Local and non local projective measurements of product observables on two qubits

The two examples below are taken from the supplementary material of [1], that will appear online on Feb 18 2016.

A. Notation

We consider the measurement of the operator $\Omega = \sum_k \omega_k \Pi_k$ where $\{\Pi_k\}$ are orthogonal projection operators and $\{\omega_k\}$ are distinct eigenvalues of Ω. Note that if Ω is degenerate, there is at least one Π_k of rank > 1.

The projective Lüders measurement of Ω induces the channel M on the state of the system ρ.

$$M(\rho) = \sum_k \Pi_k \rho \Pi_k.$$

The measurement on ρ will produce the result corresponding to ω_k with probability given by the Born rule $P(\omega_k) = tr[\Pi_k \rho]$. If the measurement produces the result ω_k, the resulting sub-channel is $M_k(\rho) = \Pi_k \rho \Pi_k$. Note that we use the convention where the sub-channel is not trace preserving.

In what follows we will sometimes use the fact that the sub-channels are projections so that we can use pure state notation. In those cases we replace the notation $M_k(|\psi\rangle\langle\psi|)$ with

$$M_k(|\psi\rangle) = \Pi_k |\psi\rangle \tag{1}$$

B. Measurements of $\Omega = \sigma_z^A \otimes \sigma_z^B$

Consider the measurement of $\Omega = \sigma_z^A \otimes \sigma_z^B$. The Krauss operators for the Lüders channel are $\Pi_{+1} = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $\Pi_{-1} = |10\rangle\langle 10| + |01\rangle\langle 01|$. Now take an initial separable state $|\psi\rangle = \frac{1}{2}[|0\rangle + |1\rangle] \otimes [|0\rangle + |1\rangle] = \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$. Let us assume we perform the non-local measurement of Ω and get the result $+1$, the outgoing state will be (up to normalization)

$$\Pi_{+1} |\psi\rangle = \frac{1}{2}[|00\rangle + |11\rangle] \tag{2}$$

which is entangled.

1. *The corresponding local measurement*

Consider now the joint measurement broken into two local measurements σ_z^A and σ_z^B. The joint measurement has four possible outcomes $\{(+1,+1),(+1,-1),(-1,+1),(-1,-1)\}$, it induces a channel $\tilde{M}$ with Krauss operators $\tilde{\Pi}_{+,+} = |00\rangle\langle 00|$, $\tilde{\Pi}_{+,-} = |01\rangle\langle 01|$, $\tilde{\Pi}_{-,+} = |10\rangle\langle 10|$ and $\tilde{\Pi}_{-,-} = |11\rangle\langle 11|$. Consequently, the sub-channels are entanglement breaking.

We can coarse grain the measurement by forgetting the local results so that the sub channels will be

$$\tilde{M}_{+1}(\rho) = |00\rangle\langle 00|\rho|00\rangle\langle 00| + |11\rangle\langle 11|\rho|11\rangle\langle 11|$$

and

$$\tilde{M}_{-1}(\rho) = |01\rangle\langle 01|\rho|01\rangle\langle 01| + |10\rangle\langle 10|\rho|10\rangle\langle 10|$$

These sub-channels are still entanglement breaking.

C. The Lüders measurement $\Omega = |11\rangle\langle 11|$ is not localizable

A bipartite channel is said to be localizable (in the sense of [2]) if it cannot be implemented using local operations and shared entanglement (in other words, it cannot be implemented instantaneously). A bipartite channel is called causal if it does not allow transmission of information between the two parts. Causality is a pre-requisite for localizability. In what follows we show that the Lüders measurement of $\Omega = |11\rangle\langle 11|$ can be used to transmit information between Alice and Bob, consequently it is neither causal nor localizable.

Consider the measurement of $\Omega = |11\rangle\langle 11|$. The Krauss operators for the channel $M(\rho)$ are $\Pi_1 = |11\rangle\langle 11|$ and $\Pi_0 = |00\rangle\langle 00| + |10\rangle\langle 10| + |01\rangle\langle 01|$. To show that this channel can produce entanglement, consider the same initial state $|\psi\rangle$ as the previous section. The sub-channel corresponding to the result 0 will produce an entangled state.

To show that the channel $M(\rho)$ can be used to transmit information, consider the following strategy for Alice to signal Bob. Alice and Bob prepare the initial state $|Zero\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes [|0\rangle + |1\rangle]$. Now Alice can signal Bob by either doing nothing, in which case $M(|Zero\rangle) = |Zero\rangle$, or by changing her state locally to $|1\rangle$ so that global state reads $\frac{1}{\sqrt{2}}|1\rangle \otimes [|0\rangle + |1\rangle] = |One\rangle$. Now

$$M(|One\rangle\langle One|) = \frac{1}{2}[|10\rangle\langle 10| + |11\rangle\langle 11|] \tag{3}$$

Bob's local state is $\frac{1}{2}[|0\rangle\langle 0| + |1\rangle\langle 1|]$. What we see is

$$Tr_A[M(|Zero\rangle\langle Zero|)] \neq Tr_A[M(|One\rangle\langle One|)] \tag{4}$$

Alice can thus use the channel M together with local operations on her side to change Bob's state, consequently the channel is not causal (and not localizable).

[1] A. Brodutch, and E. Cohen, Phys. Rev. Lett (2016). arXiv:1409.1575
[2] D. Beckman, D. Gottesman, M. A. Nielsen, and J. Preskill, Phys. Rev. A **64**, 052309 (2001). arXiv:quant-ph/0102043