1. No (micro- or macro-) system is ever truly isolated \[ U = S + E \] (1)

\[ U = \text{“universe”}, \ S = \text{“system”}, \ E = \text{“environment”}, \ C = \text{“control”}. \] All of these are in principle quantum-mechanical. However:

2. When can we treat \( C \) (and possibly part of \( E \)) as a deterministic classical object? Crudely speaking, when according to a full quantum-mechanical calculation a (substantial) change in the state of \( S \) is not accompanied by any substantial change in \( C \).

Ex: Creation of polarization of an atom. \[ \langle s | \Pi | s \rangle = \langle p | \Pi | p \rangle = 0. \] (2)

\[ \Rightarrow \text{must create a quantum superposition} \ \alpha |s\rangle + \beta |p\rangle, \text{more generally} \] \[ \hat{\Pi} = \text{Tr} \left( \hat{\rho} \hat{\Pi} \right). \] (4)

solution:

\[ \Psi_{\text{in}} = |\text{coh}\rangle |s\rangle \] (5)

then

\[ \Psi_{\text{f}} = \alpha |\text{coh}\rangle |s\rangle + \beta'(|\text{coh}\rangle |p\rangle + (|\perp\rangle)) \] (6)

\[ \simeq |\text{coh}\rangle (\alpha |s\rangle + \beta |p\rangle) \Rightarrow \langle \Pi \rangle \neq 0. \] (7)

where \( |\perp\rangle \) is orthogonal to \( |\text{coh}\rangle \). If \( C \) is approximately in a coherent state, then \( \langle \perp | a |\text{coh}\rangle \sim N^{-1/2} \). (Ex: for typical radio wave, \( N \sim 10^{30} \)).

3. When can we treat \( E \) in terms of a random classical force? Suppose \( E \) is known to be harmonic (e.g. blackbody radiation field) and moreover to be in thermal equilibrium. Then there exists a theorem:
Any thermal equilibrium state of a simple harmonic oscillator, or set of simple harmonic oscillators, can be represented as an incoherent mixture of coherent states [Digression: $P$-representation of quantum oscillators].

If $k_B T \gg \hbar \omega$, then coherent states represented in mixture are mainly $\alpha \gg 1$ (large-amplitude), so they can be treated classically. Thus, we can specify the effects of $E$ by a classical statistical representation.

**NB:** It is a sufficient set of conditions that (a) $E$ is harmonic, and (b) $k_B T \gg \hbar \omega$, $\forall$ relevant $\omega$. It is not clear that the first condition is necessary.

**Ex:** classical Brownian motion (Einstein, 1905):

$$M \ddot{x} + \eta \dot{x} \left( + M \omega_0^2 x \right) = F(t) \quad \leftarrow \text{classical fluctuating force} \quad (8)$$

Here, $\eta \dot{x}$ is a steady force due to $E$. For consistency (want to ensure $M \ddot{x}^2 = k_B T$ in thermal equilibrium) we must set

$$\langle F(t) \rangle = 0, \quad \langle F(t) F(t') \rangle = 2 \eta k_B T \delta(t - t'). \quad (9)$$

Another example: spin subject to a fluctuating magnetic field (note in this case $S$ is definitely quantum!).

4. **Quantum system in contact with a quantum environment**

**NB:** Attempts in literature to describe $S$ by phenomenological equation valid only (if at all) for a simple harmonic oscillator. (“most semiclassical system”). In general, we must treat $S$ and $E$ as interacting, highly entangled, systems. In principle:

$$\begin{cases} 
\text{start with } \hat{\rho}_U (\text{possibly } \hat{\rho}_S \otimes \hat{\rho}_E, \text{though see below}), \\
\text{evolve } \hat{\rho}_U \text{ with } \hat{U}(t) \equiv \exp -i \hat{H} t, \quad \hat{H} \equiv \hat{H}_S + \hat{H}_E + \hat{H}_{SE}, \\
\text{trace over } E \text{ to obtain the reduced density matrix } \hat{\rho}_S. 
\end{cases} \quad (10)$$

In practice, there are two major problems:

(1) do we know $\hat{H}_{SE}$ (or for that matter, $\hat{H}_S$)?
(2) can we carry out the calculation explicitly?

For the moment, assume we do know $\hat{H}_{SE} \equiv \hat{V}$.

5. **The weak-coupling limit: The quantum master equation**

Situation characteristic of quantum oscillators: may apply to some condensed-matter systems (e.g. nuclear spins coupled to phonons). Return below to how “weak” coupling $\hat{V}$ must be
The usual choice of initial density matrix of \( U \): \( \dot{\rho}_U = \rho_E (t) \rho_S (t) \), \( \dot{\rho}_E = \) equilibrium thermal density matrix corresponding to \( \dot{\hat{H}}_E \). \( \dot{\rho}_S (t) \) is arbitrary. With this factorization approximation, the reduced density matrix is \( \dot{\rho}_S (0) \equiv \dot{\rho}_S^{(i)} \).

Consider evaluation of \( \dot{\rho}_U (t) \):

\[
\dot{\rho}_U (t) = \dot{\rho}_U (0) + \frac{1}{i\hbar} \int_0^t dt' \left[ \dot{V} (t'), \dot{\rho}_U (0) \right] - \frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} dt'' \left[ \dot{V} (t'), \left[ \dot{V} (t''), \dot{\rho}_U (t'') \right] \right]
\]

(13)

and take the derivative with respect to \( t \):

\[
\frac{d\dot{\rho}_U}{dt} = \frac{1}{i\hbar} \left[ \dot{V} (t), \dot{\rho}_U (0) \right] - \frac{1}{\hbar^2} \int_0^t dt' \left[ \dot{V} (t'), \left[ \dot{V} (t''), \dot{\rho}_U (t'') \right] \right]
\]

(14)

This is still exact, and we can take the trace over \( E \) to obtain an equation for \( \dot{\rho}_S \). (Note that the time derivative and the trace over \( E \) commute!). For most practical cases, we find that \( \text{Tr}_E \dot{V} (t) \dot{\rho}_U (0) = \text{Tr}_E \left( \dot{V} (t) \dot{\rho}_E \right) \dot{\rho}_S^{(i)} \) vanishes (\( \text{Tr}_E \left( \dot{V} (t) \dot{\rho}_E \right) = \text{Tr}_E \left( \dot{\rho}_E \dot{V} \right) \)) since \( \dot{\rho}_E (t) = \dot{\rho}_E \), so:

\[
\frac{d\dot{\rho}_S}{dt} = -\frac{1}{\hbar^2} \text{Tr}_E \int_0^t dt' \left[ \dot{V} (t'), \left[ \dot{V} (t''), \dot{\rho}_U (t'') \right] \right]
\]

(15)

We now make two important approximations:

(1) Weak-coupling approximation: we can neglect entanglement between \( S \) and \( E \), i.e., put

\[
\dot{\rho}_U (t') \simeq \dot{\rho}_S (t') \dot{\rho}_E (t') \simeq \dot{\rho}_S (t) \dot{\rho}_E (t) \quad \text{(since bath is “large”)}
\]

(16)

\[
\Rightarrow \frac{d\dot{\rho}_S}{dt} = -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr}_E \left[ \dot{V} (t'), \left[ \dot{V} (t''), \dot{\rho}_E \dot{\rho}_S (t'') \right] \right]
\]

(17)

Typically, the interaction \( \dot{V} (t) \) will involve a sum of terms each of which is a product of some operator \( \dot{Q}_i \) of the system and an operator \( \dot{\Omega}_i \) of the environment. Because the trace of a commutator vanishes identically, most of the terms
arising from the double commutator in Eq. (17) are zero, and one ends up very schematically with an equation of the form

\[
\frac{d\hat{\rho}_S}{dt} = -\frac{1}{\hbar^2} \sum_{ij} \int_0^t dt' \langle \Omega_i(t)\Omega_j(t') \rangle \left[ \dot{\Omega}_i(t), \left[ \dot{\Omega}_j(t'), \hat{\rho}_S(t') \right] \right]
\]

(18)

where \( \langle \hat{A}\hat{B} \rangle \equiv \text{Tr} \hat{A}\hat{B}\hat{\rho}_E \).

(2) Markovian approximation: we suppose that the quantity \( \langle \Omega_i(t)\Omega_j(t') \rangle \) has some characteristic range \( \tau_c \) in \( (t-t') \). Then if \( \hat{\rho}_S(t) \) is assumed to vary slowly over \( \tau_c \), we can approximate \( \hat{\rho}_S(t') \) by \( \hat{\rho}_S(t) \) and extend the integral over \( t' \) to \( \infty \). Thus, again schematically,

\[
\frac{d\hat{\rho}_S}{dt} = -\frac{1}{\hbar^2} \sum_{ij} K_{ij} \left[ \dot{\Omega}_i, \left[ \dot{\Omega}_j, \hat{\rho}_S(t) \right] \right]
\]

(19)

where \( \dot{\Omega}_i \) and \( \dot{\Omega}_j \) are now the Schrödinger-representation (time-independent) operators and

\[
K_{ij} \sim \int_0^\infty dt' \langle \Omega_i(0)\Omega_j(t') \rangle \exp(-i\omega_{ij} t') \quad \text{[needs amplification]}
\]

(20)

The best-known example in quantum optics is the situation of a single oscillator of frequency \( \Omega \) coupled to cavity modes: then the double commutator gives rise to a characteristic structure

\[
\frac{d\hat{\rho}_S(t)}{dt} \sim a_0\hat{\rho}_S a_0^\dagger - \frac{1}{2}a_0 a_0^\dagger \hat{\rho}_S - \frac{1}{2} \hat{\rho}_S a_0 a_0^\dagger + \cdots
\]

(21)

Note that in general no evolution equation of the form of the quantum master equation need exist.

6. What if we do not know the form of \( \hat{H}_{SE} \) (or perhaps of \( \hat{H}_E \)) in detail? Let \( q, p \) denote the coordinate and canonical momentum of \( S \), and \( \xi_i, \pi_i \) those of \( E \)

\[
\hat{H}_S \equiv \hat{H}_S(p, q) \quad \text{(22)}
\]

\[
\hat{H}_E \equiv \hat{H}_E(\xi_i, \pi_i) \quad \text{(23)}
\]

\[
\hat{H}_{SE} \equiv \hat{H}_{SE}(p, q; \xi_i, \pi_i) \quad \text{(24)}
\]

Suppose that for “relevant” values of \( p \) and \( q \), only one degree-of-freedom of the environment is only weakly perturbed. Then, trivially, (cf. 19th-century “oscillator” model of the atom)

\[
\hat{H}_E \to \sum_i \left( \frac{\hat{p}_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2\hat{x}_i^2 \right)
\]

(25)
What about $\hat{H}_{SE}$? In this limit, quite generally,

$$\hat{H}_{SE} = \sum_i \{ F_i(p,q) \dot{x}_i + G_i(p,q) \dot{p}_i \} + \Phi(p,q)$$  \hspace{1cm} (26)$$

$\Phi(p,q)$ may depend on oscillator parameters, but not on $\{x_i, p_i\}$. ($F_i, G_i$ are related to the master equations of the original interaction).

We can motivate the form of $F_i(p,q), G_i(p,q)$

(a) from symmetries: e.g. if time-reversal invariant, then (can choose degrees of freedom of $x_i, p_i$ such that) $F_i(p,q)$ is even under time reversal, $G_i(p,q)$ odd.

(b) from requirements on the classical equations of motion: e.g. suppose the classical damped equation of motion is of the form

$$M \ddot{q} + \eta \dot{q} + V'(q) = 0$$  \hspace{1cm} (27)$$

then “barring pathology” $G_i(p,q) = 0$ and $F_i(p,q) = q C_i$. This then leads to a simple form

$$\hat{H}_{SE} = q \sum_i C_i x_i \hspace{1cm} (+\text{counterterm})$$  \hspace{1cm} (28)$$

Generalization:
(a) $\eta = \eta(q)$ but not frequency-dependent, then $C_i \rightarrow f_i(q)$.
(b) $\eta$ is frequency-dependent, but not $\eta(q)$, then a simple form is OK (but $J(\omega)$ [see below] is more complicated).

If $\eta$ is both frequency- and amplitude-dependent, this is a can of worms!

(What do we do for a bath when one degree-of-freedom is not necessarily weakly perturbed? No general procedure known).

**Important observation:**

All effects of linearly-responding environments on the motion of a system are encapsulated in a single function

$$J(\omega) \equiv \frac{\pi}{2} \sum_i \frac{C_i^2}{m_i \omega_i} \delta(\omega - \omega_i).$$  \hspace{1cm} (29)$$

The quantity $J(\omega)$ can often be read off from the equation of motion obeyed by the system in the classical limit. In particular, for linear frequency-independent friction described by a term $-\eta \dot{q}$ in the equation of motion,

$$J(\omega) = \eta \omega$$  \hspace{1cm} (30)$$
In the general case of a shunting “black box” described by an admittance $Y(\omega)$, we have (AJL, Phys. Rev. B 30, 1208 (1984))

$$J(\omega) = \text{Im} (i\omega Y(\omega))|_{\text{Im}\omega \to 0}$$

(31)

7. Calculations based on the oscillator-bath model

Any number of possibilities: e.g. to obtain classical damped equation of motion, simply write down the equations of motion for $U$ and eliminate $E$ (i.e., the oscillator bath).

Most flexible is probably the Feynman-Vernon functional integral technique (applicable to real-time and imaginary-time (i.e. tunneling) problems). E.g. for simple ohmic dissipation the action which goes in the imaginary-time functional integral is simply

$$S_{\text{eff}} \{q(\tau)\} = \int_0^t d\tau \left\{ \frac{1}{2} m \dot{q}^2 + V(q) \right\} + \frac{\eta}{4\pi} \int \int d\tau d\tau' \frac{(q(\tau) - q(\tau'))^2}{(\tau - \tau')^2}$$

(32)

so the WKB exponent $S_{\text{eff}}/\hbar$ is always increased by the dissipation.

Other results: spin-boson model, more general two-state systems ($T_1, T_2$)

Further references: