Lesson

THE TWO GOLDEN RULES OF QUANTUM MECHANICS WITH LIGHT POLARIZATION

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Outline

THE TWO GOLDEN RULES OF QUANTUM MECHANICS

ACTIVITY GOAL:
Investigate the superposition and measurement principle using the polarization of light

LEARNING OBJECTIVES

- The quantum nature of light polarization.
- Mutually exclusive states and measurement choice.
- The superposition and measurement principles in quantum mechanics
- Probabilistic nature of quantum mechanics and probability amplitudes.

ACTIVITY OUTLINE

We start by defining **mutually exclusive states** of polarization, and linking the wave and particle pictures through Malus' law.

CONCEPT:
Quantum measurement is a probabilistic process.

We then show how any polarization state can be broken down into components and thought of as a **superposition** of other polarization states.

CONCEPT:
Superposition is relative to the measurement context.

By introducing more polarizers, we observe that measurements can change a quantum state, and explain it using the idea of **wave collapse**.

CONCEPT:
Measuring an unknown quantum state will change it.

PREREQUISITE KNOWLEDGE

Light is made of particles called **photons.**
Light is an **electromagnetic wave** with a **polarization.**
Cartesian co-ordinates and trigonometry.

SUPPLIES REQUIRED

3 polarizers with labelled orientations

Lesson

THE TWO GOLDEN RULES OF QUANTUM MECHANICS

PHOTONS AND THE WAVE-PARTICLE DEBATE

In classical physics, we can neatly separate objects into two categories: particles and waves. In quantum mechanics, that distinction no longer holds. In its simplest form, quantum mechanics is based on the idea that **everything is made of particles that have wave-like properties** – this is the **wave-particle duality.** In order to fully appreciate wave-particle duality, we need to highlight some fundamental differences between the two:

Particles	Waves
Exist at one place (localized)	Exist over a large space (delocalized)
Have well-defined properties like mass and volume	Have well-defined properties like wavelength and frequency
Have kinetic collisions	Show wave interference
Are countable	Are continuous

Light is a perfect example of how quantum mechanics doesn't always allow for things to fit into such neat categories. Light behaves like a wave, in that it has a well-defined frequency, can spread out over a large space, and exhibits interference rather than kinetic collisions with other light.

But, as learned from Planck's studies of blackbody radiation and Einstein's explanation of the photoelectric effect, light is made up of indivisible, **countable** "chunks" called **photons**.

In this lesson, we'll consider what it means for a photon to behave like a wave through its **polarization**. We'll discover the rules of quantum superposition and measurement, and how they connect to the wave-particle nature of a photon.

Schrödinger's Class
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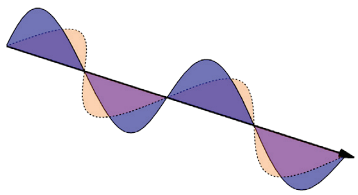
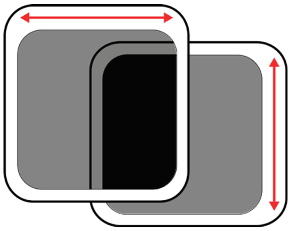
POLARIZATION AND MALUS' LAW



Recall that light is an **electromagnetic wave**, carrying an oscillating electric field perpendicular to the direction it travels in. In the image to the right, the black arrow shows the direction of the light wave. The electric field could be oscillating vertically up-and-down (blue), horizontally side-to-side (orange), or anywhere in between. The direction of the electric field is called the light's **polarization**.



Most objects don't care about the polarization of light that they interact with. **Polarizers** are special sheets of material that absorb or transmit light depending on its polarization. If we hold a polarizer vertically, it will absorb all horizontally polarized light and transmit all vertically polarized light. If we place another polarizer after the first one, we'll find that all the light after the first is now vertically polarized.

For classical polarized light, such from a laser pointer or an LCD computer monitor, we can calculate the intensity of light that makes it through the polarizer using **Malus' Law**. It states that the intensity of light I_{out}, in Watts, that makes it through the polarizer is given by:

$$I_{out} = I_{in} \cos^2 \theta,$$

where I_{in} is the intensity of the incoming light and θ is the angle between the polarization of the incoming light and the axis of the polarizer. The polarizer absorbs the rest of the light.

This is consistent with the **wave** picture of light, as the wave component parallel to the polarizer transmits, and the one perpendicular to the polarizer is absorbed. But we know that ultimately, light is made up of indivisible "chunks" – photons. The intensity of light is directly related to the number of photons per second, but what if we send a **single** photon of light to a polarizer?

Consider a single photon at a polarizer. There are only two possible things that can happen:

A. The photon goes through;
B. The photon is absorbed

However, when we have many photons at the same time, we must get Malus' law back. We can resolve this problem if each photon has a certain **probability** of going through the polarizer, and a probability of being absorbed. If the photon's polarization is an angle θ from the polarizer, the probability of being transmitted is:

$$Prob(transmit) = \cos^2 \theta$$

1. What is the probability that the photon is absorbed by the polarizer?

2. Suppose horizontally polarized **classical light** is sent to a polarizer at an angle θ from the horizontal axis. What is the ratio of the input to output intensity?

θ	$0°$	$45°$	$-45°$	$-30°$	$60°$	$90°$
I_{out}/I_{in}						

3. Suppose horizontally polarized **photons** are sent to a polarizer at an angle θ from horizontal. What is the probability of the photon transmitting or being absorbed?

θ	$0°$	$45°$	$-45°$	$-30°$	$60°$	$90°$
Probability transmitted						
Probability absorbed						

Malus' law is a consequence of the superposition principle and the probabilistic nature of quantum mechanics. By the end of this activity, we'll be able to answer the key question: **just what does a superposition mean**?

To answer this, we'll need to describe light polarization using **exclusive states**.

MUTUALLY EXCLUSIVE STATES

The **state** of an object defines its properties at a specific time. For example, if we flip a coin, we might find it lands in the state “heads”.

Two states are **mutually exclusive** if being found in one state means you definitely aren’t in another. In other words, an object can’t be in both states at the same time. For example, if we find the coin in the state “heads”, we know with certainty it is not in the state “tails”. “Heads” and “Tails” are therefore mutually exclusive states.

1. Brainstorm other states that are mutually exclusive to each other.
2. Take a vertical polarizer and look through it. Take a second polarizer rotated at 90° and put it front of the first one. How much light coming out of the vertical polarizer makes it through the horizontal one?
3. Are horizontal and vertical polarizations mutually exclusive? Why or why not?
4. Are vertical and diagonal (45°) polarizations mutually exclusive? Can you test this experimentally?
5. Is there any polarization state, other than horizontal, that is mutually exclusive to the vertical state?
6. Can you find a state that is mutually exclusive to the 45° polarization state? Test your prediction using the polarizers.
7. Can you think of other collections of states that would be mutually exclusive to each other?

MEASUREMENT BASIS

As we just saw, two perpendicular polarizations form a pair of mutually exclusive states. Let's re-examine the effect of the polarizer with this in mind.

If a horizontally polarized photon hits a horizontal polarizer, it will definitely pass through. If a vertical photon hits the same polarizer, it will definitely be absorbed. By looking for photons after the polarizer, we can tell if they were horizontally or vertically polarized.

Essentially, the horizontal polarizer is asking the photons a question: **are you horizontally or vertically polarized**?

In general, a polarizer performs a **measurement** that distinguishes between two mutually exclusive states!

1. What measurement does a vertical polarizer perform?
2. What measurement does a 45° polarizer perform?
3. What measurement does a -30° polarizer perform?

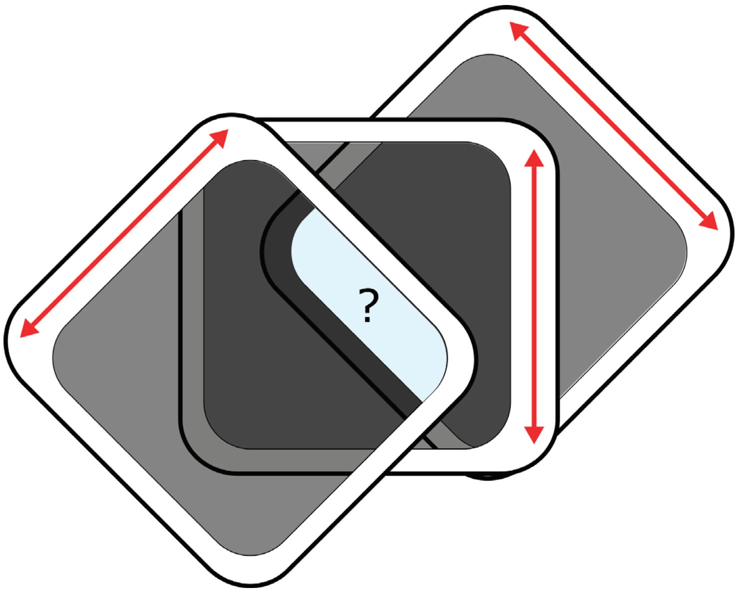
A pair of mutually exclusive states are referred to as a **measurement basis**.

For example, the horizontal and vertical polarizers both perform a measurement in the horizontal/vertical basis, often written in short-form as the HV basis. The 45° polarizer performs a measurement in the +45°/-45° basis, sometimes called the diagonal basis.

A SURPRISING PHENOMENON

If you hold a +45° and -45° polarizer together, you should see that no light goes through.

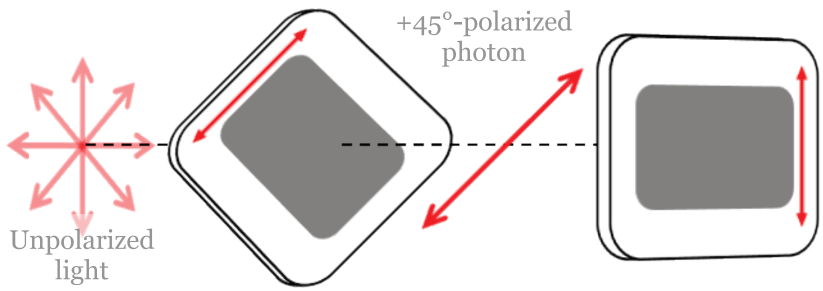
What happens when we introduce a third polarizer into the experiment?



1. What do you observe when you place a vertical polarizer **in front** of the diagonal polarizers? Does this match your expectation?
2. What do you observe when you place a vertical polarizer **in between** the two diagonal polarizers?
3. Can you explain this effect? With your group members, come up with a hypothesis.

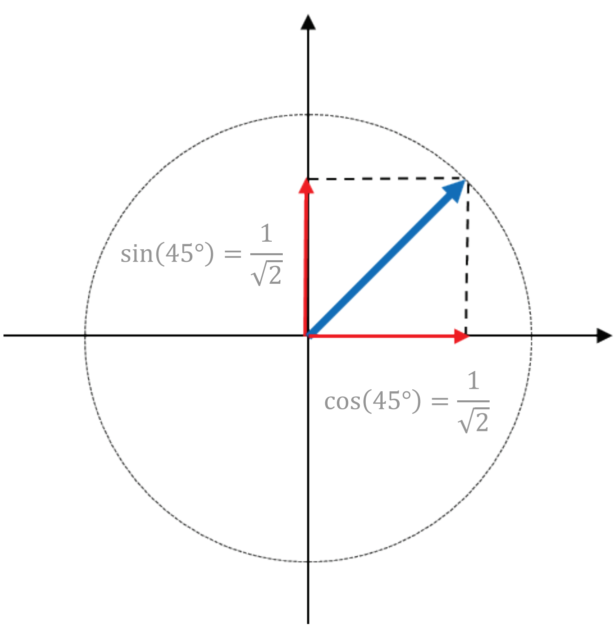
SUPERPOSITION AND MEASUREMENT

Let's analyze the situation below using what we've learned about photons, polarizers, and mutually exclusive states. In the first step, a +45° polarizer is used to prepare a +45°-polarized photon, which we then measure with a vertical polarizer.



We know that, after the first polarizer, all the photons will be +45°-polarized. The second polarizer performs a measurement in the HV basis. In other words, it asks the photons the question "**Are you in the horizontal or the vertical state?**"

The question cannot be answered immediately because the photon is +45°-polarized, which is neither horizontal or vertical. But any polarization state can be described using a combination of horizontal and vertical **components**.



We can describe +45° polarization as a unit vector, shown in blue.

It has an equal horizontal and vertical component, shown in red.

Let's investigate further using the Cartesian plane. We know that we can describe any point on the plane using only two coordinates: how much of "x" and how much of "y", or if you prefer, how much of a "**horizontal component**" and how much of a "**vertical component**".

We can directly relate polarization to the Cartesian plane. Think of the polarization state of the photon as a vector of length one starting at zero, or as a point on the unit circle centred on the origin.

We can describe polarization in any direction by decomposing that vector into horizontal and vertical components. For example, +45° polarization has a horizontal component of cos(45°), and a vertical component of sin(45°).

By breaking it into components, it is possible to describe +45°-polarized light as a combination of horizontally and vertically polarized light:

$$\nearrow = \frac{1}{\sqrt{2}} \rightarrow + \frac{1}{\sqrt{2}} \uparrow$$

A wave can have components oscillating in different directions at the same time. Recall, however, that the light is made of individual, indivisible photons. For the following questions, think about the situation as if we sent one single photon to the polarizers.

1. Using Malus' law, what is the probability that a +45°-polarized photon will make it through the vertical polarizer? What is the probability that it is absorbed?

2. Can you find a mathematical relationship between the vertical component of the +45° vector and the probability of passing through the vertical polarizer?

3. If a photon makes it through the vertical polarizer, what is its polarization state afterward?

Because it has both a horizontal and a vertical component, we call +45° polarization a **superposition** of horizontal and vertical polarizations. However, since there is only one photon, it can only be horizontal or vertical when we ask it which it is with the vertical polarizer.

When we measure the superposition in the HV basis, the results are impossible to predict! It will randomly **collapse** to either horizontal or vertical polarization, with the probabilities given by Malus' law.

We can summarize this with two rules that describe the behaviour of photons, electrons, and all objects that obey quantum mechanics: the **two golden rules of quantum mechanics**.

Rule #1: Superposition

A quantum object can be in a superposition of multiple states at once.

For example, a +45°-polarized photon is in a superposition of horizontal and vertical polarization states

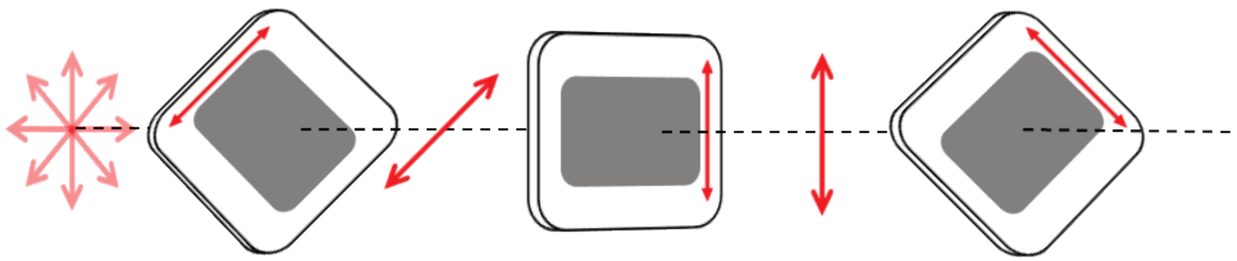
Rule #2: Measurement

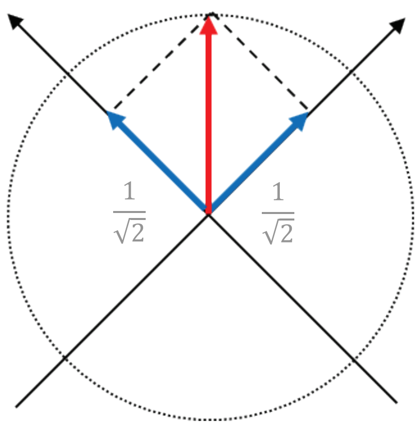
Rule #1 works as long as you don't look!

The act of measuring the superposition will collapse it and change the state.

For example, when we measure the +45°-polarized photon in the HV basis, it must collapse to either horizontal or vertical polarization

The third polarizer





The light that makes it through the first two polarizers is now vertically polarized. The final polarizer performs a measurement in the ±45° basis. In other words, it asks the photons the question: “**Are you in the +45° or -45° state?**”

We are in a situation similar to before, except this time we need to decompose the vertical polarization into +45° and -45° components, as in the plane on the left.

Just as with the HV basis in the previous step, we can describe vertically polarized light in terms of its components in the ±45° basis as:

$$\uparrow = \frac{1}{\sqrt{2}} \nearrow - \frac{1}{\sqrt{2}} \searrow$$

1. Using the quantum superposition rule (Rule #1), can you relate vertical polarization to its decomposition in the ±45° basis?
2. What is the probability of the vertical photon making it through the -45° polarizer, and how does it relate to its -45° vector component?
3. If a photon makes it through the -45° polarizer, what is its polarization state afterward? Can you relate this to the measurement principle (Rule #2)?

4. Inserting the vertical polarizer between the +45° and the -45° polarizers suddenly allowed light to pass through. Using the measurement principle (Rule #2), explain why the presence of an extra measurement changed the experiment.

5. The negative sign in front of the -45° component tells us the **phase** between the two components. For polarization, we can think of the negative sign flipping the arrow's direction.

   What polarization state does the following superposition represent?

$$\frac{1}{\sqrt{2}} \nearrow \ + \ \frac{1}{\sqrt{2}} \searrow = ?$$

We saw that 45° polarized light can be seen as a superposition of both horizontal and vertical polarizations. Similarly, we found that vertical polarization can be seen as a superposition of 45° and -45° polarization. This suggests that t**he concept of superposition is relative and depends on the context of the measurement.**

For example, 45° polarization must be seen as a superposition of horizontal and vertical if we are performing a measurement in the HV basis. But, if we were to perform a measurement in the ±45° basis instead, the 45° polarization state would not be considered a superposition!

Similarly, vertical polarization is not considered a superposition when we measure it in the HV basis, but must be considered a superposition when measured in the ±45° basis.

These effects can be seen in both the photon and the classical electromagnetic wave theories of light. Indeed, there is nothing "quantum" about superposition; **superposition is a natural property of waves**.

When a single photon exists in superposition, it behaves as if it is in two states at the same time. If we measure a 45°-polarized photon in the HV basis, its state is not well-defined, which is a **wave behaviour**. But once it is measured, it is definitely either horizontal or vertical. Its state becomes well-defined, which is a **particle behaviour**. This mixture of wave and particle behaviours in one quantum system is the hart of **wave-particle duality**.

FURTHER EXPLORATION

Let's generalize the concept of superposition.

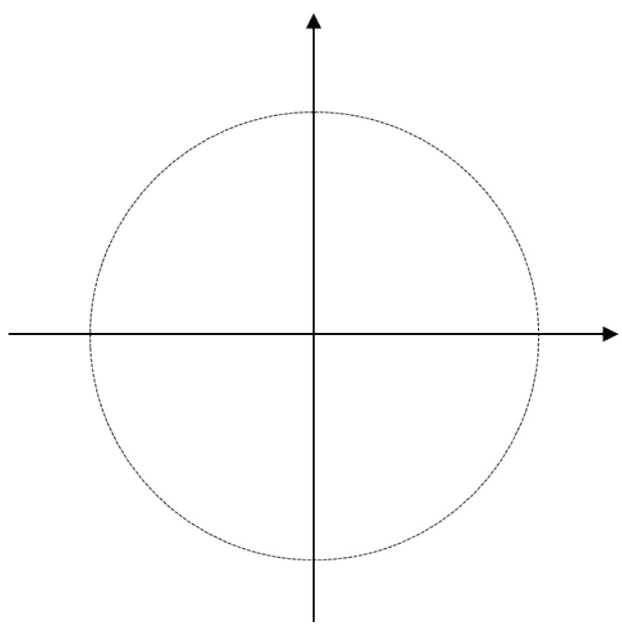
1. Suppose a single photon has a polarization vector with an angle θ with respect to the horizontal, such that $\theta = 0°$ corresponds to horizontal and $\theta = 90°$ corresponds to vertical polarization. You are measuring using a horizontal polarizer.

   Fill in the table below, using Malus' law to calculate probabilities. You can use the graph below the table to help find components.

	$\theta = 0°$	$45°$	$-45°$	$-30°$	$60°$	$90°$
H component		$1/\sqrt{2}$				
V component		$1/\sqrt{2}$				
Superposition		$\frac{1}{\sqrt{2}}\rightarrow + \frac{1}{\sqrt{2}}\uparrow$				
"H" probability		½				
"V" probability		½				

2. From observation and with the help of Malus' law, what is the relationship between the horizontal/vertical components and the probabilities of measuring the photon in the horizontal or vertical state?

3. For a photon with generic polarization angle θ, what is the probability of it being measured in the horizontal or vertical states? Give general formulas in terms of θ.



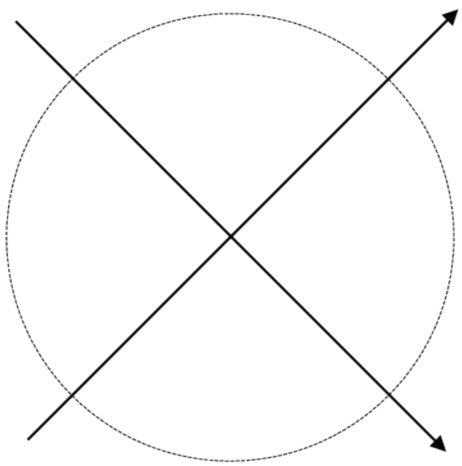
In general, it is possible to decompose any polarization state using any two mutually exclusive states. The probability of measuring each of the exclusive states is given by the absolute-value squared of the superposition component. We refer to that component as a **probability amplitude**.

4. Suppose that you measure with a polarizer aligned to the +45° diagonal instead. As before, the angle θ in each column corresponds to a different polarization state.

   Fill in the table below, once again using the axes below to help find components.

	$\theta = 0°$	$45°$	$-45°$	$-30°$	$60°$	$90°$
45° component	$1/\sqrt{2}$					
-45° component	$1/\sqrt{2}$					
Superposition	$\frac{1}{\sqrt{2}}\nearrow + \frac{1}{\sqrt{2}}\searrow$					
45° probability	½					
-45° probability	½					

5. For a photon with generic polarization angle θ relative to the horizontal, what is the probability of it being measured in the +45° or -45° states? Give general formulas in terms of θ.



THE QUANTUM WAVE FUNCTION

In this activity, we investigated the **quantum wave function**. We just saw how thinking of different polarization states in terms of their components allows us to write them as **superposition states**.

Classical waves also have components, but they represent intensity along a certain direction. In the quantum case, the components represent **probabilities** instead. If you square the coefficient (or probability amplitude) of a given component, you get the probability of measuring the photon in that state.

This idea extends beyond the polarization of photons. For example, a photon can be found in a specific location “x”, and each possible position is mutually exclusive with all others (if we find it at “x”, it is certainly not at position “y”). The position of the photon must have some **quantum uncertainty**, which is another way of saying it is in a superposition of many positions! The position wave function is defined by a function $\psi(x)$, and the square of the wave function gives the probability distribution of where to find the photon.

It also extends beyond photons to other quantum objects. Electrons have a property called **spin** which, like polarization, can be in one of two exclusive states (spin-up or spin-down). And just like polarization, these states can be put into superposition, and different measurement contexts can reveal different properties of the electrons.

APPLICATIONS AND TECHNOLOGY

Many quantum technologies rely on the fact that we can measure quantum systems in different ways.

Quantum computing works by using quantum systems built of two mutually exclusive states, like polarization, as the binary bits of a computer. These quantum bits, or **qubits**, can sometimes perform computational tasks more efficiently than classical computers by taking advantage of the fact that we can measure them in different bases (along with other properties, like quantum entanglement).

When using the three polarizers, we saw that introducing a measurement changed the state, allowing light to pass through. Let’s apply this idea to information security! If two people communicate using the polarization of single photons, and someone tries to eavesdrop by measuring in the middle, they’ll change the state of the photon, which can be measured. This idea is the key to **quantum key distribution**, which allows people to guarantee that their communications are not being read by anyone else.

QUANTUM QUIZ

1. Are each the following examples of mutually exclusive states? Why or why not?
    a. Whether a coin is heads or tails
    b. Whether a cup has a volume of 250 mL or is made of glass
    c. Whether a photon is diagonally polarized or horizontally polarized
    d. Whether a photon is +75°-polarized or -15°-polarized

2. 1000 horizontally polarized photons are sent to a polarizer at +30°. How many do you expect will make it through? Is this number exact?

3. What are the horizontal and vertical components of a photon polarized at +30° relative to the horizontal? Write the state as a superposition of horizontal and vertical components.

4. What are the +45° and -45° components of a photon polarized at +30°? Use the Cartesian plane to be sure whether the components are positive or negative, and write the photon's state as a superposition of +45° and -45° polarizations.

QUANTUM CONCEPTS

1. Noting that a laser beam is made up of many photons, relate Malus' law for the intensity of a laser beam through a polarizer to the probability rules for a single photon through a polarizer.

2. Explain how wave-particle duality is connected to the superposition and measurement principles.

3. Is a horizontally polarized photon in a superposition state? Explain why or why not.

4. Polarization can be used to encode one bit ("0" or "1") of information. Sketch how someone could two people could communicate by encoding photons and measuring their polarization.

QUANTUM LEAP: CHALLENGE QUESITONS

1. Consider the two different experiments.

   In Experiment A, we prepare a +45°-polarized photon and measure it with a -45° polarizer.

   In Experiment B, we flip a coin and prepare either a horizontally or vertically polarized photon, which we measure with a -45° polarizer.

   What is the probability of the photon passing through the -45° polarizer in each experiment? How does a superposition behave differently than randomly being in one state or another?

2. What is the probability that a horizontally polarized photon makes it through (in order) a 30° polarizer followed by a 60° polarizer, and vertical polarizer?

3. If we start with a horizontally polarized photon and pass it through N polarizers, starting at an angle of $^{90°}/_{N}$ and increasing in increments of $^{90°}/_{N}$ up to vertical (90°), what is the probability that the photon makes it through the vertical polarizer? What is the limit as N goes to infinity? This is called the "Quantum Zeno Effect".

GLOSSARY

- **Photons** are indivisible units of light. Each photon carries the smallest amount of electromagnetic energy possible, equal to
- **Polarization** is the direction the electric field oscillates in a beam of light. The polarization is always perpendicular to the direction the beam travels.
- A **state** defines the properties of an object. For example, a light switch can be in the “ON” state or the “OFF” state.
- A **quantum state** defines the properties of an object that obeys quantum mechanics, such as an electron or photon. Unlike a classical state, a quantum state may be fundamentally uncertain.
- **Mutually exclusive states** are two possible states that cannot both occur at the same time. More precisely, they are two states that can be perfectly distinguished by an ideal measurement. For example, “heads” and “tails” on a coin are mutually exclusive, as are the horizontal and vertical polarization states of a photon.
- **Measurement bases** or **measurement contexts** are different ways of measuring quantum objects that cannot be performed at the same time. For example, we cannot measure the position and the momentum of a photon at the same time, nor can we measure the horizontal/vertical and ±45° polarization at the same time. We can think of these as the different possible questions we can ask our quantum object.
- **Superposition** is the property of a quantum state having a component in more than one possible state in a set of mutually exclusive states. For example, a +45° polarized photon has both a horizontal and a vertical component, and can be considered a superposition of the two. All states are superposition states when considered in the context of different measurement bases.

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