Answer Key

Activity

THE UNCERTAINTY PRINCIPLE

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Schrödinger's Class

Answer Key

Outline

THE UNCERTAINTY PRINCIPLE

ACTIVITY GOAL: Measure and verify the uncertainty principle using laser pointers.

LEARNING OBJECTIVES

- Momentum of light.
- Tradeoff between position and momentum uncertainty.
- Using uncertainty to measure unknowns.

ACTIVITY OUTLINE

First, we'll examine the uncertainty principle in the context of single-slit diffraction, learning how to express the momentum of light.

CONCEPT:
Light carries momentum, which can have x, y, and z components.

After calibrating our equipment, we'll measure the momentum spread of light after passing through slits of known widths.

CONCEPT:
The more certain we are about position, the less certain we are about momentum.

We'll use our data to estimate the constant of proportionality in the uncertainty principle, and use that information to measure slits and barriers of unknown width.

CONCEPT:
Uncertainty itself can be used to make accurate measurements.

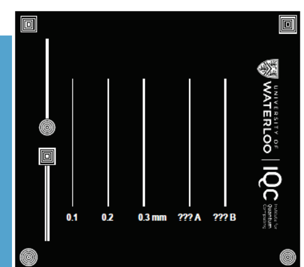
PREREQUISITE KNOWLEDGE

Diffraction and interference of light

SUPPLIES REQUIRED

Laser pointers (Red, Green, and Blue, optimally)
Diffraction grating with known lines-per-millimetre
Measuring tape and ruler
Paper, pen, and tape
Graph paper or plotting software
Uncertainty Tracing worksheet (optional)

Slits of known and unknown width, can be homemade with:
- Laser printer (best results with >600dpi resolution)
- Overhead transparency paper



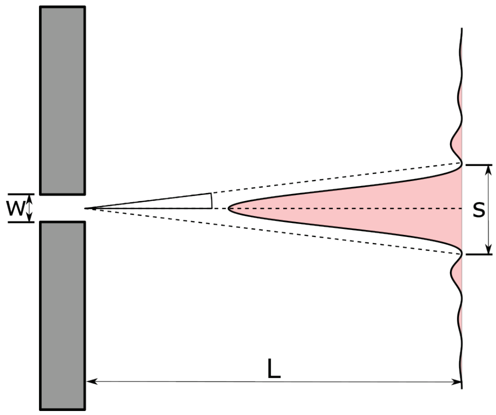
Activity

THE UNCERTAINTY PRINCIPLE

POSITION AND MOMENTUM OF LIGHT

A laser beam is a directed beam of light containing many photons. Each photon in the laser beam must obey the rules of quantum mechanics, including Heisenberg's Uncertainty Principle, which states that it is impossible to know both the position and momentum of a photon at the same time.

The photons in the laser beam have a very well-defined ***momentum***. This is why a laser beam can remain fairly small over long distances: we know precisely where the photons are going.



Light passes through a slit of width w and travels a distance L.
The diffraction pattern's central peak has a width s.

When a laser beam gets smaller, like at the focus of a lens, we can pinpoint objects more precisely, since we know the ***position*** of the photons much more accurately. However, the beam will expand much more quickly, as we lost accurate knowledge of their momentum.

If we have a large laser beam and pass it through a slit, as shown on the image to the right, we restrict the possible positions of the photons, gaining information about where they were. By doing this, we must lose information about their momentum (where they were going).

After the slit, the different possible directions (momenta) split up and form a diffraction pattern on a screen, following the distribution seen on the left. The central spot size s can be related to the uncertainty in momentum. Before passing through the slit, the photons' momentum was entirely perpendicular to the slit. After the slit, the photons may have a component both parallel and perpendicular to the slit.

This change in the possible photon momenta is a result of the uncertainty principle. By passing through the slits, the photon position is known to be within $\Delta x = w$. According to the uncertainty principle, this results in an increase in the photon momentum uncertainty as:

$$\Delta x \Delta p_x \geq 2h$$

where $h = 6.626 \times 10^{-34} \frac{m^2 kg}{s}$ is Planck's constant.

You may have seen this equation written in other ways before. The right-hand side depends on how we measure uncertainty or spread. Since we'll be using the **full width** of the slit and the diffraction pattern in our calculations, 2h is the appropriate constant.

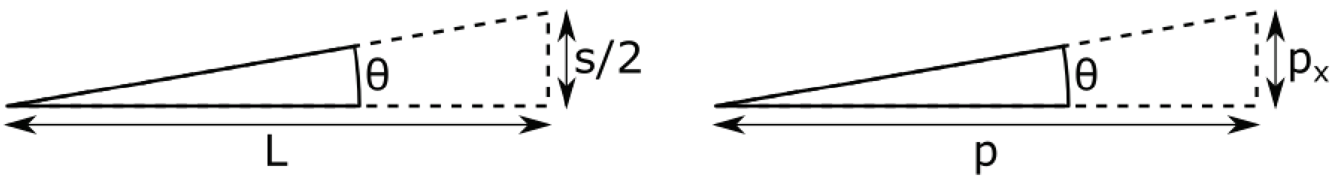
By decreasing the uncertainty in the location of the photons within the slit, we're increasing the uncertainty of the spread in their momenta. This results in the intensity pattern's width spreading as w gets smaller. This effect is only significant when the width of the slit is comparable to the wavelength of the light used.

The momentum of a photon is given by its wavelength (color) as:

$$p = \frac{h}{\lambda}$$

Therefore, blue photons have a much higher momentum than the red photons. If we change the color of the laser, we should see a change in the width of the pattern.

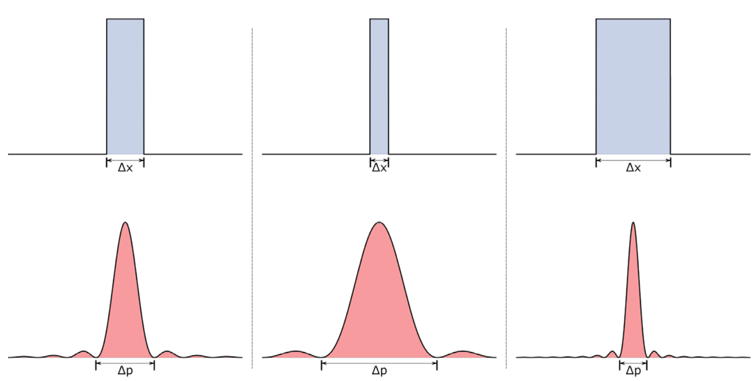
We know that $\Delta x = w$, but how do we quantify the uncertainty in the momentum Δp? We will think about the initial momentum, p, as travelling purely parallel to the initial laser beam. However, after passing through the slit, each photon's momentum now has an x component (p_x) which causes the beam to spread. This momentum spread can be found from the diffraction pattern using trigonometry and similar triangles as:



The above triangles show where a specific momentum p_x would show up on the screen, but we could have p_x values that that are smaller or negative, which would all contribute to the diffraction pattern. We're looking for the uncertainty in the momentum Δp_x. The whole width of the spot s will correspond to the uncertainty Δp_x. This finally gives us the relation:

$$\Delta p_x = \frac{s \cdot p}{L}$$

Next, we'll use this relation to confirm Heisenberg's relation and measure unknown widths for light to pass through.



Graphical representation of Heisenberg's uncertainty principle. As we gain certainty in the position "x", we lose it in momentum "p". For a square-shaped "x" distribution, we have a $\sin(p)/p$, or "sinc", distribution in momentum.

PREPARATION: CALIBRATING YOUR LASERS

To use Heisenberg's uncertainty relation, we need to know the momentum of the objects we're measuring. In the case of light, we need to know the wavelength.

All laser pointers (even if they look like the same color) have slightly different wavelengths, so it's important to first calibrate your laser by passing light through slits of known separation if you have not already done so.

Use a diffraction grating with a known line density (for example, 500 lines/mm). These cause the laser to split into multiple spots, with the separation depending on the wavelength of the light.

Carefully measure the separation between the middle spot and the one closest to it, as well as the distance from the grating to the wall. Tracing out the diffraction pattern on a piece of paper may help increase your accuracy.

$$Wavelength\ of\ Laser = \frac{Separation\ of\ Bright\ Spots}{Lines\ per\ Metre\ \times Distance\ from\ Grating\ to\ Wall}$$

What is the wavelength λ of the laser pointers?

λ_{red} = ______________________ metres

λ_{green} = ______________________ metres

λ_{blue} = ______________________ metres

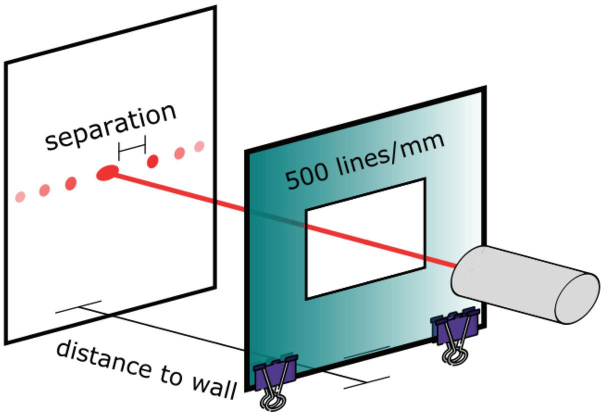
What's the momentum p of the laser pointers?

p_{red} = ______________________ $\frac{kg \cdot m}{s}$

p_{green} = ______________________ $\frac{kg \cdot m}{s}$

p_{blue} = ______________________ $\frac{kg \cdot m}{s}$

Remember to be careful about units!



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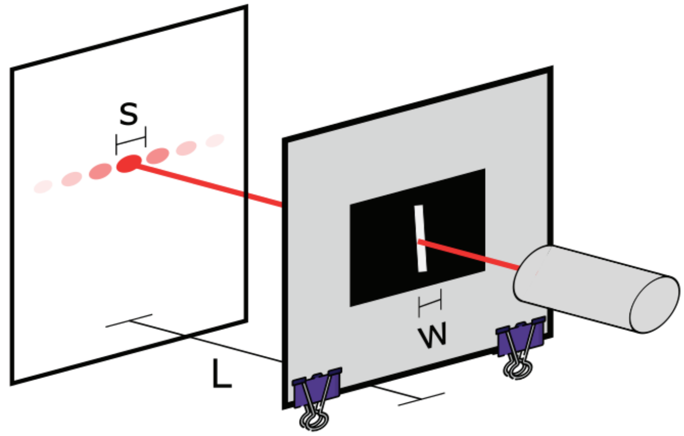
Answer Key

EXPERIMENT: THE UNCERTAINTY PRINCIPLE

1. Tape either the Uncertainty Principle Tracing worksheet or a blank piece of paper to a wall.

2. Set up your laser pointer and slit as in the image to right, illuminating the paper. Try to make the distance to the wall as long as possible (at least a few metres) to allow the light to diffract significantly. Measure the distance L between the slit and the spot, and keep this constant throughout the experiment.



3. Illuminate the widest single slit (0.3 mm) and trace the central bright spot on the paper. Measure it's width s with a ruler.

4. Repeat Step 3 for the 0.1 mm and 0.2 mm slits.

5. Repeat Steps 3 and 4 using laser pointers with different colours, if available.

6. Noting that the slit width w is the position uncertainty Δx, and finding the momentum uncertainty in the x-direction with $\Delta p_x = \frac{s \cdot p}{L}$, plot $\frac{1}{\Delta p_x}$ as a function of Δx .

7. Create a line of best fit through the points and determine the slope of the line. What is the proportionality constant between Δx and ${}^{1}/_{\Delta p_x}$? How does it compare to the known value of Planck's constant?

8. Repeat the measurements for the slits of unknown sizes. Measure Δp_x and use the equation of the line found before to the determine the width of the unknown slits.

What is the constant of proportionality between Δx and ${}^{1}/_{\Delta p_x}$?

$$\Delta x \Delta p_x \geq \_\_\_\_\_\_$$

What are the widths of the unknown slits?

Width A = _____ mm | Width B = _____ mm

1. Compare and contrast the patterns for the different slit widths. What does this tell you about the relationship between the width of the slits and the width of the pattern?

   The narrower the slit is, the wider the diffraction pattern is. The two are inversely proportional.

2. Compare and contrast the patterns for the wide slit for the three wavelengths (colours) of light. What does this tell you about the relationship between the wavelength of light and the width of the pattern?

   Light with a longer wavelength (red) spreads out more than light with a shorter wavelength.

3. How does this experiment demonstrate the uncertainty principle?

   A narrow slit means a better position certainty, hence more momentum uncertainty. Since light has a constant speed, the momentum uncertainty must come from an uncertainty in its direction of propagation.

4. Why does the beam only spread out in one direction (width)? Why does it not also spread in the y-direction? Relate your answer to the uncertainty principle.

   By passing through the slit, we gain certainty about the x-position. But since the slit is "tall", we gain no information about the y-position. The y-momentum remains fairly certain after the slit, and therefore the beam doesn't significantly expand vertically.

5. Why does the blue laser have a smaller spread than the red laser, for a given slit width?

   Since momentum in inversely proportional to the wavelength, light with a short wavelength will need less change in direction of propagation to create a big change in momentum. Put another way, the same uncertainty in momentum results in less of a direction shift for blue light than red, since the overall momentum of the blue light is much greater.

6. What does the slope in our equation of the line of best fit represent? Should this constant be different for lasers of different colours?

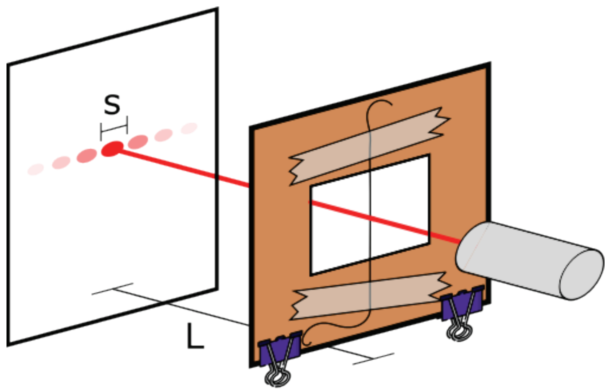
   The slope in our equation represents the minimum product of the change in position Δx and the change in momentum Δp_x. The constant should not change depending on the colour of the light, since the wavelength is accounted for when converting from spot size to momentum.

Answer Key

EXTENSION: MEASURING THE WIDTH OF YOUR HAIR

We've seen that the width of a diffraction pattern for light passing through a narrow slit depends on the width of the slit. When a laser beam illuminates a narrow barrier instead, the diffraction pattern has nearly the exact same features; this is known as **Babinet's Principle**. With the same procedure we used to measure the width of a narrow slit, we can measure the width of a thin object, like a human hair.

1. Take a single strand of hair from a team member (with their permission).
2. Mount the hair, for example with tape and a cardboard cutout (as in the image).
3. Illuminate the hair with one of the lasers you already calibrated.
4. Measure the distance s between the dark spots and the distance L from the hair to the wall.
5. Calculate the width of your hair using the uncertainty relationship, being careful of units.



1. What is the mathematical relationship between the width of your hair w, the wavelength of the laser λ, the separation of the dark spots s, and the distance to the wall L?

Using the relationships $\Delta x = w$, $\Delta p_x = \frac{sh}{\lambda L}$, and $\Delta x \Delta p_x > 2h$, the width of the hair is $w \geq \frac{2\lambda L}{s}$. Since we're using a decent source of laser light, we can approximate that $w = \frac{2\lambda L}{s}$.

2. What is the width of your hair, in μm?

Most human hairs are between 10 and 100 μm wide, although there will be variation.

3. What other objects can have their width measured in this way that may be difficult using traditional ruler-and-measuring tape methods?

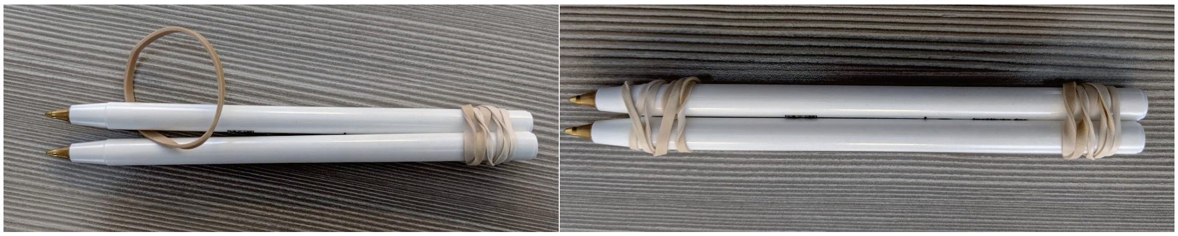
Common household examples may include sewing needles and wire. In research, this technique is used to measure things like crystal structure and the width of blood cells.

A Note on Single/Double Slits:
It is tempting to view the hair experiment as a sort of double-slit, where the light that passes around the left of the hair interferes with the light that passes around the right. This is partially true! We can think about the interference we see in a single-slit as arising from the scattered light on the left edge interfering with that on the right edge. But true double-slit interference has a much richer structure, and we can't directly use the formulas derived for single-slit or single-barrier interference for it.

NO SLITS? NO PROBLEM!

What about if we don't have slits handy? Knowing Heisenberg's relation in advance, we can use it to find the width of other narrow passages that light can travel through.

You'll need two round pens and two rubber bands. Take the pens and secure them tightly together on one end with the elastic. Do the same on the other end, but loop the elastic band between the pens once to create a gap.



Voila! You've created a variable single slit, which gets wider towards the looped end and narrower towards the tight end. Using Heisenberg's uncertainty relation and the experimental setup from before, can you measure how wide it is at each end?

WHAT ELSE CREATES DIFFRACTION PATTERNS?

If using the IQC diffraction cards, see what happens when you shine light through the double-slits, as well as the circular and square patterns. Just like in this experiment, the momentum components spread. However, when we have more complicated patterns, those different momentum components can *interfere* as well, resulting in more complex patterns on the screen.

The relationship between the pattern on the slit and the diffraction pattern generated is called the **Fourier transformation**. It is used in generated holographic images as well as analyzing signals in physics, electrical engineering, and many other fields. On modern Canadian bills, this is used for security. Shine a laser through the clear maple leaf at a far-away wall and see what you see.

UNCERTAINTY BEYOND POSITION/MOMENTUM

Position and momentum are what physicists call a **conjugate pair of variables**. This means that they share an uncertainty relation; the more I learn about one variable, the less I know about the other. It also means that when I make a **measurement** of the position state, I disturb the momentum state, and vice versa.

Position and momentum are not the only two variables that obey the uncertainty principle. It is similarly impossible to precisely know both the colour/wavelength of a pulse of light and it's time of arrival. To create ultrashort pulses of laser light, they need to be made of many different colours through processes like supercontinuum generation. Other examples of conjugate variables include the number of photons in and the phase of a beam of light, the angular momentum and orientation of an object, and the electric charge and electric potential in a quantum circuit.

A familiar example may be in the **polarization** of light. The more we know about the polarization in the HV basis, the less we know about the polarization in the ±45° basis. This fundamental uncertainty is what makes technologies like quantum key distribution possible.

Answer Key

QUANTUM CONCEPTS

1. Our measurement of the Uncertainty Principle constant of proportionality is limited by how accurately we make measurements in the lab. What are some possible sources of error in your experiment?

   Answers will vary, but may include the resolution of the ruler/measuring tape, the resolution of the printer used to print the "known" slits, the accuracy of the lines-per-mm reading of the diffraction grating (or of the laser wavelength given by the manufacturer if it was not calibrated), and the imprecision in measuring the width of the central spot due to blurry edges.

2. Say that we illuminate a 0.2-mm-width slit with photons one-at-a-time. What can we say about the "width" of the photons that go through the slit?

   The photons that make it through the slit have a position uncertainty equal to the width of the slit. We can think of the photons as "0.2 -mm wide", but it is more accurate to think of the photons as being in a superposition of infinitely many possible positions within a 0.2-mm width.

3. If we measure the photons with a photon-sensitive camera after allowing them to diffract, what will we measure? How will it compare to the experiment we performed with lasers?

   We will measure the exact same pattern, but instead of seeing it immediately projected on a wall, it will build up one photon at a time. The sinc-shaped diffraction pattern defines the probability of measuring a photon at any one point, and after detecting many photons we'll be able to trace it out in the same way as the laser experiment.

4. If we send a photon through a 0.1 mm slit followed immediately by a 0.3 mm slit, what pattern will we see on the wall? What about if we reverse the slit order?

   Regardless of the order, we will see the pattern corresponding to the smaller slit, since it defines our position uncertainty.

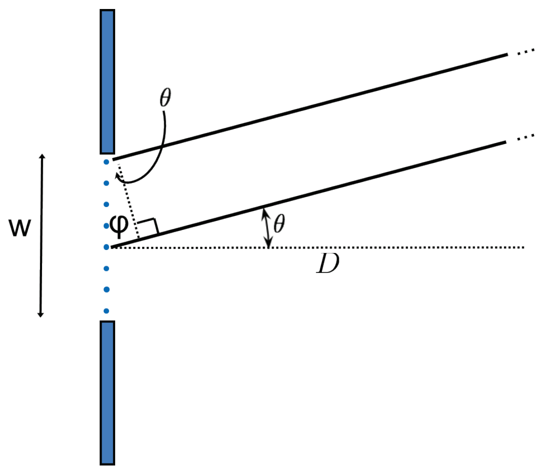
QUANTUM LEAP: CHALLENGE QUESTIONS

1. To focus sunlight to a smaller spot or gain higher resolution in a camera or telescope, it helps to use a larger lens. How does this relate to the uncertainty principle? Think about what a lens does to the momentum/direction of a ray light that hits it.

   When a ray of light hits a lens, it changes its direction (and therefore its momentum) depending on where it hits the lens; a ray travelling perpendicular to the lens and striking the centre is not altered, whereas one hitting the edge of the lens is dramatically tilted. By using a larger lens, there are more possible "tilts" for the rays hitting the lens, resulting in more momentum components. This larger uncertainty in momentum after the lens results in a greater possible certainty in position, meaning that the beam can be focused to a smaller point. The same idea holds for gaining resolution in a camera/telescope. The benefit of using larger lenses is partially that you can collect more light, but it also allows us to make smaller focused spots through the uncertainty principle.

GLOSSARY

- The **uncertainty principle** states that it is impossible to know everything about a system at once. For example, we can only make precise statements about the position of a quantum object if we have little to no knowledge of its momentum.
- **Conjugate variables** are two variables whose estimation precision are limited by the uncertainty principle. Examples include position-momentum and energy-time. The probability distributions of two conjugate variables are linked by the **Fourier transformation**.
- **Diffraction** is a phenomenon experienced by waves, such a light, when they encounter an obstacle or opening, causing them to spread out. This results in regions of constructive and destructive interference, which can be understood by deconstructing the wave passing through the opening as a series of point sources, as below.



Single-Slit Diffraction Quick Review:

Diffraction interference through a single slit can be understood by deconstructing the plane wave into a series of point sources spanning the width if the slit.

In this image, we decompose it into 8 points, and by symmetry, the interference between the 1st and 5th point source will be the same as between the 2nd and 6th, the 3rd and 7th, and the 4th and 8th.

We see destructive interference when the waves from the 1st point (at $x = 0$) and the 5th point (at $x = \frac{w}{2}$) are φ=π out of phase, meaning that their optical path length difference φ is $\frac{w}{2}\sin\theta_n = \frac{n\lambda}{2}$, for any odd value of n.

Answer Key

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The Institute for Quantum Computing (IQC) is a world-leading research centre in quantum information science and technology at the University of Waterloo. IQC's mission is to develop and advance quantum information science and technology through interdisciplinary collaboration at the highest international level. Enabled by IQC's unique infrastructure, the world's top experimentalists and theorists are making powerful new advances in fields spanning quantum computing, communications, sensors and materials. IQC's award-winning outreach opportunities foster scientific curiosity and discovery among students, teachers and the community.

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TRACING WORKSHEET: THE UNCERTAINTY PRINCIPLE

Slit Width	Red	Green	Blue
0.3 mm			
0.2 mm			
0.1 mm			
Unknown A			
Unknown B			

TRACING WORKSHEET:
THE UNCERTAINTY PRINCIPLE

If you do not have access to laser pointers, you can use the below pre-filled traces for the activity.

We assume a distance of 3 metres from the grating to the wall. If printed on US letter paper, the traces should be to scale. If you cannot print, resize your screen until the reference square below is 5 cm wide.

The wavelengths of the laser pointers are given for this example.

Slit Width	Red = 650 nm	Green = 532 nm	Blue = 405 nm
0.3 mm			
0.2 mm			
0.1 mm			
Unknown A			
Unknown B			

Reference
5 cm
square

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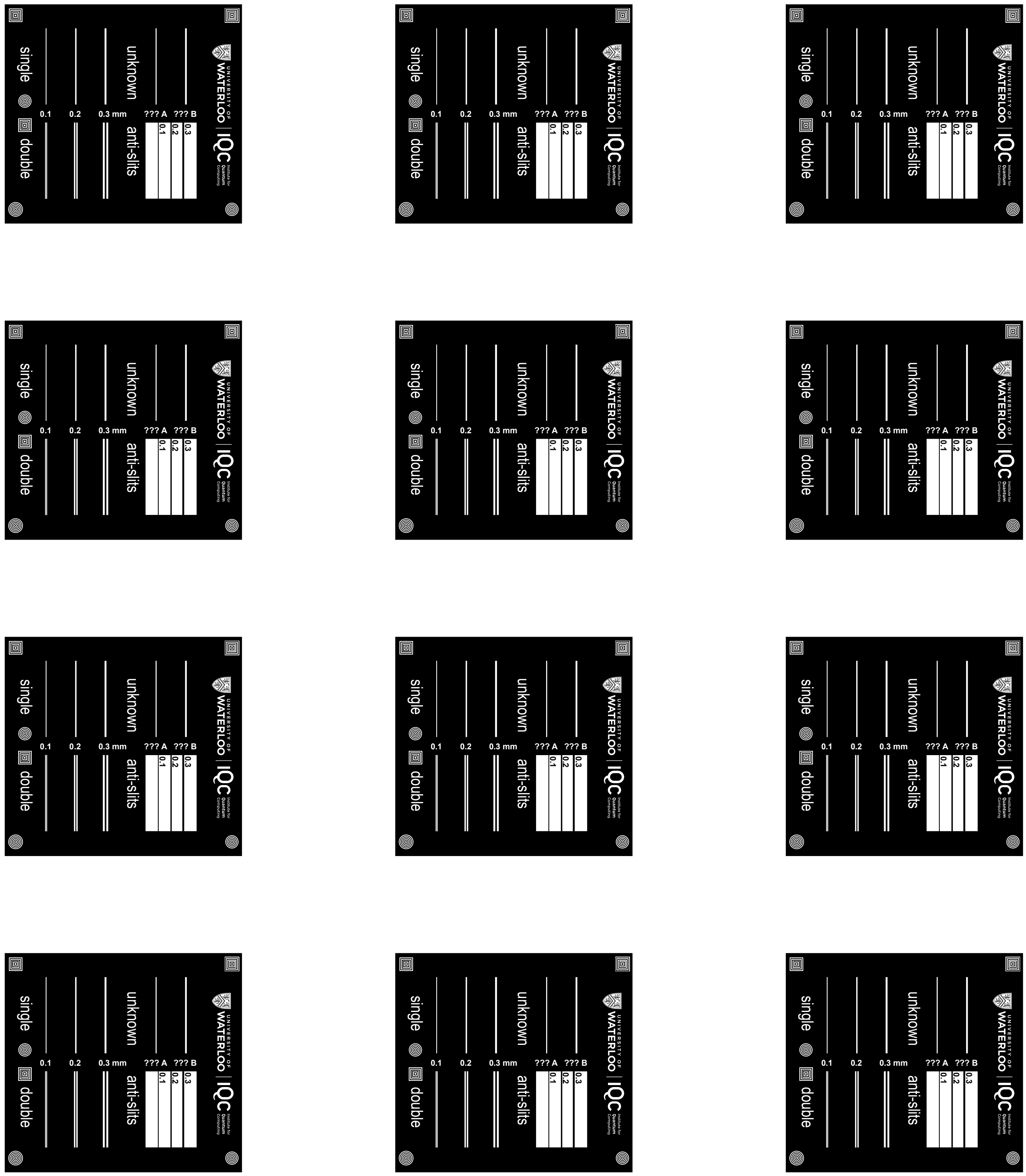
Uncertainty Principle & Double Slit Cards: Kitchen Sink Edition



unknown widths

A = 0.137 mm

B = 0.271 mm

Double-slits: space between equal to width

Use projector paper and a laser printer

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