Supplementary Materials

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I. LANGEVIN EQUATION IN THE INTERACTION PICTURE

Our model is composed of two cavity modes a_1 and a_2 and one mechanical mode b_m coupling via the optomechanical forces in the form of $\sum \hbar G_i a_i^\dagger a_i (b_m + b_m^\dagger)$. This general model can be realized in various experimental systems. For example, it can be two whispering gallery modes with distinctly different frequencies both coupling with the same mechanical mode in a microsphere system [1]. It can also involve one microwave cavity mode and one optical cavity mode coupling with the same mechanical mode [2]. In our discussions in the main paper, we choose arbitrary units for the parameters so that this general model can be applied to a wide range of systems with very different parameters. The strong coupling regime studied in this work (see Sec. II) has recently been demonstrated in experiments in the opto/electro-mechanical system [3, 4].

The total linearized Hamiltonian in the rotating frame is

$$H_{rot} = \sum_{i=1,2} -\hbar\Delta_i a_i^\dagger a_i + \hbar g_i (a_i^\dagger b_m + b_m^\dagger a_i) + \hbar\omega_m b_m^\dagger b_m + H_{diss} \tag{1}$$

where $\Delta_i = \omega_{di} - \omega_i$ is the laser detuning, g_i is the effective linear coupling with $g_i = G_i a_{i,ss}$ and $a_{i,ss}$ being the steady state amplitude of the cavity modes (we assume $a_{i,ss}$ to be real for simplicity of discussion), and H_{diss} describes the system-bath coupling. Here,

$$\begin{aligned} H_{diss} &= \sum_{i=1,2} \int d\omega \hbar(\omega - \omega_{di}) b_i^\dagger(\omega) b_i(\omega) + i\hbar \int d\omega g_i(\omega)(a_i^\dagger b_i(\omega) - b_i^\dagger(\omega) a_i) \\ &+ \int d\omega \hbar\omega c^\dagger(\omega) c(\omega) + i\hbar \int d\omega g_m(\omega)(b_m^\dagger c(\omega) - c^\dagger(\omega) b_m) \end{aligned} \tag{2}$$

where $b_i(\omega)$ $(c(\omega))$ are the bath modes for a_i (b_m) and $g_i(\omega)$ $(g_m(\omega))$ are the corresponding coupling constants. Here, the bath Hamiltonian is also in the rotating frame with the frequencies of the cavity bath modes shifted to $(\omega - \omega_{di})$. For slowly-varying bath spectrum, we assume $g_i(\omega) = \sqrt{\kappa_i/2\pi}$ and $g_m(\omega) = \sqrt{\gamma_m/2\pi}$ in terms of the damping rates.

In the following sections, we study adiabatic quantum state conversion and traveling pulse transmission in the interaction picture of the Hamiltonian H_0 with

$$H_0 = \hbar\omega_m \left(a_1^\dagger a_1 + a_2^\dagger a_2 + b_m^\dagger b_m + \sum_{i=1,2} \int d\omega b_i^\dagger(\omega) b_i(\omega) + \int d\omega c^\dagger(\omega) c(\omega) \right). \tag{3}$$

In this interaction picture, our system is thus governed by the Hamiltonian

$$H_I = \sum_{i=1,2} -\hbar\delta_i a_i^\dagger a_i + \hbar g_i (a_i^\dagger b_m + b_m^\dagger a_i) + H_{I,diss} \tag{4}$$

where $\delta_i = \Delta_i + \omega_m$ and $H_{I,diss}$ has the same form as H_{diss} with the frequency of the $b_i(\omega)$ modes replaced by $\omega - \omega_{di} - \omega_m$ and the frequency of the $c(\omega)$ modes replaced by $\omega - \omega_m$. We choose $\delta_i = 0$ for the adiabatic scheme in our discussions in the following sections.

Using the Hamiltonian H_I, the time evolution of the system operators can be derived as

$$i\dot{a}_i(t) = -\delta_i a_i(t) + g_i b_m(t) + i \int d\omega g_i(\omega) b_i(\omega, t), \tag{5a}$$

$$i\dot{b}_m(t) = \sum_i g_i a_i(t) + i \int d\omega g_m(\omega) c(\omega, t). \tag{5b}$$

The time dependence of the bath operators can be derived as

$$b_i(\omega,t)=b_i(\omega,t_0)e^{-i(\omega-\omega_i-\delta_i)(t-t_0)}-g_i(\omega)\int_{t_0}^{t}dt'e^{-i(\omega-\omega_i-\delta_i)(t-t')}a_i(t'), \tag{6a}$$

$$c(\omega,t)=c(\omega,t_0)e^{-i(\omega-\omega_m)(t-t_0)}-g_m(\omega)\int_{t_0}^{t}dt'e^{-i(\omega-\omega_m)(t-t')}b_m(t'). \tag{6b}$$

The input operators and the output operators for the cavity modes can then be written as

$$a_{in}^{(i)}(t)=\frac{1}{\sqrt{2\pi}}\int d\omega b_i(\omega,t_0)e^{-i(\omega-\omega_i-\delta_i)(t-t_0)} \tag{7a}$$

$$a_{out}^{(i)}(t)=\frac{1}{\sqrt{2\pi}}\int d\omega b_i(\omega,t_1)e^{-i(\omega-\omega_i-\delta_i)(t-t_1)} \tag{7b}$$

and the input operator and the output operator for the mechanical mode can be written as

$$b_{in}(t)=\frac{1}{\sqrt{2\pi}}\int d\omega c(\omega,t_0)e^{-i(\omega-\omega_m)(t-t_0)} \tag{8a}$$

$$b_{out}(t)=\frac{1}{\sqrt{2\pi}}\int d\omega c(\omega,t_1)e^{-i(\omega-\omega_m)(t-t_1)} \tag{8b}$$

for time $t_0<t<t_1$. With $\vec{v}(t)=[a_1(t),b_m(t),a_2(t)]^{\mathrm{T}}$ and $\vec{v}_{in}(t)=[a_{in}^{(1)}(t),b_{in}(t),a_{in}^{(2)}(t)]^{\mathrm{T}}$, the Langevin equation in the interaction picture can be derived using the above results:

$$id\vec{v}(t)/dt=M(t)\vec{v}(t)+i\sqrt{K}\vec{v}_{in}(t) \tag{9}$$

with the dynamic matrix

$$M(t)=\begin{pmatrix}-\delta_1-i\frac{\kappa_1}{2} & g_1(t) & 0\\ g_1(t) & -i\frac{\gamma_m}{2} & g_2(t)\\ 0 & g_2(t) & -\delta_2-i\frac{\kappa_2}{2}\end{pmatrix} \tag{10}$$

and the diagonal matrix $K=\mathrm{diag}(\kappa_1,\gamma_m,\kappa_2)$. Similarly, the input-output relations $\vec{v}_{out}(t)=\vec{v}_{in}(t)-\sqrt{K}\vec{v}(t)$ for both the cavity modes and the mechanical mode can also be derived.

In the main text and in Sec. III, we choose the time correlations for the above noise operators to be $\langle a_{in}^{(i)}(t)a_{in}^{(i)\dagger}(t')\rangle=\delta(t-t')$ and $\langle b_{in}(t)b_{in}^{\dagger}(t')\rangle=(n_{th}+1)\delta(t-t')$ with n_{th} being the thermal phonon number. Here, for simplicity of discussion, we assume the mechanical noise to be Markovian noise with the correlation function being a delta-function which is valid at high temperature $k_BT_0\gg\hbar\omega_m$ for quantum Brownian motion. We want to emphasize that the specific form of the noise correction function does not affect the main results of this work. For the fidelity of the state conversion scheme, the form of the noise correlation only affects the term f_2 in Eq. (31b) by a numerical factor on the order of unity.

In our model in Eq. (1), we applied the rotating wave approximation to the system modes to obtain the effective linear coupling, which requires that $\omega_m\gg g_i$. We also applied the rotating wave approximation to the system-bath coupling terms. In particular, the coupling terms between the mechanical mode and its bath modes are now in the standard quantum optics form similar to that for a cavity mode, which requires that $\omega_m\gg\gamma_i n_{th}$ and directly results in the input-output relation derived above for the mechanical mode. This condition can be well satisfied in the strong coupling regime.

II. MECHANICAL DARK MODE AND THE ADIABATIC CONDITION

We study the adiabatic scheme under the conditions: (1) $\Delta_1=\Delta_2=-\omega_m$ (i.e. $\delta_i=0$) and (2) $\kappa_i,\gamma_m\ll g_0$ with $g_0=\sqrt{g_1^2(t)+g_2^2(t)}$ which is the so-called strong coupling regime in optomechanical systems. We start with the simple case of zero dampings $\kappa_i=\gamma_m=0$. The eigenmodes of the matrix $M(t)$ are then

$$\psi_1=\begin{bmatrix}-g_2/g_0\\ 0\\ g_1/g_0\end{bmatrix},\ \psi_2=\frac{1}{\sqrt{2}}\begin{bmatrix}g_1/g_0\\ -1\\ g_2/g_0\end{bmatrix},\ \psi_3=\frac{1}{\sqrt{2}}\begin{bmatrix}g_1/g_0\\ 1\\ g_2/g_0\end{bmatrix} \tag{11}$$

and the eigenvalues for these modes are

$$\lambda_1 = 0,\ \lambda_2 = -g_0,\ \lambda_3 = g_0. \tag{12}$$

The mode ψ_1 is a mechanical dark mode only involving the two cavity modes. This mode is immune from the mechanical noise and is also separated from the two other modes by an energy gap g_0. The gap protects the mechanical dark mode from mixing with the two other eigenmodes when the system parameters are adiabatically varied.

Assume that the system is initially prepared in the mechanical dark mode ψ_1. As the effective couplings $g_i(t)$ are varied, the eigenmodes evolve with the couplings. From the Landau-Zener theory, when the time-dependence of the couplings satisfies the adiabatic condition:

$$\left|\frac{\hbar}{\Delta E}\frac{dg_i}{dt}\right| \ll \frac{\Delta E}{\hbar} \tag{13}$$

with ΔE being the minimum energy separation between the eigenmodes, transitions (mixings) from the initial mode to the two other modes can be neglected. In our system, $\Delta E = \hbar g_0$ and the adiabatic condition becomes $|dg_i/dt| \ll g_0^2$. In our scheme, we let the coupling g_1 be adiabatically increased from zero to g_0 and the coupling g_2 be adiabatically varied from $-g_0$ to zero during a total time T. The adiabatic condition hence requires that $T \gg 1/g_0$. The system can then be preserved in the mechanical dark mode as the couplings are varied. In realistic systems, to achieve high fidelity for the state conversion, it also requires that $\kappa_i T \ll 1$, as will be shown in Sec. III. Hence, the time duration of this scheme needs to satisfy the condition $1/\kappa_i \gg T \gg 1/g_0$. This condition can be realized in the strong coupling regime in the optomechanical systems [3, 4]. In Fig. 1 in the main paper, we choose the parameters $g_0 = 5$, $T = \pi/2$, and $\kappa_i \sim 0.2$ in arbitrary units, which satisfy this condition.

Next, we consider finite damping rates but with $\kappa_i, \gamma_m \ll g_0$. The damping terms in Eq. (10)

$$\delta M = \text{diag}(-i\kappa_1/2, -i\gamma_m/2, -i\kappa_2/2) \tag{14}$$

can be treated as perturbation to the eigenmodes in Eq. (11). Using a perturbation theory approach, we derive the eigenvalues:

$$\lambda_1 = -i\left(\frac{g_1^2}{2g_0^2}\kappa_2 + \frac{g_2^2}{2g_0^2}\kappa_1\right) \tag{15a}$$

$$\lambda_{2,3} = \mp g_0 - i\left(\frac{\gamma_m}{4} + \frac{g_1^2}{4g_0^2}\kappa_1 + \frac{g_2^2}{4g_0^2}\kappa_2\right) \tag{15b}$$

to the first order of the perturbation. The energy separations between these modes are not significantly affected by the perturbation, and hence the adiabatic condition remains unaffected. The mechanical dark mode becomes

$$\psi_1 = \left[\ -\tfrac{g_2}{g_0},\ \ -\tfrac{i(\kappa_1-\kappa_2)}{2g_0}\tfrac{g_1 g_2}{g_0^2},\ \ \tfrac{g_1}{g_0}\ \right]^{\mathrm{T}} \tag{16}$$

which contains a first-order component from the mechanical mode proportional to κ_i/g_0. Hence, the mechanical dark mode is not totally "dark" any more. The thermal noise of the mechanical mode can generate a small effect on the adiabatic quantum state conversion process through this component which will be discussed in Sec. III.

Another factor to consider is the offset in laser detunings. Assume small offsets $\delta_1 = \Delta_1 + \omega_m$ and $\delta_2 = \Delta_2 + \omega_m$ with $|\delta_{1,2}| \ll g_0$. We treat the offsets also as perturbation to the eigenmodes in addition to the damping terms with

$$\delta M = diag(-\delta_1 - i\kappa_1/2, -i\gamma_m/2, -\delta_2 - i\kappa_2/2). \tag{17}$$

The eigenvalues are now

$$\lambda_1 = -i\left(\frac{g_1^2}{2g_0^2}\kappa_2 + \frac{g_2^2}{2g_0^2}\kappa_1\right) - \left(\frac{g_1^2}{g_0^2}\delta_2 + \frac{g_2^2}{g_0^2}\delta_1\right) \tag{18a}$$

$$\lambda_{2,3} = \mp g_0 - i\left(\frac{\gamma_m}{4} + \frac{g_1^2}{4g_0^2}\kappa_1 + \frac{g_2^2}{4g_0^2}\kappa_2\right) - \left(\frac{g_1^2}{2g_0^2}\delta_1 + \frac{g_2^2}{2g_0^2}\delta_2\right) \tag{18b}$$

and the mechanical dark mode becomes

$$\psi_1 = \left[\ -\tfrac{g_2}{g_0},\ \ -\left(\tfrac{i(\kappa_1-\kappa_2)}{2g_0} + \tfrac{(\delta_1-\delta_2)}{g_0}\right)\tfrac{g_1 g_2}{g_0^2},\ \ \tfrac{g_1}{g_0}\ \right]^{\mathrm{T}} \tag{19}$$

including an extra term proportional to δ_i/g_0 from the mechanical component when compared with Eq. (16). The effect of the detuning offsets on the quantum state conversion can be studied following the same approach as is used in studying the effect of the damping terms. In fact, to add the contribution of the detuning offsets to the conversion fidelity, one can simply replace the $[(\kappa_1-\kappa_2)/2g_0]^2$ factor in Eq. (31b) (see Sec. III) by the factor

$$[(\kappa_1-\kappa_2)/2g_0]^2+[(\delta_1-\delta_2)/g_0]^2. \tag{20}$$

In our discussions in Sec. III, IV, V, we let $\delta_i=0$ for simplicity.

A. Strong damping and (or) large detuning offset limit

The proposed adiabatic scheme does not work in the regime of strong damping rates and (or) large detuning offsets when $\kappa_i, |\delta_i| \gg g_0$. In this regime, we treat the effective couplings as perturbation term, while treat the δM term in Eq. (17) as the non-perturbed Hamiltonian. At $g_i=0$, the eigenmodes of δM are $\psi_1=a_1$, $\psi_2=b_m$, and $\psi_3=a_2$, with the corresponding eigenvalues

$$\lambda_1=-\delta_1-i\kappa_1/2,\ \lambda_2=-i\gamma_m/2,\ \lambda_3=-\delta_2-i\kappa_2/2. \tag{21}$$

At small effective couplings g_i, we apply a perturbation theory approach to derive the eigenmodes:

$$\psi_1=\begin{bmatrix}1\\ \frac{2ig_1}{(\kappa_1-2i\delta_1-\gamma_m)}\\ 0\end{bmatrix},\ \psi_2=\begin{bmatrix}-\frac{2ig_1}{(\kappa_1-2i\delta_1-\gamma_m)}\\ 1\\ -\frac{2ig_2}{(\kappa_2-2i\delta_2-\gamma_m)}\end{bmatrix},\ \psi_3=\begin{bmatrix}0\\ \frac{2ig_2}{(\kappa_2-2i\delta_2-\gamma_m)}\\ 1\end{bmatrix}, \tag{22}$$

which include small deviations from the non-perturbed eigenmodes to the first order of the effective couplings. Hence, the adiabatic scheme cannot convert one cavity mode to the other by varying the couplings. Quantum states cannot be converted between the cavities using the adiabatic scheme in this limit.

B. Raman-like scheme

In the proposed scheme, we choose $-\Delta_i=\omega_m$. A different situation is for $|\omega_m+\Delta_i|\gg g_0$ but still under the two-photon resonance condition $\Delta_1=\Delta_2$. The matrix $M(t)$ is then

$$M(t)=\begin{pmatrix}-\Delta-i\frac{\kappa_1}{2} & g_1(t) & 0\\ g_1(t) & -i\frac{\gamma_m}{2} & g_2(t)\\ 0 & g_2(t) & -\Delta-i\frac{\kappa_2}{2}\end{pmatrix} \tag{23}$$

with $\Delta=\omega_m+\Delta_i$. The transitions between the cavity modes and the mechanical mode are suppressed by the large energy difference $|\Delta|$ in the diagonal matrix elements of $M(t)$. With $\kappa_i\ll g_0$, the two cavity modes are connected by an effective Rabi coupling in the form of $\left(\Omega_R a_1^\dagger a_2+\Omega_R^\star a_2^\dagger a_1\right)$ with $\Omega_R\approx -g_1g_2/|\Delta|$. This can be derived by eliminating the mechanical mode from the total system. Quantum state conversion between the two cavity modes can then be realized via a Rabi flip using this effective Rabi coupling.

III. FIDELITY FOR GAUSSIAN STATES

Solving the Langevin equation in the adiabatic limit, we derive

$$\vec{\alpha}(t)=e^{-i\int_0^t dt'\Lambda(t')}\vec{\alpha}(0)+\int_0^t dt' e^{-i\int_{t'}^t dt''\Lambda(t'')}\vec{\beta}(t') \tag{24}$$

with $\vec{\alpha}(t)=U^{-1}\vec{v}(t)$, $\vec{\beta}(t)=U^{-1}\sqrt{K}\vec{v}_{in}(t)$, and $U=[\psi_1,\psi_2,\psi_3]$. At time $t=0$ with $g_1=0$,

$$U^{-1}(0)\approx\begin{pmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix} \tag{25}$$

after neglecting the small terms due to the dampings, and $[\vec{\alpha}(0)]_1 = a_1(0)$. The damping terms only generate small corrections in the 2×2 block for the operators b_m and a_2, and do not affect $[\vec{\alpha}(0)]_1$. At time $t = T$ with $g_2 = 0$, we have

$$U^{-1}(T) = \begin{pmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \tag{26}$$

after neglecting the small terms due to the dampings, and $[\vec{\alpha}(T)]_1 = a_2(T)$. Again, the damping terms only generate small corrections in the 2×2 block at the lower-left corner and do not affect $[\vec{\alpha}(T)]_1$. At an intermediate time $0 < t < T$, the matrix $U^{-1}(t)$ contains terms that mix the cavity modes with the mechanical mode to the first order of the damping rates. Using Eq. (24), we derive

$$a_2(T) = e^{-f(0,T)} a_1(0) + \int_0^T dt' e^{-f(t',T)} \beta_1(t') \tag{27}$$

with the function

$$f(t,T) = i\int_t^T dt' \lambda_1(t') = \int_t^T dt' \left(\frac{g_1^2(t')}{2g_0^2}\kappa_2 + \frac{g_2^2(t')}{2g_0^2}\kappa_1 \right), \tag{28}$$

and the noise operator

$$\beta_1(t) = \frac{g_2}{g_0}\sqrt{\kappa_1} a_{in}^{(1)}(t) + \frac{i(\kappa_1-\kappa_2)g_1 g_2}{2g_0^3}\sqrt{\gamma_m} b_{in}(t) + \frac{g_1}{g_0}\sqrt{\kappa_2} a_{in}^{(2)}(t) \tag{29}$$

which contains a contribution from the mechanical noise $b_{in}(t)$ in the first-order of κ_i/g_0.

Let the covariances of two operators A and B be $\mathrm{cov}(A,B) = \langle AB\rangle - \langle A\rangle\langle B\rangle$. The covariances of the system operators can be derived from Eq. (27). For the operators $a_2^\dagger$ and a_2, we derive

$$\mathrm{cov}\left(a_2(T)a_2(T)\right) = e^{-2f(0,T)}\mathrm{cov}\left(a_1(0)a_1(0)\right) \tag{30a}$$
$$\mathrm{cov}\left(a_2(T)a_2^\dagger(T)\right) = e^{-2f(0,T)}\mathrm{cov}\left(a_1(0)a_1^\dagger(0)\right) + f_1 \tag{30b}$$
$$\mathrm{cov}\left(a_2^\dagger(T)a_2(T)\right) = e^{-2f(0,T)}\mathrm{cov}\left(a_1^\dagger(0)a_1(0)\right) + f_2 \tag{30c}$$
$$\mathrm{cov}\left(a_2^\dagger(T)a_2^\dagger(T)\right) = e^{-2f(0,T)}\mathrm{cov}(a_1^\dagger(0)a_1^\dagger(0)) \tag{30d}$$

with the functions

$$f_1 = \int_0^T dt' e^{-2f(t',T)} \left(\frac{g_1^2(t')}{g_0^2}\kappa_2 + \frac{g_2^2(t')}{g_0^2}\kappa_1 \right) + f_2\left(n_{th}+1\right)/n_{th} \tag{31a}$$
$$f_2 = \int_0^T dt' e^{-2f(t',T)} \left[\frac{(\kappa_1-\kappa_2)}{2g_0}\right]^2 \frac{g_1^2(t')}{g_0^2}\frac{g_2^2(t')}{g_0^2}\gamma_m n_{th}. \tag{31b}$$

Using Eqs. (30a, 30b, 30c, 30d), we can derive the covariance matrix for the quadrature variables as defined in [5].

The fidelity of the quantum state conversion is defined in the main text. For gaussian states, the conversion fidelity can be calculated analytically using the covariance matrix for the quadrature variables. Here, we consider the squeezed states $D(\alpha)\exp(\frac{1}{2}(\epsilon^\star a_1^2 - \epsilon(a_1^\dagger)^2))|0\rangle$ initially prepared in cavity mode a_1, where $\epsilon = r\exp(2i\phi)$, r is the squeezing the parameter, α is the coherent amplitude of state, and $D(\alpha)$ is the shift operator with $D^\dagger(\alpha)a_1 D(\alpha) = a_1 + \alpha$. For this state, $\langle a_1^\dagger a_1\rangle = \alpha^\star\alpha + \sinh^2 r$. We also define $q_0 = (\alpha+\alpha^\star)/\sqrt{2}$, $p_0 = (\alpha-\alpha^\star)/\sqrt{2}i$ with $q_0^2 + p_0^2 = 2|\alpha|^2$. As studied in [5], the conversion fidelity can be written as $F = F_1 F_2$ with

$$F_1 = \frac{2}{\sqrt{(f_1+f_2)^2 + (e^{-2f(0,T)}+1)^2 + 2(f_1+f_2)(e^{-2f(0,T)}+1)\cosh(2r)}} \tag{32a}$$
$$F_2 = \exp\left(-\frac{\left(1-e^{-f(0,T)}\right)^2 \left[2|\alpha|^2(f_1+f_2) + (1+e^{-2f(0,T)})y(\alpha,r)\right]}{(f_1+f_2)^2 + (e^{-2f(0,T)}+1)^2 + 2(f_1+f_2)(e^{-2f(0,T)}+1)\cosh(2r)} \right). \tag{32b}$$

To the lowest order of the damping rates (first order for F_1 and second order for F_2),

$$F_1 \approx 1 - f(0,T)(\cosh(2r) - 1) - f_s \cosh(2r) \tag{33a}$$

$$F_2 \approx 1 - f^2(0,T) y(\alpha, r)/2 \tag{33b}$$

with the function $f_s = f_2 \left(2n_{th} + 1\right)/2n_{th}$ and

$$y(\alpha, r) = 2|\alpha|^2 \cosh^2(2r) + (q_0^2 - p_0^2)\cos(2\phi)\sinh^2(2r) - 2p_0 q_0 \sin(2\phi)\sinh^2(2r) \tag{34}$$

which depends on the coherent amplitude α and $\alpha^\star$ quadratically. For a coherent state $|\alpha\rangle$ (at $r = 0$), $F_1 \approx 1 - f_s$ and $F_2 \approx 1 - f(0,T)^2 |\alpha|^2$.

IV. TRANSMISSION OF TRAVELING PHOTON PULSES

Photon transmission can be realized in our system via the optomechanical couplings. First, we consider the situation of time-independent effective couplings, i.e. g_i is a constant during the photon transmission. Here, the Langevin equation can be solved in the frequency domain. For an arbitrary operator $\hat{o}(t)$, we can write its frequency components as

$$\hat{o}(\omega) = \int \frac{dt}{\sqrt{2\pi}} \hat{o}(t) e^{i\omega t}. \tag{35}$$

The Langevin equation in Eq. (9) can be transformed to the frequency domain as

$$\vec{v}(\omega) = i\left(I\omega - M\right)^{-1} \sqrt{K} \vec{v}_{in}(\omega). \tag{36}$$

Together with the input-output relation, we derive that $\vec{v}_{out}(\omega) = \widehat{T}(\omega)\vec{v}_{in}(\omega)$ with the transmission matrix $\widehat{T}(\omega) = (I - i\sqrt{K}(I\omega - M)^{-1}\sqrt{K})$. Note that in the interaction picture defined in Sec. I, the input and output operators in Eqs. (7a, 7b) can be rewritten as

$$a_{in}^{(i)}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \left[b_i(\omega + \omega_i, t_0) e^{i\omega t_0}\right] e^{-i\omega t} \tag{37a}$$

$$a_{out}^{(i)}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \left[b_i(\omega + \omega_i, t_1) e^{i\omega t_1}\right] e^{-i\omega t} \tag{37b}$$

which give the frequency components $a_{in}^{(i)}(\omega) = b_i(\omega + \omega_i, t_0)e^{i\omega t_0}$ and $a_{out}^{(i)}(\omega) = b_i(\omega + \omega_i, t_1)e^{i\omega t_1}$, clearly defined relative to the cavity resonances.

The output operator $a_{out}^{(2)}(\omega)$ can then be expressed as

$$a_{out}^{(2)}(\omega) = \widehat{T}_{31}(\omega) a_{in}^{(1)}(\omega) + \widehat{T}_{32}(\omega) b_{in}(\omega) + \widehat{T}_{33}(\omega) a_{in}^{(2)}(\omega) \tag{38}$$

in terms of the frequency components of the input operators. To transmit the input $a_{in}^{(1)}(\omega)$ to the output $a_{out}^{(2)}(\omega)$, two conditions need to be satisfied: 1. the information in $a_{in}^{(1)}(\omega)$ can be accurately transmitted to cavity a_2; and 2. the noise in $a_{in}^{(2)}(\omega)$ and $b_{in}(\omega)$ can be effectively blocked from entering cavity a_2. The first condition requires that the transmission matrix element $\widehat{T}_{31}(\omega) \to 1$ and the second condition requires that the transmission matrix elements $\widehat{T}_{32}(\omega), \widehat{T}_{33}(\omega) \to 0$. It can be shown that the transmission matrix $\widehat{T}$ is a unitary matrix with $\widehat{T}^{-1} = \widehat{T}^{\dagger}$. Hence, the above two conditions are intrinsically correlated with each other.

The transmission matrix element $\widehat{T}_{31}(\omega)$ can be derived as

$$\widehat{T}_{31}(\omega) = \frac{i g_1 g_2 \sqrt{\kappa_1 \kappa_2}}{g_2^2(\omega + i\frac{\kappa_1}{2}) + g_1^2(\omega + i\frac{\kappa_2}{2}) - (\omega + i\frac{\kappa_1}{2})(\omega + i\frac{\kappa_2}{2})(\omega + i\frac{\gamma_m}{2})}. \tag{39}$$

At $\omega = 0$, this equation gives

$$\widehat{T}_{31}(0) = 8 g_1 g_2 \sqrt{\kappa_1 \kappa_2} / \left(4g_1^2 \kappa_2 + 4g_2^2 \kappa_1 + \gamma_m \kappa_1 \kappa_2\right), \tag{40}$$

which reaches maximum at the optimal transmission condition $g_1^2\kappa_2 = g_2^2\kappa_1$. With $\gamma_m \ll g_0$, we have

$$\max(\widehat{T}_{31}(0)) = \frac{1}{1+\gamma_m\sqrt{\kappa_1\kappa_2}/8g_1g_2} \approx 1. \tag{41}$$

The transmission of noise, on the other hand, is determined by the matrix elements $\widehat{T}_{32}(\omega)$ and $\widehat{T}_{33}(\omega)$. It can be shown that $\widehat{T}_{32}(\omega), \widehat{T}_{33}(\omega) \to 0$ at $\omega \to 0$. For example,

$$\widehat{T}_{32}(\omega) = \frac{ig_2\sqrt{\gamma_m\kappa_2}(\omega+i\frac{\kappa_1}{2})}{g_2^2(\omega+i\frac{\kappa_1}{2})+g_1^2(\omega+i\frac{\kappa_2}{2})-(\omega+i\frac{\kappa_1}{2})(\omega+i\frac{\kappa_2}{2})(\omega+i\frac{\gamma_m}{2})}, \tag{42}$$

which gives the ratio

$$\frac{\widehat{T}_{32}(0)}{\widehat{T}_{31}(0)} = \frac{i\sqrt{\gamma_m\kappa_1}}{2g_1} \to 0. \tag{43}$$

Another important feature of the transmission matrix element is the transmission half-width $\Delta\omega$ which can be defined using the relation: $|\widehat{T}_{31}(\Delta\omega)| = |\widehat{T}_{31}(0)|/2$. In Eq. (39), we can see that the half-width is on the same order of magnitude as the cavity damping rates with $\Delta\omega \sim \kappa_i \ll g_i$. Using this observation, we derive the half-width as

$$\Delta\omega \approx \frac{\sqrt{3}(g_1^2\kappa_2+g_2^2\kappa_1+\gamma_m\kappa_1\kappa_2/4)}{2(g_1^2+g_2^2)}. \tag{44}$$

Given the above results, input components at $|\omega| \ll \Delta\omega$ can be transmitted with high fidelity, where $a_{out}^{(2)}(\omega) \approx a_{in}^{(1)}(\omega)$ with $\widehat{T}_{31}(\omega) \to 1$ and $\widehat{T}_{32}(\omega), \widehat{T}_{33}(\omega) \to 0$. Hence, for quantum input pulses that have spectral width σ_ω much narrower than the transmission half-width $\sigma_\omega \ll \Delta\omega$, the quantum pulses can be transmitted to the output channel with high fidelity.

The average of output frequency components $\langle a_{out}^{(2)}(\omega)\rangle$ can be expressed in terms of the average of the input frequency components $\langle a_{in}^{(1)}(\omega)\rangle$ as $\langle a_{out}^{(2)}(\omega)\rangle = \widehat{T}_{31}(\omega)\langle a_{in}^{(1)}(\omega)\rangle$ with the noise operators $\langle a_{in}^{(2)}(\omega)\rangle = \langle b_{in}(\omega)\rangle = 0$. Hence, the output pulse shape is

$$\left\langle a_{out}^{(2)}(t)\right\rangle = \int \frac{d\omega}{\sqrt{2\pi}}\widehat{T}_{31}(\omega)\left\langle a_{in}^{(1)}(\omega)\right\rangle e^{-i\omega t}. \tag{45}$$

The pulse fidelity can be defined as

$$F_p = \frac{\left|\int dt\langle a_{in}^{(1)}(t)\rangle\langle a_{out}^{(2)}(t)\rangle^\star\right|^2}{\int dt\left|\langle a_{in}^{(1)}(t)\rangle\right|^2\int dt\left|\langle a_{out}^{(2)}(t)\rangle\right|^2} \tag{46}$$

which compares the input and the output pulse shapes. This expression is equivalent to

$$F_p = \frac{\left|\int d\omega\langle a_{in}^{(1)}(\omega)\rangle\langle a_{out}^{(2)}(\omega)\rangle^\star\right|^2}{\int d\omega\left|\langle a_{in}^{(1)}(\omega)\rangle\right|^2\int d\omega\left|\langle a_{out}^{(2)}(\omega)\rangle\right|^2}, \tag{47}$$

when written in terms of the frequency components. The above pulse fidelity only reaches $F_p = 1$ at $\langle a_{in}^{(1)}(\omega)\rangle = c\langle a_{out}^{(2)}(\omega)\rangle$ with c being a constant number. In our system, with $\widehat{T}_{31}(\omega) \to 1$ as $\omega \to 0$ at the optimal transmission condition, which gives $c \to 1$. The pulse fidelity is determined solely by the property of the matrix element $\widehat{T}_{31}(\omega)$ which is intrinsically correlated with the other matrix elements $\widehat{T}_{32}(\omega), \widehat{T}_{33}(\omega)$. When high pulse fidelity is achieved, it indicates $\widehat{T}_{31}(\omega) \to 1$, and hence $\widehat{T}_{32}(\omega), \widehat{T}_{33}(\omega) \to 0$. Thus, high pulse fidelity clearly indicates the possibility of high-fidelity in the transmission of traveling quantum pulses in this system.

V. PULSE ENGINEERING IN THE OUTPUT CHANNEL

With time-dependent couplings $g_i(t)$, the output pulse can be manipulated and the output pulse shape can be engineered. Given the Eq. (24) for adiabatically adjusted couplings and the input-output relation $\vec{v}_{out}(t) - \vec{v}_{in}(t) = -\sqrt{K}\vec{v}(t)$, the time-dependence of the output operators can be derived. By applying the transformation $U(t)$ (as defined in Sec. II) to Eq. (24), we have

$$\vec{v}_{out}(t) = \vec{v}_{in}(t) - \sqrt{K}U(t)e^{-i\int_0^t dt'\Lambda(t')}U^{-1}(0)\vec{v}(0) - \int_0^t dt'\sqrt{K}U(t)e^{-i\int_{t'}^t dt''\Lambda(t'')}U^{-1}(t')\sqrt{K}\vec{v}_{in}(t') \quad (48)$$

where $\vec{v}(0)$ is the initial condition for the system modes. The output operators at time t depend on the input operators at all earlier times $0 \le t' \le t$. In the adiabatic limit with slowly varying $g_i(t)$, the contributions that contain the fast oscillating terms $e^{-i\int dt'\lambda_{2,3}}$ can be neglected. Consider the case of $g_1(0) = 0$ and $g_2(0) = -g_0$ where the above equation does not contain $a_2(0)$ and $b_m(0)$ terms. It can be shown that

$$\begin{aligned} a_{out}^{(2)}(t) &= a_{in}^{(2)}(t) - \frac{\sqrt{\kappa_2}g_1(t)}{g_0(t)}e^{-f(0,t)}a_1(0) - \frac{\sqrt{\kappa_2}g_1(t)}{g_0(t)}\int_0^t dt'e^{-f(t',t)}\frac{\sqrt{\kappa_2}g_1(t')a_{in}^{(2)}(t') + \sqrt{\kappa_1}g_2(t')a_{in}^{(1)}(t')}{g_0(t)'} \\ &+ \frac{\sqrt{\kappa_2}g_1(t)}{2g_0(t)}\int_0^t dt'e^{-f(t',t)}\frac{i\sqrt{\gamma_m}g_1(t')g_2(t')b_{in}(t')(\kappa_1-\kappa_2)}{g_0(t')^3} \end{aligned} \quad (49)$$

which includes contributions from the input pulse and the noise operators. It is interesting to note that in the adiabatic limit, the contribution from the mechanical noise $b_{in}(t)$ is again suppressed by the small ratio $(\kappa_1 - \kappa_2)/g_0$, similar to that in Eq. (27).

To demonstrate the idea of pulse engineering, we consider the simple example of the output pulse shape. With $\langle a_{in}^{(2)}(t)\rangle = \langle b_{in}(t)\rangle = 0$ and $\langle\vec{v}(0)\rangle = 0$, the output pulse shape can be expressed as

$$\langle a_{out}^{(2)}(t)\rangle = \sqrt{\kappa_2\kappa_1}\int_0^t dt'e^{-f(t',t)}\frac{g_1(t)}{g_0(t)}\frac{g_2(t')}{g_0(t')}\left\langle a_{in}^{(1)}(t')\right\rangle \quad (50)$$

which is an integral function of the input pulse shape $\left\langle a_{in}^{(1)}(t)\right\rangle$ and the time-dependent couplings. By varying the time-dependence of $g_i(t)$, the output pulse shape can be manipulated in a wide range.

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