1. (a) An atom in an excited state can interact with the radiation field in the process of decaying and thus become entangled with it. Suppose that the atom is initially described by a reduced density matrix $\hat{\rho}_{at}$ which is diagonal in the (isolated atom) energy representation and such that the probabilities of the three lowest states is in the ratio 1:0:1. Describe qualitatively how the entropy $S_{at} (\equiv -\text{Tr} \hat{\rho}_{at} \ln \hat{\rho}_{at})$ varies as a function of time, and in particular find an upper limit on the maximum value it attains. Is $dS_{at}/dt$ always positive? If not, is any paradox involved? Why (not)?

(b) Crudely speaking, for a two-level system interacting with a bath of oscillators phase coherence between the two states will tend to be destroyed by either emission or absorption of a bath quantum\(^1\) (cf. part (a)), while energy relaxation will be governed by the competition of emission and absorption. Use this consideration to show that for a two-level system, with usual definitions, $T_1$ and $T_2$ must be of the same order of magnitude.

(c) Now consider a simple quantum harmonic oscillator of frequency $\omega_0$ with a bath of other oscillators. Assume the coupling between the system and bath to be linear in both system and bath coordinates. The initial state is a superposition of two narrow wave packets (width $\sim$ the zero-point spread $x_{\text{rms}}$) momentarily stationary and displaced relative to each other by $\Delta x \gg x_{\text{rms}}$. Defining $T_1$ as the relaxation time of the energy and $T_2$ as the time for destruction of phase coherence between the two packets, use an argument similar to that in part (b) to estimate the (order-of-magnitude) ratio of $T_2$ to $T_1$ as a function of $\Delta x$ and temperature.

2. Consider a dust particle of mass $10^{-9}$ grams; to make order-of-magnitude estimates, use a “typical” density and the fact that the viscosity of air under ‘standard’ conditions is $1.8 \times 10^{-5} \text{kg/s} \cdot \text{m}$. We will be interested in the off-diagonal elements $\rho(X, X': t)$ in position representation, which are a measure of the ‘coherence’ between the behavior of the (center of mass of the) particle at different points in space.

(a) Find the form (up to normalization) of $\rho(X, X': t)$ if the particle is in thermal equilibrium at temperature $T$ [Hint: $\rho(X, X': t) = \rho(X-X': t)$ is the Fourier transform of the momentum distribution]. Estimate $\langle (X - X')^2 \rangle$ at room temperature and compare with the size of the particle.

(b) Suppose that the initial value of $\langle (X - X')^2 \rangle$ is much larger than the result obtained in (a); then the initial behavior of $\rho(X, X': t)$ is adequately described by the equation (derived in 1.2): \[ \frac{d\hat{\rho}}{dt} = -\frac{\eta k_B T}{\hbar^2} \left[ \hat{X}, \left[ \hat{X}, \hat{\rho} \right] \right] \] Show that the initial rate of decay of $\rho(X, X': t)$ is proportional to $\langle (X - X')^2 \rangle$ and find the constant of proportionality.

\(^1\)Assuming we can neglect “pure phase” decoherence, that is the decoherence associated with oscillators whose energy tends to zero (any such effect will be in addition to the one calculated here).
3. In standard Landau Fermi liquid theory, one considers a set of identical nonrelativistic fermions of spin 1/2 described by a Hamiltonian $\hat{T} + \hat{V}$, where $\hat{T}$ is the kinetic energy and

$$\hat{V} \equiv \frac{1}{2} \sum_{ij} V(|\mathbf{r}_i - \mathbf{r}_j|).$$

Some conclusions of the theory are (where $\delta n(p\sigma)$ is the quasiparticle distribution, see lecture 3):

1) $\delta N = \sum_{p\sigma} \delta n(p\sigma)$

2) $S_z = \frac{1}{2} \sum_{p\sigma} \sigma \delta n(p\sigma)$

3) $P = \sum_{p\sigma} p \delta n(p\sigma)$

4) $\chi = \text{independent of } T$

5) $c_v = \text{const.} \times T$

Which of these conclusions would you expect to remain true if the Hamiltonian is modified so as to incorporate

(a) a periodic lattice potential,

(b) scattering by spinless impurities,

(c) spin-orbit scattering?

(give a very brief justification in each case. Assume in each case that the system remains a Fermi liquid (no band filling or localization)).