Lecture 11: Topological quantum computing

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Topological Quantum Computing

A. General Notions

For an ‘ideal’ quantum computer, we would like a Hilbert space s.t.:

1. in absence of intervention from experimentalist,
   \[ \hat{H} = k \hat{I} \]

2. result of intervention robust against small errors in implementation

Why might topological considerations be relevant?

In (strictly) 2D:

→ For everyday-life objects, no effect!

Quantum states?
**Berry Phase (general, not 2D)**

(a) Scalar: external parameter
\[
\psi_0(z) \equiv \psi_0(z : \lambda) \quad , \quad \lambda = \lambda(t), \quad \lambda(T) = \lambda(0)
\]
\[
df : \quad \varphi_B \equiv \text{Im} \int d\lambda \left( \Psi(\lambda), \frac{\partial \Psi}{\partial \lambda} \right)
\]

standard example: spin \(1/2\) particle in magnetic field
\[
\propto \hat{n}(t), \quad \hat{n}(t) \equiv (\theta, \varphi(t)) \Rightarrow \lambda(t) \longleftrightarrow \varphi(t)
\]
\[
|\psi_0\rangle = \begin{pmatrix}
\cos \left( \frac{\theta}{2} \right) \\
\sin \left( \frac{\theta}{2} \right) \exp i\varphi
\end{pmatrix}
\]

but,
\[
\varphi_B = \pi (1 - \cos(\theta_0)) \neq 0 \quad !
\]
Berry phase is "average" phase change around loop in GS.

(b) Matrix:
\[
\psi_0(x : \lambda) \text{ may be degenerate. Then df.}
\]
\[
K_{mn} \equiv \oint \langle \psi_m(\lambda) | \frac{\partial \psi_n(\lambda)}{\partial \lambda} \rangle d\lambda \quad (K_{mm} \equiv i\varphi_{B_m})
\]

example: spin-1 particle with \(\hat{H} = \alpha(S \cdot \vec{B})^2\), \(\alpha > 0\)
Magn. field \(\vec{B}\) carried around circuit ("parallel transport")

\[
\psi_{0,a} = (-\sin \varphi, \cos \varphi, 0) \equiv \underline{a} \\
\psi_{0,b} = (-\cos \theta \cos \varphi, -\sin \theta \sin \varphi, -\sin \theta) \equiv \underline{b}
\]

as \(\vec{B}\) carried around circuit, \(\underline{a}\) and \(\underline{b}\) rotate through \(\Delta \chi = \varphi_0(1 - \cos(\theta_0))\)
THE BRAID GROUP
The braid group $B_n$ for $n$ objects is constructed by numbering them arbitrarily 1, 2, ......$n$ and defining $\hat{T}_i =$directed interchange of $i$ and $i+1$. (e.g. $i$ over $i+1$)

Note that $\hat{T}_i \neq \hat{P}_{i,i+1}$ and in particular we do not have $\hat{T}_i^2 = 1$:

It is immediately obvious that the $\hat{T}_i$ satisfy

$$[\hat{T}_i, \hat{T}_j] = 0 \quad \text{for} |i - j| > 1$$

Slightly less obviously,

$$\hat{T}_i\hat{T}_j\hat{T}_i = \hat{T}_j\hat{T}_i\hat{T}_j \quad \text{for} |i - j| = 1$$

(1) and (2) may be taken as the definition of the braid group
It is of course perfectly possible to have 1-dimensional representations of the braid group (ex.: bosons, fermions, abelian anyons). In this case, the wave function satisfies
\[ \hat{T}_{ij}\psi = \exp i\alpha \psi \]

Also, it is (trivially) possible to have reducible representations which can be broken down into 1D representations \[ \hat{T}_{ij}\psi_n = \exp i\alpha_n \psi_n \]

Nonabelian anyons correspond to (nontrivial) irreducible representations of the braid group. By braiding the anyons we can in principle carry out TQC.

Example: Majorana fermions on half-quantum vortices in a \((p+ip)\) Fermi superfluid. ("Ising" anyons). In this case, an MF is shared between two vortices, (but the assignment is arbitrary), so each pair is a qubit, with basis \(|1MF\rangle\) and \(|0MF\rangle\) ⇒ dimension of Hilbert space of 2n vortices = \(2^n\) (not \(2^{2n}\))

For 2 vortices:
\[ \hat{T} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \]
i.e. \(\hat{T}_{12} \equiv \exp i\pi/4 \hat{\sigma}_z\)

For 4 vortices (2 qubits) with \(Q_1 \equiv (1,2), Q_2 \equiv (3,4)\):
\[ \hat{T}_{12} \equiv \exp i\pi/4 \hat{\sigma}_z^{(1)} \]
\[ \hat{T}_{34} \equiv \exp i\pi/4 \hat{\sigma}_z^{(2)} \]

but \(\hat{T}_{23} = \frac{1}{\sqrt{2}} (1 + \hat{\sigma}_y^{(1)} \hat{\sigma}_x^{(2)})\)

↑
entangling!
So far, we have been able to perform one 2-qubit operation (which, when supplemented by a 1-qubit operation, becomes CNOT), but only one single-qubit operation, namely $\exp i\hat{\sigma}_z \frac{\pi}{4}$. By adding further qubits we can realize also $\exp i\hat{\sigma}_j \frac{\pi}{4}$, ($j = x, y$), but this is still not enough for universal TQC. ("Pauli group is not dense in SU(2")).

The simplest set of anyons which will permit universal TQC is Fibonacci anyons. These are objects s.t. the minimum no. of anyons which can simulate a qubit is 3 (dimension of Hilbert space of n anyons $\sim \text{Fib}(n)$). For the states of such a qubit we have

$$\hat{T}_{12} = \begin{pmatrix}
e^{-4\pi i/5} & 0 \\
0 & -e^{-4\pi i/5}\end{pmatrix}$$

$$\hat{T}_{23} = \begin{pmatrix}
\eta e^{-\pi i/5} & -i\eta^{1/2}e^{-\pi i/10} \\
-i\eta^{1/2}e^{-\pi i/10} & -\eta
\end{pmatrix}$$

$$\eta \equiv \frac{2}{1 + \sqrt{5}}$$

Since the group of rotations through $\frac{\pi}{10}$ around perpendicular axes is dense in SU(2), this is adequate in principle for completely topologically protected quantum computation.

Need for gap: Nothing in df. of ”anyon” requires it to have a nonzero excitation energy. But, for the purposes of TQC, essential to avoid uncontrolled/unwanted anyons.
THE FRACTIONAL QUANTUM HALL EFFECT:
THE CASES OF $\nu = 5/2$ AND $\nu = 12/5$

Reminder re QHE:

Occurs in (effectively) 2D electron system (2DES) (e.g. inversion layer in GaAs - GaAlAs heterostructure) in strong perpendicular magnetic field, under conditions of high purity and low ($\lesssim 250\text{mK}$) temperature.

If df. $l_m \equiv \sqrt{\frac{\hbar}{eB}}$ (magnetic length) then area per flux quantum $\frac{\hbar}{e}$ is $2\pi l_m^2$, so, no. of flux quanta $= \frac{A}{2\pi l_m^2}$ ($A \equiv$ area of sample). If total no. of electrons is $N_e$, define

$$\nu \equiv \frac{N_e}{N_\Phi}$$

(”filling factor”)

QHE occurs at and around (a) integral values of $\nu$ (integral QHE) and (b) fractional values $p/q$ with fairly small ($\lesssim 13$) values of $q$ (fractional QHE). At $\nu$th step, Hall conductance $\sum_{xy}$ quantized to $\nu e^2/h$ and longitudinal conductance $\sum_{xx} \approx 0$

Nb : (1) Fig. shows IQHE only

(2) expts usually plot $R_{xy}$ vs $B(\propto \frac{1}{\nu})$

so general pattern is same but details different.
**SYSTEMATICS OF FQHE**

FQHE is found to occur at and near $\nu = p/q$, where $p$ and $q$ are mutually prime integers. By now, $\sim 50$ different values of $(p,q)$. Generally, FQHE with large values of $q$ tend to be more unstable against disorder and temperature. 

eg. plateaux narrower, $\rho_{xx} \to 0$

Possible approaches to identification of phases:

1. analytic, trial wf (eg Laughlin)
2. numerical, few-electron (typically $N \simeq 18$)
3. via CFT $\leftrightarrow$ conformal field theory
4. experiment:

   alas, cannot usually measure much other than electrical props ! ideally, would at least like to know total spin of sample, but........

The simplest FQHE states (Laughlin states) : reminders

The Laughlin states have $p = 1$, $q = \text{odd integer}$, i.e. $\nu = 1/(2m+1), (m \text{ integral}, \nu = 1/3, 1/5, \ldots)$

These are well accounted for by the Laughlin w.f.

$$\psi_N = \prod_{i<j}^N (z_i - z_j)^q \exp \left[ -\sum_i \frac{|z_i|^2}{4l_m^2} \right]$$

$q = \frac{1}{\nu} = 2m + 1$

$z \equiv x + iy$

Elementary excitations are quasiholes generated by multiplying GSWF by $\prod_{i=1}^N (z_i - \eta_0)$ (hole at $\eta_0$). They have charge $e^* = \nu e$ and are abelian anyons:

$$\psi(1,2) = \exp \left[ i\pi \nu \right] \psi(2,1)$$

Fairly convincing evidence for fractional charge ($\nu = 1/3$), some evidence for fractional statistics.
\( \nu = \frac{5}{2} \) FQHE STATE

Only even-denominator FQHE state (reliably) seen so far: fairly robust \((\Delta \sim 500 \text{ mK})\) excitation gap as deduced from T-dep. of \(R_{xx}\).

Various possible identifications, of which most popular (among theorists!) is the Moore-Read Pfaffian state:

\[
\Psi_N = \Psi_N^{(L)} \cdot \text{Pf} \left( \frac{1}{z_i - z_j} \right), \quad \Psi_N^{(L)} \equiv \prod_{i<j}^N (z_i - z_j)^2 \exp \left[ -\sum_i \frac{|z_i|^2}{l_m^2} \right]
\]

"Laughlin" state for \( \nu = \frac{1}{2} \)

\[
\frac{1}{z_1 - z_2} - \frac{1}{z_3 - z_4} + \ldots \quad \text{"("Pfaffian")}
\]

Ansatz for single quasihole (analogous to that for Laughlin states):

\[
\Psi_{qh} = \prod_{i=1}^N (z_i - \eta_0) \Psi_N
\]

believed to have charge \( e^* = e/4 \)

For \( n \) quasiholes \( 2^{n-1} \) linearly independent states. With Nayak-Wilczek construction for \( n \)-quasiholes states, nonabelian (Ising) statistics.

\( \nu = \frac{12}{5} \) FQHE STATE

Seen to date in only one experiment: very fragile

Could be simply \( n = 1 \) analog of the \( \nu = \frac{2}{5} \) state, thus would fit into composite-fermion scheme and have only abelian qps.

Speculation (Read-Reyazi): "parafermion" state, with qps Fibonacci Anyons
**SPECIFIC SYSTEMS, cont.**

**p+ip FERMI SUPERFLUIDS**

Required: 2D system of fermions in Cooper-paired ("superfluid") state with pair order parameter of form

$$F(r) = (x + iy) f(|r|)$$

or in momentum space

$$F(p) = (p_x + ip_y) f(|p|)$$

Need either spinless system (e.g. Fermi alkali gas near p-wave Feshbach resonance) with ordinary (Abrikosov) vortices, or spinful system (e.g. $^3$He-A, Sr$_2$RuO$_4$) with half-quantum vortices. (HQV)

Established belief: a system of $2n$ HQV can sustain $n$ Majorana fermions (each M.F. is "shared" between 2 HQV's). The vortices then behave as Ising anyons: e.g. if for $2n = 4$ we associate one (possible) M.F. with the pair (1,2) ("qubit 1") and the other with the pair (3,4) ("qubit 2"), then ($\hat{T}_{ij} \equiv$ braiding of i and j), then

$$\hat{T}_{12} = \exp i \frac{\pi}{4} \hat{\sigma}_y^{(1)}$$
$$\hat{T}_{34} = \exp i \frac{\pi}{4} \hat{\sigma}_y^{(2)}$$

but,

$$\hat{T}_{23} = \frac{1}{\sqrt{2}}(1 + \hat{\sigma}_y^{(1)} \hat{\sigma}_x^{(2)})$$
(nb: not completely topologically protected)

1. The General Idea

Required: System with groundstate (typically highly entangled) separated from all (relevant) excitations by a nonzero gap, and with (some) excitations nonabelian anyons. Ex.: 2D (p+ip) Fermi superfluid. (well, maybe . . .)

<table>
<thead>
<tr>
<th>Object</th>
<th>Not(^n)</th>
<th>p+ip version</th>
</tr>
</thead>
<tbody>
<tr>
<td>Groundstate</td>
<td>1</td>
<td>&quot;vacuum&quot;</td>
</tr>
<tr>
<td>Anyon</td>
<td>(\sigma)</td>
<td>half-quantum (anti-) vortex</td>
</tr>
<tr>
<td>?</td>
<td>(\psi)</td>
<td>fermion</td>
</tr>
</tbody>
</table>

Note: 1 (Bogoliubov) fermion (=2 Majorana fermions) must always be shared between 2 vortices. So for 2n vortices, \(2^n\) possibilities \(\Rightarrow\) dimension of Hilbert space = \(2^n\) (not \(2^{2n}\)!)
("quantum dimension = \(\sqrt{2}\)\).

At large distances, states of 2 vortices corr. presence or absence of fermion are exactly degenerate \((\Delta E \sim \exp{-L/\xi})\).
Note: In the literature on TQC in \((p+ip)\) Fermi superfluids, 2 assumptions always implicit:

1. States differing in \(N\) (total particle no.) by \(2n\) are equivalent.

   Justification: BCS wf is of form
   \[
   \Psi = \prod_m \left( u_m + v_m a_m^\dagger a_m^\dagger \right) |\text{truevac}\rangle
   \]

   so does not conserve \(N\).

   \(\uparrow\): states of different parity are not equivalent.

2. States differing only by presence of \(n\) full-quantum vortices are equivalent.

   Justification: such vortices do not sustain Majorana fermions, etc.

   \(\Rightarrow\) can treat HQV and anti-HQV as equivalent (since 2 HQV \(\rightarrow\) 1 FQV, HQV+HQV \(\rightarrow\) vac)

   \(\uparrow\): For the moment, assume neutral system (e.g. \(^3\text{He-A}\)).
But of course also

Hence, "fusion rule" ("1")

\[ \sigma \times \sigma = 1 + \psi \]

2 Bogoliubov fermions can annihilate to vac. ("mod 2"), so:

\[ \psi \times \psi = 1 \]

and a (zero-E) fermion cannot be associated with a single vortex, so:

\[ \psi \times \sigma = \sigma \quad (\text{e.g. } \sigma \times \sigma \times \sigma = \sigma) \]
BRAIDING OF ANYONS

From the arguments of Ivanov*, if an \( E = 0 \) fermion is "shared" by 2 vortices and they are exchanged, MBWF changed by a factor of \( \exp i\pi/2 \equiv i \), while if there is no fermion, this factor is 1. Hence, for the single-qubit system formed by 2 vortices,

\[
\hat{T}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \text{(or } \exp i\frac{\pi}{4} \hat{\sigma}_z \text{)}
\]

Case of 4 vortices:

At first sight, 2 qubits, e.g. associated with (1,2) and (3,4). \( \Rightarrow \) 4D Hilbert space. Then:

\[
\hat{T}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}_1 \hat{1}_2, \quad \hat{T}_{34} = \hat{1}_1 \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}_{34}
\]

but

\[
\hat{T}_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left( 1 + \hat{\sigma}_y \hat{1}_1 \hat{\sigma}_{x2} \right) \quad \text{(entangling)}
\]

However, this is a bit misleading, because all operations preserve parity of state. (as do all "real-life" physical operations). Hence, preferable to fix the parity and regard 4-anyon system as single qubit associated with e.g. anyons 1 and 2: e.g. for odd \( N \)

\[
|0\rangle = |\text{no MF on (1,2), MF on (3,4)}\rangle \\
|1\rangle = |\text{MF on (1,2), no MF on (3,4)}\rangle
\]

* PRL 86 268(2001)
With the above convention
\[
|0\rangle \equiv |(MF)_{12}, 0_{34}\rangle, \quad |1\rangle \equiv |0_{12}, (MF)_{34}\rangle
\]
The braiding matrices are (all × arb. overall Abelian phase factors)
\[
\hat{R}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \hat{R}_{34} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and}
\hat{R}_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \exp -i \frac{\pi}{4} \hat{\sigma}_x
\]
i.e., multiplying \( R_{12}, \ R_{34} \) by \( \exp \pm i\pi/4 \),
\[
\hat{R}_{12} = \hat{R}_{34}^{-1} = \exp i \frac{\pi}{4} \hat{\sigma}_z, \quad \hat{R}_{23} = \exp -i \frac{\pi}{4} \hat{\sigma}_x
\]
The R’s so defined trivially satisfy the first braid-group relation,
\[
[\hat{R}_i, \hat{R}_j] = 0 \quad \text{for } |i - j| \geq 2 \quad R_i \equiv R_{i,i+1}
\]
Do they satisfy the second relation,
\[
\hat{R}_i \hat{R}_{i+1} \hat{R}_i = \hat{R}_{i+1} \hat{R}_i \hat{R}_{i+1} \quad ?
\]
Yes! \( \hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \frac{1}{\sqrt{2}} (\hat{\sigma}_x - \hat{\sigma}_z) = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23} \)

2 QUBITS

The simplest way of generating 2 qubits is to use \( 2n = 6 \) anyons, and associate qubit 1 with the states of the anyon pair (1,2) and qubit 2 with pair (5,6). For definite, e.g. odd, parity the resulting Hilbert space is \( 2^{n-1} = 4 \) dim’l, as it should be. The pair (3,4) is used, as before, to “soak up” any remaining parity. So, e.g., we can take the 4 states to be, for odd \( N \),
\[
0_10_2 \equiv |0, 1, 0\rangle, \quad 1_10_2 \equiv |1, 0, 0\rangle,
0_11_2 \equiv |0, 0, 1\rangle, \quad 1_11_2 \equiv |1, 1, 1\rangle
\]
What is the effect of braids \( \hat{R}_{i,i+1} \)?

As before,

\[
\hat{R}_{12} = \exp i \frac{\pi}{4} \hat{\sigma}^{(1)}_z \quad \text{and sim.} \quad \hat{R}_{56} = \exp i \frac{\pi}{4} \hat{\sigma}^{(2)}_z
\]

Moreover, just as before

\[
\hat{R}_{23} = \exp -i \frac{\pi}{4} \hat{\sigma}^{(1)}_x \quad \text{and sim.} \quad \hat{R}_{45} = \exp i \frac{\pi}{4} \hat{\sigma}^{(2)}_x
\]

But what is the effect of \( \hat{R}_{34} \)? Acting on the "pseudoqubit" (3,4)

\[
\hat{R}_{34} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} (\times \text{abelian phase})
\]

Hence:

\[
\hat{R}_{34}|1\ 0\ 0\rangle = |1\ 0\ 0\rangle, \quad \hat{R}_{34}|0\ 0\ 1\rangle = |0\ 0\ 1\rangle,
\]

but \( \hat{R}_{34}|0\ 1\ 0\rangle = i|0\ 1\ 0\rangle, \quad \hat{R}_{34}|1\ 1\ 1\rangle = i|1\ 1\ 1\rangle,
\]

Thus in "2-qubit" basis \( |1\ 0\ 0\rangle \rightarrow |1\ 0\ 2\rangle \), etc.

\[
\hat{R}_{34} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \equiv \exp i \frac{\pi}{2} (\hat{\sigma}_1 \hat{\sigma}_2 + 1)
\]

\[\uparrow \]

2-particle entangling gate!

\[\Rightarrow\] 6 anyons permit large set (but not complete one) of 1- and 2-qubit operations.
Consider the Laughlin ansatz formally corresponding to $\nu = \frac{1}{2}$:

$$\Psi_N^{(L)} \prod_{i<j} (z_i - z_j)^2 \exp - \sum_i |z_i|^2 / 4l_m^2$$

This cannot be correct as it is symmetric under $i \leftrightarrow j$. So must multiply it by an antisymmetric function. On the other hand, do not want to ”spoil” the exponent 2 in numerator, as this controls the relation between the LL states and the filling.

Inspired guess (Moore & Read, Greiter et. al.): ($N = \text{even}$)

$$\Psi_N = \Psi_N^{(L)} \times \text{Pf} \left( \frac{1}{z_i - z_j} \right)$$

$$\text{Pf} (f(ij)) \equiv f(12)f(34) \ldots - f(13)f(24) \ldots + \ldots (\equiv \text{Pfaffian})$$

$\uparrow$

antisymmetric under $ij$

With this GS, a single quasihole is postulated to be created, just as in the Laughlin state, by the operation

$$\Psi_q h = \left( \prod_{i=1}^N (z_i - \eta_0) \right) \cdot \Psi_N$$

It is routinely stated in the literature that ”the charge of a quasihole is -$e/4$”, but this does not seem easy to demonstrate directly: the arg’ts are usually based on the BCS analogy (quasihole $\leftrightarrow \hbar/2e$ vortex, extra factor of 2 from usual Laughlin-like considerations) or from CFT (conformal field theory). 2 qps are more interesting.
Ansatz for 2 - quasihole states (as usual, a guess!)

\[ \Psi_{2qh} = \Psi_N^{(L)} \text{Pf} \left\{ \frac{(z_i - \eta_1)(z_j - \eta_2) + (\eta_1 \leftrightarrow \eta_2)}{z_i - z_j} \right\} \]

(\neq \text{polynomial} \left\{z_i\right\} \times \text{GS} (\text{unlike Laughlin case}))

General belief (from CFT, numerics *):
adiabatic exchange \( \eta_1 \leftrightarrow \eta_2 \) gives Berry phase \( \pi/4 \).

Convenient to incorporate explicitly:

\[ \Psi_{2qh} = (\eta_1 - \eta_2)^{1/4} \Psi_N^{(L)} \text{Pf} \ldots \]

Case of 4 quasiholes

Start with \( N = 2 \), then a possible w.f. is

\[ \Psi_{4qh} = \Psi^{(L)}(z_1 - z_2)^{-1} \times \{ (z_1 - \eta_1)(z_1 - \eta_2)(z_2 - \eta_3)(z_2 - \eta_4) + (z_1 \leftrightarrow z_2) \equiv (12)(34) \]}

But at first sight \( \exists \) two other possibilities, namely (13)(24) and (14)(23). However, now we note the identity

\[ (12)(34) - (13)(24) = (z_1 - z_2)^2(\eta_1 - \eta_3)(\eta_2 - \eta_3) \]

from which it follows that

\[ (12)(34)(\eta_1 - \eta_2)(\eta_3 - \eta_4) + (13)(24)(\eta_1 - \eta_3)(\eta_2 - \eta_4) + (14)(23)(\eta_1 - \eta_4)(\eta_2 - \eta_3) = 0 \]

i.e. only 2 linearly independent functions. This result also holds for the generalization for \( N > 2 \): we define

\[ \Psi_{(13)(24)} = \Psi_N^{(L)} \cdot \text{Pf} \left\{ (z_i - \eta_1)(z_i - \eta_3)(z_j - \eta_2)(z_j - \eta_4) + (i \leftrightarrow j)(z_i - z_j)^{-1} \right\} \]

Again, the three ”permutations” \( \Psi_{(13)(24)} \), \( \Psi_{(12)(34)} \) and \( \Psi_{(14)(23)} \) are not linearly ind\( t \): there are only 2 ind\( t \) linear combinations.

A convenient choice of basis is

\[
|0\rangle = \frac{(\eta_{13}\eta_{24})^{1/4}}{\sqrt{1 \pm \sqrt{x}}} (\Psi_{(13)(24)} \pm \sqrt{x}\Psi_{(14)(23)})
\]

\[
\eta_{ij} \equiv \eta_i - \eta_j, \quad x \equiv \frac{\eta_{14}\eta_{23}}{\eta_{13}\eta_{24}}
\]

Effect of braiding:
\(\hat{R}_{12}\) (or \(\hat{R}_{34}\)) effects \(x^{-1}\), \(\Psi_{(13)(24)} \leftrightarrow \Psi_{(14)(23)}\)

Hence:

\[
\hat{R}_{12}|0\rangle = |0\rangle \quad \Rightarrow \hat{R}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}
\]

Calculation of \(\hat{R}_{23}\) requires the linear-dependence relation

\[
(12)(34) - (14)(23) = x \{ (12)(34) - (13)(24) \}
\]

which gives

\[
\hat{R}_{23} = \exp i\frac{\pi}{4} \exp i\frac{\pi}{4} \hat{\sigma}_x \equiv \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}
\]

Thus: almost* exact equivalence to (p+ip) Fermi superfluid with definite parity!

Generalization to 6 (or 2n) anyons proceeds similarly†:
e.g. for 6 anyons (2 qubits) only entangling braid is

\[
\hat{R}_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[\text{cf : } \hat{R}_{34} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}\]

* \(\hat{R}_{12} = \hat{R}_{34}\) not \(\hat{R}_{34}^{-1}\), as in (p+ip)