but not uniformly continuous.*

Proof This is an input bisimulation, since the trajectories starting from the initial state (x_0, y_0) and input signal u in $\mathcal{H}_3$ is related by R_1 to the trajectory starting from the initial state (x_0, y_0) ($e^{x_0} = 1$, since $x_0 = 0$) and the input signal u in $\mathcal{H}_4$. Further, R_2 and R_2^{-1} are identity functions and hence, uniformly continuous, where as, R_1 is an exponentially increasing function and is continuous, but not uniformly continuous. It is trivial to check that the consistence conditions hold, since 0 is mapped only to 0 by R_1, R_2, R_1^{-1} and R_2^{-1}. □

Propositions 4 and 5 imply that Theorems 3 and 4 will not hold if the uniformity condition on the continuity of the relation between the state-spaces is relaxed.

8 Examples of concrete abstraction functions

In this section, we argue that the pre-orders introduced in the paper for hybrid input-to-state stability are not too stringent by exhibiting concrete abstraction functions which satisfy the constraints imposed by the uniformly continuous input simulations/bisimulations. We show that the proofs of stability based on Lyapunov functions can be interpreted as constructing simpler hybrid input transition systems which are hybrid input-to-state stable and the Lyapunov function serves as a uniformly continuous input simulation relation. The case of linear dynamical systems was illustrated in Section 7.1. Here, we provide more general results.

8.1 Lyapunov functions for input-to-state stability

In this section, we focus on input-to-state stability of dynamical systems with respect to an equilibrium point. First, we define a Lyapunov's theorem for analyzing input-to-state stability.

Consider a continuous dynamical system defined by System (1). $\bar{0}$ is an equilibrium point, that is, $f(\bar{0}, \bar{0}) = \bar{0}$.

Definition 14 (Sontag 2006) A continuously differentiable function $V : X \to \mathbb{R}_{\geq 0}$ is said to be an *ISS* Lyapunov function for the System (1) if there exist class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and α_4 such that:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \forall x \in X, \tag{6}$$

$$\frac{\partial V(x)}{\partial x} f(x, u) \leq -\alpha_3(\|x\|) + \alpha_4(\|u\|), \forall x \in X, \forall u \in U. \tag{7}$$

Theorem 5 (Sontag and Wang 1995) *(ISS Theorem) Let $V : X \to \mathbb{R}_{\geq 0}$ be an ISS Lyapunov function for the System* (1). *Then System* (1) *is input-to-state stable.*

First, recall that System (1) is input-to-state stable if and only if $\mathcal{H}_{f,X,X_0,U}$ is hybrid input-to-state stable with respect to $\mathcal{T}_{X_0,U}$. Next, we show that an *ISS* Lyapunov function is essentially a uniformly continuous input simulation from System (1) to an input-to-state stable one-dimensional system obtained by the application of the Lyapunov function to System (1). Hence, Theorem 3 gives an alternative proof of *hISS* of System (1).

Let us say that a function $F : \mathbb{R}^n \to \mathbb{R}$ has *non-zero differential* if there exists a neighborhood Y of $\bar{0}$ such that the gradient of F at any point $y \neq \bar{0}$ in Y, $\nabla F(y)$, is non-zero. Following theorem formulates Lyapunov analysis in our framework:

Theorem 6 *Let V be an ISS Lyapunov function for System (1), and let $\ell : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be the function $u \mapsto |u|$. Let V have non-zero differential. Then there exists a compact set $\Omega \subseteq X$ and a compact set $U' \subseteq U$ containing an open ball around $\bar{0}$ such that:*

1. *$(V[\Omega], \ell)(\mathcal{H}_{f,X,X_0,U})$ input simulates $\mathcal{H}_{f,X,X_0,U}$.*
2. *$V[\Omega]$, $V[\Omega]^{-1}$, $\ell[U']$ and $\ell[U']^{-1}$ are uniformly continuous relations.*
3. *$(V[\Omega], \ell[U'])$ is consistent with $\mathcal{T}_{X_0,U}$ and $(V[\Omega], \ell[U'])(\mathcal{T}_{X_0,U})$.*
4. *$(V[\Omega], \ell[U'])(\mathcal{H}_{f,X,X_0,U})$ is hISS with respect to $(V[\Omega], \ell[U'])(\mathcal{T}_{X_0,U})$.*

Hence $\mathcal{H}_{f,X,X_0,U}$ is hISS with respect to $\mathcal{T}_{X_0,U}$.

Proof From Eq. 7, for any compact set $U' \subseteq U$, there exists a compact set $\Omega \subseteq X$ that contains $\bar{0}$ in its interior and is positively invariant, that is, all the trajectories starting from Ω remain within Ω for all future time. The uniform continuity of $V[\Omega]$ and $\ell[U']$ follows from the fact that they are continuous functions on a compact set. The uniform continuity of $V[\Omega]^{-1}$ and $\ell[U']^{-1}$ follows from the non-zero differentiability. It is trivial to check the consistency of $(V[\Omega], \ell[U'])$. To prove the fourth property, we need to show that (D1)–(D3) in Definition 8 are satisfied for $(V[\Omega], \ell[U'])(\mathcal{H}_{f,X,X_0,U})$. Equations 6 and 7 imply that

$$\frac{\partial V(x)}{\partial x} f(x, u) \leq -\alpha_3 \circ \alpha_2^{-1}(V(x)) + \alpha_4(\|u\|), \forall x \in X, \forall u \in U. \tag{8}$$

Using a standard comparison principle (Sontag and Wang 1995), it follows from Eq. 8 that there exists some $\mathcal{KL}$ function β and $\mathcal{K}_\infty$ function γ such that

$$V(x(t; \mathbf{u})) \leq \beta(V(x(0; \mathbf{u})), t) + \gamma(\|\mathbf{u}\|_\infty), \tag{9}$$

where $\mathbf{u}$ is an input signal and $x(t; \mathbf{u})$ is the solution under this input signal. From Eq. 9, it is straightforward to verify (D1)–(D3) in Definition 8 are satisfied for $(V[\Omega], \ell[U'])(\mathcal{H}_{f,X,X_0,U})$. The conclusion follows from the four properties using Theorem 3. □

8.2 Multiple Lyapunov functions for input-to-state stability

In this section, we show that proving input-to-state stability of switched systems with state jumps using multiple Lyapunov functions can be recast into a problem of preserving input-to-state stability using uniformly continuous simulations.

We consider a switched system with state jumps defined in Eq. 2. It is assumed that $f_p(\bar{0}, \bar{0}) = \bar{0}$ for all $p \in [N]$.

Definition 15 A family of continuously differentiable functions $\{V_p : X \to \mathbb{R}_{\geq 0}\}_{p \in [N]}$ is said to be *multiple ISS Lyapunov functions* for System (2) if there exist positive constants λ and $\mu \geq 1$ and class $\mathcal{K}_\infty$ functions α_1, α_2, and γ such that:

$$\alpha_1(\|x\|) \leq V_p(x) \leq \alpha_2(\|u\|), \forall x \in X, \forall p \in [N], \tag{10}$$

$$\frac{\partial V_p(x)}{\partial x} f_p(x, u) \leq -\lambda V_p(x) + \gamma(\|u\|), \forall x \in X, \forall p \in [N], \forall u \in U, \tag{11}$$

$$V_p(x) \leq \mu V_q(x) + \gamma(\|u\|), \forall x \in X, \forall p, q \in [N], \forall u \in U. \tag{12}$$

Definition 16 (Hespanha and Morse 1999) A switching signal $\alpha = (\{t_i\}_{i \in \mathbb{N}}, \{\omega_i\}_{i \in \mathbb{N}})$ is said to have an *average dwell-time* τ, if exists N_0 such that:

$$N_\alpha(T, t) \leq N_0 + \frac{T - t}{\tau}, \forall T > t \geq 0,$$

where $N_\alpha(T, t)$ is the number of switches in the interval $[t, T)$, that is, the number of i satisfying $t \leq t_i \leq T$.

Theorem 7 (Vu et al. 2007; Liu et al. 2012) *(ISS Theorem) Let* $\{V_p : X \to \mathbb{R}_{\geq 0}\}_{p \in [N]}$ *be a family of multiple ISS Lyapunov functions for the System* (2)*. Then System* (2) *is input-to-state stable, provided that the switching signals have an average dwell-time* $\tau_a > \log(\mu)/\lambda$.

We note that the input-to-state stability of System (2) is the same as the hybrid input-to-state stability of $\mathcal{H}_{f_1,\cdots,f_N,g_1,\cdots,g_N,X,X_0,U,\alpha}$ with respect to $\mathcal{T}_{X_0,U,\alpha}$ as in Vu et al. (2007) and Liu et al. (2012). We can write a family of multiple Lyapunov functions in a vector form as $\bar{V} = (V_1, \cdots, V_N)$. Given a switching signal $\alpha = (\{t_i\}_{i\in\mathbb{N}}, \{\omega_i\}_{i\in\mathbb{N}})$, define a function $\bar{V}_\alpha : \mathbb{N} \times X \to \mathbb{N} \times \mathbb{R}_{\geq 0}$ by $\bar{V}_\alpha(i, x) = (i, V_{\omega_i}(x))$.

The above ISS theorem for switched systems with state jumps can again be formulated as establishing a uniformly continuous input simulation from system (2) to a simpler system obtained by the application of the multiple Lyapunov functions $\bar{V}$ to system (2), such that the simpler system is *hISS*. Hence, Theorem 3 gives an alternative proof of *ISS* of system (2).

Theorem 8 *Let* $\bar{V} = (V_1, \cdots, V_N)$ *be a family of ISS multiple Lyapunov functions for System* (2)*, and let* $\ell : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be the function* $u \mapsto |u|$*. Let* $(V_1, \cdots, V_N)$ *have non-zero differential. Then there exist compact sets* $\Omega_1, \cdots, \Omega_N$ *and* U' *which contain an open ball around* $\bar{0}$*, such that,* $\bar{W} = (V_1[\Omega_1], \cdots, V_N[\Omega_N])$ *satisfies:*

1. $(\bar{W}_\alpha, \ell[U'])(\mathcal{H}_{f_1,\cdots,f_N,g_1,\cdots,g_N,X,X_0,U,\alpha})$ *input simulates* $\mathcal{H}_{f_1,\cdots,f_N,g_1,\cdots,g_N,X,X_0,U,\alpha}$.
2. $\bar{W}_\alpha, (\bar{W}_\alpha)^{-1}, \ell[U']$ *and* $\ell[U']^{-1}$ *are uniformly continuous.*
3. $(\bar{W}_\alpha, \ell[U'])$ *is consistent with* $\mathcal{T}_{X_0,U,\alpha}$ *and* $(\bar{W}_\alpha, \ell[U'])(\mathcal{T}_{X_0,U,\alpha})$.
4. $(\bar{W}_\alpha, \ell[U'])(\mathcal{H}_{f_1,\cdots,f_N,g_1,\cdots,g_N,X,X_0,U,\alpha})$ *is hISS with respect to* $(\bar{W}_\alpha, \ell[U'])(\mathcal{T}_{X_0,U,\alpha})$.

Hence $\mathcal{H}_{f_1,\cdots,f_N,g_1,\cdots,g_N,X,X_0,U,\alpha}$ *is hISS with respect to* $\mathcal{T}_{X_0,U,\alpha}$.

Proof The proof for properties (1)–(3) are similar. For property (4), instead of Eq. 9, we show that

$$V_{\omega_i}(x(t; \alpha, \mathbf{u})) \leq \beta(V_{\omega_0}(x(0; \alpha, \mathbf{u})), t) + \gamma(\|\mathbf{u}\|_\infty), \quad \forall t \in [t_i, t_{i+1}], \quad \forall i \geq 0, \tag{13}$$

where α is the given switching signal, $\mathbf{u}$ is an input signal, and $x(t; \alpha, \mathbf{u})$ is the solution under this input signal and the switching signal α. From Eq. 13, it is straightforward to verify (D1)–(D3) in Definition 8 are satisfied. The conclusion follows from the four properties using Theorem 3. □

9 Conclusions

In this paper, we investigated pre-orders for reasoning about input-to-state stability properties of hybrid systems. We introduced the notion of uniformly continuous input simulations and bisimulations as pre-orders which preserve input-to-state stability of hybrid systems. We showed that the notion is a reasonable pre-order to consider by establishing several concrete functions for input-to-state stability analysis from Lyapunov function based analysis. In the future, we intend to develop concrete techniques for constructing abstractions based on uniformly continuous input simulations and bisimulations analogous to predicate abstraction. Such techniques have already been considered for stability without input (Prabhakar and Soto 2013).

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Appendix A: Proof of the super-position Theorem 1

Proof $\delta ISS \Rightarrow (C1) - (C3)$: It is straightforward to check that δISS implies conditions $(C1) - (C3)$. In fact, choosing $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}$ and δ such that $\beta(\delta, 0) < \epsilon$ in Eq. 3 implies

$$|\mathbf{x}(\zeta_1, \mathbf{u})(t) - \mathbf{x}(\zeta_2, \mathbf{u})(t)| \leq \beta(|\zeta_1 - \zeta_2|, t) < \beta(\delta, 0) < \epsilon,$$

provided that $|\zeta_1 - \zeta_2| < \delta$. This shows $(C1)$ is true. Moreover, since

$$|\mathbf{x}(\zeta_1, \mathbf{u})(t) - \mathbf{x}(\zeta_2, \mathbf{u})(t)| \leq \beta(|\zeta_1 - \zeta_2|, t) \to 0$$

as $t \to \infty$, for any given ϵ and ζ_1, ζ_2, we can choose T independent of $\mathbf{u}$ such that

$$|\mathbf{x}(\zeta_1, \mathbf{u})(t) - \mathbf{x}(\zeta_2, \mathbf{u})(t)| < \epsilon$$

for all $t > T$. This shows $(C2)$ is true. Finally, choosing $\zeta_1 = \zeta_2 = \zeta$ and δ such that $\gamma(\delta) < \epsilon$ in Eq. 3 implies

$$|\mathbf{x}(\zeta, \mathbf{u})(t) - \mathbf{x}(\zeta, \mathbf{u})(t)| \leq \gamma(|\mathbf{u}_1 - \mathbf{u}_2|_\infty) < \gamma(\delta) < \epsilon,$$

provided that $\|\mathbf{u}_1 - \mathbf{u}_2\|_\infty < \delta$. This shows $(C3)$ is true.

$(C1) - (C3) \Rightarrow \delta ISS$: The proof for the opposite implication essentially follows from the proof of Lemma 4.5 in Khalil (1996). Therefore, the detailed argument is omitted and the following is an outline of the proof. First, by (C3), we can prove there exists a $\mathcal{K}_\infty$ function γ such that

$$|\mathbf{x}(\zeta, \mathbf{u}_1)(t) - \mathbf{x}(\zeta, \mathbf{u}_2)(t)| \leq \gamma(\|\mathbf{u}_1 - \mathbf{u}_2\|_\infty), \quad t \geq 0, \tag{14}$$

holds for all initial states ζ_1, ζ_2. Second, conditions $(C1)$ and $(C2)$ imply that there exist a $\mathcal{KL}$ function β such that

$$|\mathbf{x}(\zeta_1, \mathbf{u})(t) - \mathbf{x}(\zeta_2, \mathbf{u})(t)| \leq \beta(|\zeta_1 - \zeta_2|, t), \quad t \geq 0, \tag{15}$$

holds for all input trajectory u. Now given any pair of initial states $\zeta_1,\ \zeta_2$ and any pair of input trajectories $u_1,\ u_2$, it follows from Eqs. 14 and 15 that

$$\begin{aligned} |\mathbf{x}(\zeta_1, \mathbf{u}_1)(t) - \mathbf{x}(\zeta_2, \mathbf{u}_2)(t)| &\leq |\mathbf{x}(\zeta_1, \mathbf{u}_1)(t) - \mathbf{x}(\zeta_2, \mathbf{u}_1)(t) + \mathbf{x}(\zeta_2, \mathbf{u}_1)(t) - \mathbf{x}(\zeta_2, \mathbf{u}_2)(t)| \\ &\leq \beta(|\zeta_1 - \zeta_2|, t) + \gamma(\|\mathbf{u}_1 - \mathbf{u}_2\|_\infty), \quad t \geq 0. \end{aligned}$$

This completes the proof. □

Appendix B: Proof of Theorem 4

Proof Let us assume $\mathcal{H}_2$ is *hISS* with respect to $\mathcal{T}_2$. We need to show that $\mathcal{H}_1$ is *hISS* with respect to $\mathcal{T}_1$. We will show that $\mathcal{H}_1$ satisfies conditions $(D1) - (D3)$.

Proof of satisfaction of Condition $(D1)$ Let us fix an $\epsilon_1 > 0$. We need to find a $\delta_1 > 0$ such that Condition $(D1)$ holds in $\mathcal{H}_1$ and $\mathcal{T}_1$. Let ϵ_2 be the uniformity constant of R_1^{-1} corresponding to ϵ_1. Let δ_2 be the constant satisfying Condition $(D1)$ for $\mathcal{H}_2$ corresponding to ϵ_2. Set δ_1 to be the uniformity constant of R_1 corresponding to δ_2.

Let us fix an input signal $\sigma^u{}_1$ and states $\eta \in First(Exec(\mathcal{H}_1)|_{\sigma^u{}_1})$ and $\hat{\eta}_1 \in First(\mathcal{T}_1|_{\sigma^u{}_1})$ such that $d^s(\hat{\eta}_1, \eta_1) < \delta_1$. Let $\sigma^s{}_1$ be such that $(\sigma^u{}_1, \sigma^s{}_1) \in Exec(\mathcal{H}_1)$ and $First(\sigma^s{}_1) = \eta_1$. We need to find $\hat{\sigma}_1^s$ such that $(\sigma^u{}_1, \hat{\sigma}_1^s) \in \mathcal{T}$, $First(\hat{\sigma}_1^s) = \hat{\eta}_1$ and $d^s(\sigma^s{}_1, \hat{\sigma}_1^s) < \epsilon_1$.

First, we establish that $R_2(\sigma^u{}_1)$ is a singleton set. This follows from $(A3)$. Let $R_2(\sigma^u{}_1) = \{\sigma^u{}_2\}$. Hence, from $(A1)$, we have that there exists $\hat{\eta}_2 \in First(\mathcal{T}_2|_{\sigma^u{}_2})$ such that $R_1(\hat{\eta}_1, \hat{\eta}_2)$. From uniform continuity, we have $R_1(B_{\delta_1}(\eta_1)) \subseteq B_{\delta_1}(R_1(\eta_1))$. Hence, $\hat{\eta}_2 \in B_{\delta_2}(\eta_2)$ for some $\eta_2 \in R_1(\eta_1)$. From the definition of input simulation and uniqueness of the choice of $\sigma^u{}_2$, we have that there exists $\sigma^s{}_2$ such that $First(\sigma^s{}_2) = \eta_2$, $(\sigma^u{}_2, \sigma^s{}_2) \in Exec(\mathcal{H})$ and $R_1(\sigma^s{}_1, \sigma^s{}_2)$. Then, from the *hISS* of $\mathcal{H}_2$ with respect to $\mathcal{T}_2$, we obtain that there exists $(\sigma^u, \hat{\sigma}_2^s) \in \mathcal{T}_2$ such that $First(\hat{\sigma}_2^s) = \hat{\eta}_2$ and $d^s(\hat{\sigma}_2^s, \sigma^s{}_2) < \epsilon_2$. From $(A2)$, there exists $\hat{\sigma}_1^s$ such that $First(\hat{\sigma}_1^s) = \hat{\eta}_1$, $(\sigma^u{}_1, \hat{\sigma}_1^s) \in \mathcal{T}_1$ and $R_1(\hat{\sigma}_1^s, \hat{\sigma}_2^s)$.

It remains to show that $d^s(\sigma^s{}_1, \hat{\sigma}_1^s) < \epsilon_1$. Note that $d^s(\sigma^s{}_2, \hat{\sigma}_2^s) < \epsilon_2$ and $R_1^{-1}(\hat{\sigma}_2^s) = \{\hat{\sigma}_1^s\}$ (a singleton, from $(A4)$). To show that $d^s(\sigma^s{}_1, \hat{\sigma}_1^s) < \epsilon_1$, we need to show that $d^s(gr(\sigma^s{}_1), gr(\hat{\sigma}_1^s)) < \epsilon_1$. Consider $(t_1, i_1, x_1) \in gr(\sigma^s{}_1)$, we need to find $(t_1', i_1', x_1') \in gr(\hat{\sigma}_1^s)$ such that $d^s((t_1, i_1, x_1), (t_1', i_1', x_1')) < \epsilon_1$. Since $R_1(\sigma^s{}_1, \sigma^s{}_2)$, there exists $(t_2, i_2, x_2) \in gr(\sigma^s{}_2)$, such that $R_1((t_1, i_1, x_1), (t_2, i_1, x_2))$. Note, in fact, that $t_1 = t_2$ and $i_1 = i_2$, and $R_1(x_1, x_2)$. Since, $d^s(\sigma^s{}_2, \hat{\sigma}_2^s) < \epsilon_2$, there exists $(t_2', i_2', x_2') \in gr(\hat{\sigma}_2^s)$ such that $d^s((t_2, i_2, x_2), (t_2', i_2', x_2')) < \epsilon_2$. Then, from the continuity of R_1^{-1}, (t_2, i_2, x_2) is within ϵ_1 of some element in $R_1^{-1}(t_2', i_2', x_2')$. But, since, the latter is a singleton set (from $(A4)$), namely, (t_1', i_1', x_1'), we obtain that $d^s((t_1, i_1, x_1), (t_2', i_2', x_2')) < \epsilon_1$. Note that $t_1' = t_2'$, and $i_1' = i_2'$. The other part where given $(t_1', i_1', x_1') \in gr(\hat{\sigma}_1^s)$, we need to find $(t_1, i_1, x_1) \in gr(\sigma^s{}_1)$ such that $d^s((t_1, i_1, x_1), (t_1', i_1', x_1')) < \epsilon_1$ is similar.

Proof of satisfaction of Condition $(D2)$ Let us fix an $\delta_1, \epsilon_1 > 0$ and $T_1 \geq 0$, such that Condition $(D2)$ holds in $\mathcal{H}_1$ and $\mathcal{T}_1$. Let ϵ_2 be the uniformity constant of R_1^{-1} corresponding to ϵ_1. Let δ_2 be the constant satisfying Condition $(D1)$ for $\mathcal{H}_2$ corresponding to ϵ_2. Set δ_1 to be the uniformity constant of R_1 corresponding to δ_2. Let $T_2 = T_1$. We will show that Condition $(D2)$ holds in $\mathcal{H}_2$ and $\mathcal{T}_2$ for δ_1, ϵ_2 and T_2.

The proof is similar to that of Condition $(D1)$. Here we need to show that $d^s(\sigma^s{}_1|_{T_1}, \hat{\sigma}_1^s|_{T_1}) < \epsilon_1$ instead of $d^s(\sigma^s{}_1, \hat{\sigma}_1^s) < \epsilon_1$. Note that in the previous proof the choice of the times when we move from one system to the other are the same. That is, $t_1 = t_2$ and $t_1' = t_2'$. We mainly need to check if the triples being chosen at different steps belong to the time restricted signals rather than the complete signals as required in the previous proof. Hence, if we consider the triple $(t_1, i_1, x_1) \in gr(\sigma^s{}_1|_{T_1})$, the corresponding triple (t_2, i_2, x_2) will belong to $gr(\sigma^s{}_1|_{T_1})$, since, $T_1 = T_2$ and the time stamp $t_2 \geq T_2$. Here, we can choose the triple (t_2', i_2', x_2') such that its time stamp $t_2' \geq T_2$, because we have $d^s(\sigma^s{}_2|_{T_2}, \hat{\sigma}_2^s|_{T_2}) < \epsilon_2$ instead of $d^s(\sigma^s{}_2, \hat{\sigma}_2^s) < \epsilon_2$. Finally, the triple (t_1', i_1', x_1') will have time stamp $t_1' = t_2' \geq T_2 = T_1$. Hence, $(t_1', i_1', x_1') \in gr(\hat{\sigma}_1^s|_{T_1})$.

Proof of satisfaction of Condition $(D3)$ Let us fix an $\epsilon_1 > 0$. We need to find a $\delta_1 > 0$ such that Condition $(D3)$ holds in $\mathcal{H}_1$ and $\mathcal{T}_1$. Let ϵ_2 be the uniformity constant of R_1^{-1} corresponding to ϵ_1. Let δ_2 be the constant satisfying Condition $(D3)$ for $\mathcal{H}_2$ corresponding to ϵ_2. Set δ_1 to be the uniformity constant of R_2 corresponding to δ_2.

Let us fix input signals $\sigma^u{}_1, \hat{\sigma}_1^u$, state $\eta_1 \in \mathcal{T}_1|_{\hat{\sigma}^u}$. Let $d(\sigma^u{}_1, \hat{\sigma}_1^u) < \delta_1$ and $(\sigma^u{}_1, \sigma^s{}_1) \in Exec(\mathcal{H})$ with $First(\sigma^s{}_1) = \eta_1$. We need to find $\hat{\sigma}_1^s$ such that $(\hat{\sigma}_1^u, \hat{\sigma}_1^s) \in \mathcal{T}_1$, $First(\hat{\sigma}_1^s) = \eta_1$ and $d^s(\sigma^s{}_1, \hat{\sigma}_1^s) < \epsilon_1$.

Since $\eta_1 \in \mathcal{T}_1|_{\hat{\sigma}_1^u}$, we obtain from $(A1)$ that there exists $\hat{\sigma}_2^u$ and $\eta_2 \in \mathcal{T}_2|_{\hat{\sigma}_2^u}$ such that $R_1(\eta_1, \eta_2)$ and $R_2(\hat{\sigma}_1^u, \hat{\sigma}_2^u)$. Again, since, $R_1(\eta_1, \eta_2)$, from the definition of simulation, there exists $(\sigma^u{}_2, \sigma^s{}_2) \in Exec(\mathcal{H}_2)$ such that $First(\sigma^s{}_2) = \eta_2$, $R_1(\sigma^s{}_1, \sigma^s{}_2)$ and $R_2(\sigma^u{}_1, \sigma^u{}_2)$. From the choice of δ_1, we have that $\sigma^u{}_2 \in B_{\delta_1}(R_2(\hat{\sigma}_1^u))$. However, since, $R_2(\hat{\sigma}_1^u)$ is unique, namely, $\hat{\sigma}_2^u$, we obtain that $d^u(\hat{\sigma}_2^u, \sigma^u{}_2) < \delta_2$. Therefore, from the *hISS* of $\mathcal{H}_2$ with respect to $\mathcal{T}_2$, there exists $\hat{\sigma}_2^s$ such that $(\hat{\sigma}_2^u, \hat{\sigma}_2^s) \in \mathcal{T}_2$, $First(\hat{\sigma}_2^s) = \eta_2$ and $d^s(\sigma^s{}_2, \hat{\sigma}_2^s) < \epsilon_2$. Finally, from $(A2)$, there exists $\hat{\sigma}_1^s$ such that $First(\hat{\sigma}_1^s) = \eta_1$, $R_1(\hat{\sigma}_1^s, \hat{\sigma}_2^s)$ and $(\hat{\sigma}_1^u, \hat{\sigma}_1^s) \in \mathcal{T}_1$.

It remains to show that $d^s(\sigma^s{}_1, \hat{\sigma}_1^s) < \epsilon$. The argument is the same as that in the proof of part $D1$. □

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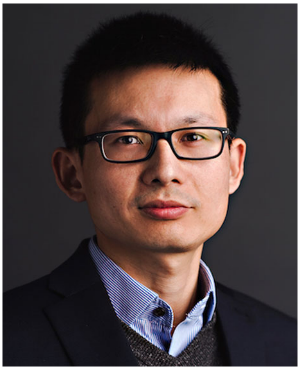

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