Qudit versions of the $\pi/8$ gate: Applications in fault-tolerant QC and nonlocality

Mark Howard[1] **& Jiri Vala**[1,2]

October 2012

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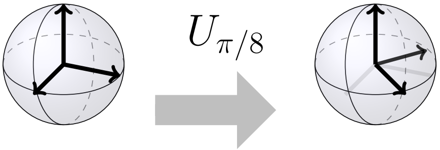
IRC Empower Fellowship & SFI Grant No. 10/IN.1/I3013

The $U_{\pi/8}$ gate and its uses



$$U_{\pi/8} = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix} \propto \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

- $\mathrm{UQC} = \langle \text{Cliffords}, U_{\pi/8} \rangle$
  $\mathrm{UQC} \neq \langle \text{Cliffords} \rangle$

P. O. Boykin, T. Mor, M. Pulver, V. Roychowdhury and F. Vatan.
A new universal and fault-tolerant quantum basis
Information Processing Letters **75**, 3 pp. 101–107, (2000).

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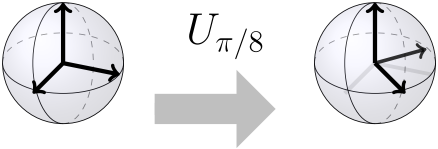

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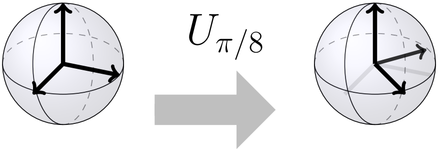

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B. Zeng, H. Chung, A. Cross and I. Chuang,
Local unitary versus local Clifford equivalence of stabilizer and graph states,
Phys. Rev. A **75**, 032325 (2007).

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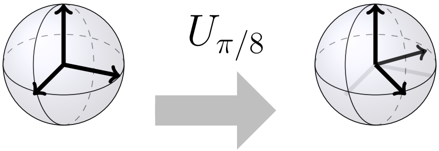

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S. Bravyi and R. Koenig
Classification of topologically protected gates for local stabilizer codes,
arXiv:1206.1609 (2012).

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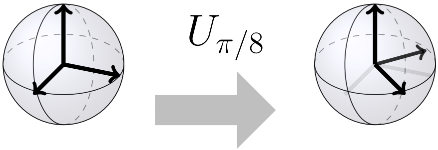

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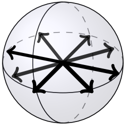

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A. M. Childs,
Secure assisted quantum computation
Quantum Info. Comput.**5**, pp. 456, (2005).

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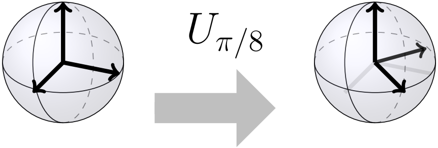

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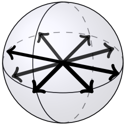

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- Measurement-based UQC with graph states

M. Silva, V. Danos, E. Kashefi and H. Ollivier,
A direct approach to fault-tolerance in measurement-based quantum computation via teleportation
New Journal of Physics **9**, 6 pp. 192, (2007).

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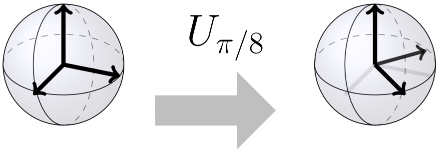

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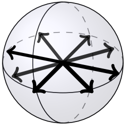

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- Optimal CHSH game with $(|00\rangle + |11\rangle)/\sqrt{2}$

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Nonlocality as a benchmark for universal quantum computation in Ising anyon topological quantum computers,
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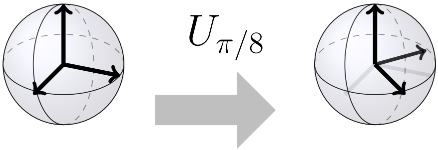

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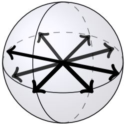

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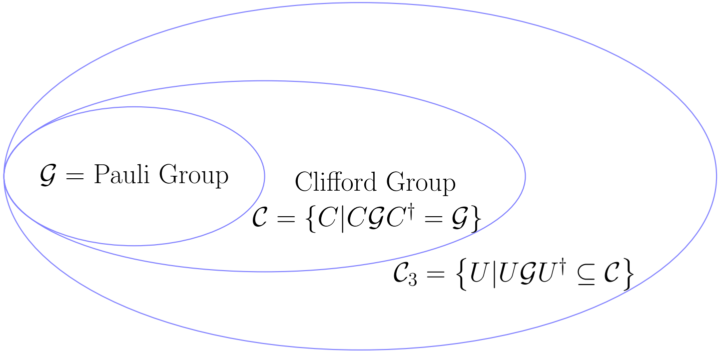
- Blind UQC

A. Broadbent, J. Fitzsimons and E. Kashefi,
Universal blind quantum computation,
Annual IEEE Symposium on Foundations of Computer Science, pp. 517 –526, (2009).

Structure of Pauli/Clifford groups



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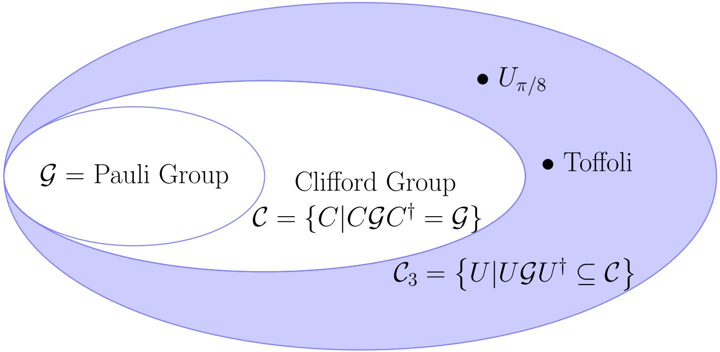

We will focus on **single, p-level** particles

- Generalized σ_x/σ_z : $X|j\rangle = |j+1 \bmod p\rangle \quad Z|j\rangle = \omega^j|j\rangle \quad (\omega = e^{2\pi i/p})$
- Displacement operators, $D_{(x|z)} = \omega^{2^{-1}xz}X^xZ^z$, form Pauli group $\mathcal{G}$

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Single-Qudit Clifford Gates

$$\mathcal{C} = \{C_{(F|\vec{\chi})} \mid F \in SL(2, \mathbb{Z}_p), \vec{\chi} \in \mathbb{Z}_p^2\},$$

- $F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with unit determinant. $\vec{\chi} = \begin{pmatrix} x \\ z \end{pmatrix}$ is a vector of length 2.
- All elements of $F, \vec{\chi}$ are from $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$
- Explicit recipe for constructing a Clifford unitary (where $\tau = \omega^{2^{-1}}$):

$$C_{(F|\vec{\chi})} = D_{(x|z)} V_F$$

$$V_F = \begin{cases} \frac{1}{\sqrt{p}} \sum_{j,k=0}^{p-1} \tau^{\beta^{-1}\left(\alpha k^2 - 2jk + \delta j^2\right)} |j\rangle\langle k| & \beta \neq 0 \\ \sum_{k=0}^{p-1} \tau^{\alpha\gamma k^2} |\alpha k\rangle\langle k| & \beta = 0. \end{cases}$$

D. M. Appleby,
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$$C_{\left(\left[\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right] \middle| \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]\right)} \in \mathrm{SU}(p) \quad \forall p > 3,$$

$$\det\left(C_{\left(\left[\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right] \middle| \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]\right)}\right) = \tau^{2\gamma} \text{ for } p = 3,$$

U_v as qudit version of $U_{\pi/8}$ $(p > 3)$

Define $$U_v = U(v_0, v_1, \ldots) = \sum_{k=0}^{p-1} \omega^{v_k} |k\rangle\langle k| \quad (v_k \in \mathbb{Z}_p)$$

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Easy to show $$U_v D_{(x|z)} U_v^\dagger = D_{(x|z)} \sum_k \omega^{(v_{k+x} - v_k)} |k\rangle\langle k|$$

and in particular $$U_v D_{(1|0)} U_v^\dagger = D_{(1|0)} \sum_k \omega^{(v_{k+1} - v_k)} |k\rangle\langle k| \quad *$$

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Must have $$U_v D_{(1|0)} U_v^\dagger = \omega^{\epsilon'} C_{\left(\left[\begin{smallmatrix} 1 & 0 \\ \gamma' & 1 \end{smallmatrix}\right] \middle| \left[\begin{smallmatrix} 1 \\ z' \end{smallmatrix}\right]\right)}$$
$$= \omega^{\epsilon'} D_{(1|z')} \sum_{k=0}^{p-1} \tau^{\gamma' k^2} |k\rangle\langle k| \quad **$$

U_υ as qudit version of $U_{\pi/8}$ $(p>3)$

Define $U_\upsilon = U(\upsilon_0, \upsilon_1, \ldots) = \sum_{k=0}^{p-1} \omega^{\upsilon_k} |k\rangle\langle k| \quad (\upsilon_k \in \mathbb{Z}_p)$

Easy to show $U_\upsilon D_{(x|z)} U_\upsilon^\dagger \quad = D_{(x|z)} \sum_k \omega^{(\upsilon_{k+x} - \upsilon_k)} |k\rangle\langle k|$

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$= \omega^{\epsilon'} D_{(1|z')} \sum_{k=0}^{p-1} \tau^{\gamma' k^2} |k\rangle\langle k|$ **

Equating * and ** : $\omega^{\upsilon_{k+1} - \upsilon_k} = \omega^{\epsilon'} \tau^{z'} \omega^{kz'} \tau^{k^2\gamma'} \quad (\forall k \in \mathbb{Z}_p)$,

$\Rightarrow \upsilon_k = 12^{-1} k(\gamma' + k(6z' + (2k-3)\gamma')) + k\epsilon' \quad (\upsilon_0 = 0)$

U_v as qudit version of $U_{\pi/8}$ $(p=3)$

For $p = 3$: $\det\left(\zeta^{2\gamma'} C_{\left(\left[\begin{smallmatrix}1 & 0\\ \gamma' & 1\end{smallmatrix}\right] \middle| \left[\begin{smallmatrix}1\\ z'\end{smallmatrix}\right]\right)}\right) = 1 \quad (\zeta = e^{\frac{2\pi i}{9}})$

$$\Rightarrow U_v = \sum_{k=0}^{2} \zeta^{v_k} |k\rangle\langle k| \quad (v_k \in \mathbb{Z}_9)$$

$$v = (v_0, v_1, v_2) = (0, 6z' + 2\gamma' + 3\epsilon', 6z' + \gamma' + 6\epsilon') \mod 9$$

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Examples:

$$p=3: \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{9}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{9}} \end{pmatrix}_{[z'=1,\gamma'=2,\epsilon'=0]} \quad p=5: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{-\frac{4\pi i}{5}} & 0 & 0 & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{5}} & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{4\pi i}{5}} & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{2\pi i}{5}} \end{pmatrix}_{[z'=1,\gamma'=4,\epsilon'=0]}$$

Group Structure of U_v gates

Can create

- p^3 gates $U_v(z', \gamma', \epsilon')$ varying over $z', \gamma', \epsilon' \in \mathbb{Z}_p$.
- $p^2(p-1)$ non-Clifford U_v varying over $z', \epsilon' \in \mathbb{Z}_p$, $\gamma' \in \mathbb{Z}_p^*$.

$$U_v D_{(1|0)} U_v^\dagger = \omega^{\epsilon'} C_{\left(\left[\begin{smallmatrix} 1 & 0 \\ \gamma' & 1 \end{smallmatrix}\right] \middle| \left[\begin{smallmatrix} 1 \\ z' \end{smallmatrix}\right]\right)} \quad \text{but} \quad C_{\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \middle| \left[\begin{smallmatrix} 1 \\ z' \end{smallmatrix}\right]\right)} = D_{(1|z')}$$

Easy to show p^3 gates $\{U_v\}$ form a finite Abelian group.

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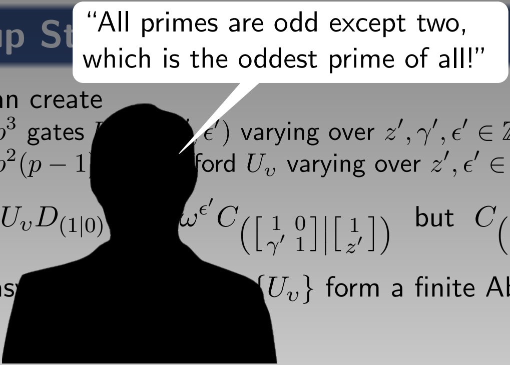
Use Fund. Thm. of finite Abelian groups to characterize $\{U_v\}$

	Group name	No. elements of order 1	p	p^2	p^3	Min. no. of generators
$p = 2$	$\mathbb{Z}_8$	1	1	2	4	1
$p = 3$	$\mathbb{Z}_9 \times \mathbb{Z}_3$	1	8	18	0	2
$p > 3$	$\mathbb{Z}_p^3$	1	$p^3 - 1$	0	0	3

Group St



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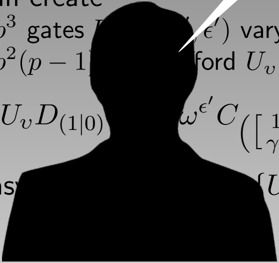

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$$U_\upsilon D_{(1|0)} U_\upsilon^\dagger = \omega^{\epsilon'} C_{\left(\left[\begin{smallmatrix}1 & 0\\ \gamma' & 1\end{smallmatrix}\right] \middle| \left[\begin{smallmatrix}1\\ z'\end{smallmatrix}\right]\right)} \quad \text{but} \quad C_{\left(\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right] \middle| \left[\begin{smallmatrix}1\\ z'\end{smallmatrix}\right]\right)} = D_{(1|z')}$$

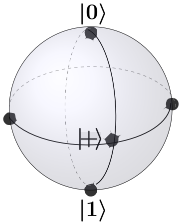
Easy to show p^3 gates $\{U_\upsilon\}$ form a finite Abelian group.

$$U_\upsilon(z_1, \gamma_1, \epsilon_1) U_\upsilon(z_2, \gamma_2, \epsilon_2) = U_\upsilon(z_1 + z_2, \gamma_1 + \gamma_2, \epsilon_1 + \epsilon_2)$$

Use Fund. Thm. of finite Abelian groups to characterize $\{U_\upsilon\}$

	Group name	No. elements of order 1	p	p^2	p^3	Min. no. of generators
$p=2$	$\mathbb{Z}_8$	1	1	2	4	1
$p=3$	$\mathbb{Z}_9 \times \mathbb{Z}_3$	1	8	18	0	2
$p>3$	$\mathbb{Z}_p^3$	1	p^3-1	0	0	3

Qubit Geometry: Magic States and $U_{\pi/8}$



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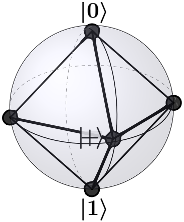

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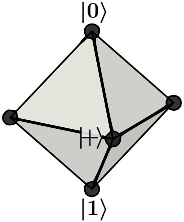

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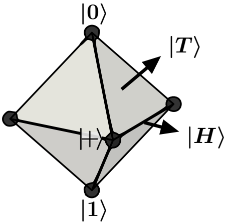
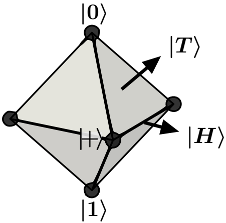


$$|T\rangle\langle T| = \frac{1}{2}\left(\mathbb{I} + \frac{\sigma_x + \sigma_y + \sigma_z}{\sqrt{3}}\right)$$

$$|H\rangle\langle H| = \frac{1}{2}\left(\mathbb{I} + \frac{\sigma_x + \sigma_y}{\sqrt{2}}\right)$$

- Both $|T\rangle$ and $|H\rangle$ are eigenvectors of Clifford gates
- $|T\rangle$ is the most non-stabilizer qubit state
- $|H\rangle$ is the most non-stabilizer qubit state in the equatorial plane
  - Note that $|H\rangle = U_{\pi/8}|+\rangle$ $= \frac{1}{\sqrt{2}}\mathrm{diag}(U_{\pi/8})$

Qubit Geometry: Magic States and $U_{\pi/8}$



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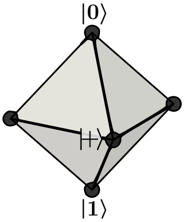
    - Note that $|H\rangle = U_{\pi/8}|+\rangle$ $= \frac{1}{\sqrt{2}}\mathrm{diag}(U_{\pi/8})$
- In [vDH'11], states $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ were found to be maximally non-stabilizer, and Clifford eigenvectors, in all odd prime dimensions (similar to $|T\rangle$?)
- We will argue that $|\psi_{U_v}\rangle \equiv U_v|+\rangle$ is the qudit analogue of $|H\rangle$.

W. van Dam and M. Howard,
Noise thresholds for higher-dimensional systems using the discrete Wigner function
Phys. Rev. A. **83**, 032310, (2011).

Geometry: $|\psi_{\mathrm{U}_v}\rangle$ as the qudit analogue of $|\mathrm{H}\rangle$



$\mathcal{STAB}$ = Convex hull of qudit stabilizer states

$$= \left\{ \rho \;\middle|\; \rho = \sum_{i=1}^{p(p+1)} q_i |\psi^{(i)}_{STAB}\rangle\langle\psi^{(i)}_{STAB}|, \sum_{i=1}^{p(p+1)} q_i = 1 \right\}$$

$$= \left\{ \rho \;\middle|\; \min_{u \in \mathbb{Z}_p^{p+1}} \mathrm{Tr}\left[A(u)\rho\right] \geq 0 \right\}$$

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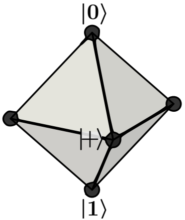

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Define $\begin{cases} U_\theta = e^{i\theta_k}|k\rangle\langle k| \;\; (\theta_k \in \mathbb{R}) \\ |\psi_{U_\theta}\rangle = \frac{e^{i\theta_k}}{\sqrt{p}}|k\rangle = U_\theta|+\rangle \end{cases}$ so that $\begin{cases} \{U_v\} \subset \{U_\theta\} \\ \{|\psi_{U_v}\rangle\} \subset \{|\psi_{U_\theta}\rangle\} \end{cases}$

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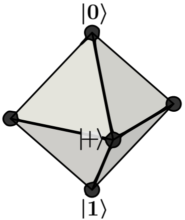

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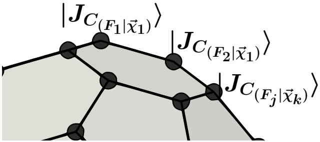
- $|\psi_{U_v}\rangle$ is farthest outside $\mathcal{STAB}$ of all $|\psi_{U_\theta}\rangle$ (for $p = 2, 3, 5, 7$ at least)
- $|\psi_{U_v}\rangle$ is also a Clifford eigenvector:

$$C_{\left(\left[\begin{smallmatrix}1 & 0\\ \gamma' & 1\end{smallmatrix}\right]\middle|\left[\begin{smallmatrix}1\\ z'\end{smallmatrix}\right]\right)}|\psi_{U(z',\gamma',\epsilon')}\rangle = \omega^{-\epsilon'}|\psi_{U(z',\gamma',\epsilon')}\rangle.$$

Geometry: U_υ as the qudit analogue of $\mathrm{U}_{\pi/8}$

For $\begin{cases} U \in \mathrm{U}(p) \\ \mathcal{E} : \rho_{in} \mapsto \rho_{out} \end{cases}$ Jamiołkowski state $= \begin{cases} |J_U\rangle \equiv (\mathbb{I} \otimes U) \sum_{j=0}^{p-1} \frac{|jj\rangle}{\sqrt{p}} \\ \varrho_{\mathcal{E}} = [\mathcal{I} \otimes \mathcal{E}] \left(\sum_{j,k=0}^{p-1} \frac{|j,j\rangle\langle k,k|}{p} \right) \end{cases}$

$\mathcal{CLIFF}$ = Convex hull of qudit Clifford gates



$$= \left\{ \varrho_{\mathcal{E}} \;\middle|\; \varrho_{\mathcal{E}} = \sum_{j,k=1}^{\substack{j=p(p^2-1)\\ k=p^2}} q_{j,k} |J_{C_{(F_j|\vec{x}_k)}}\rangle\langle J_{C_{(F_j|\vec{x}_k)}}| \right\}$$

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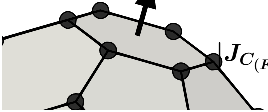

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- Seems that U_υ is the most non-Clifford $U \in \mathrm{U}(p)$ (for $p = 2, 3, 5, 7, ?$)

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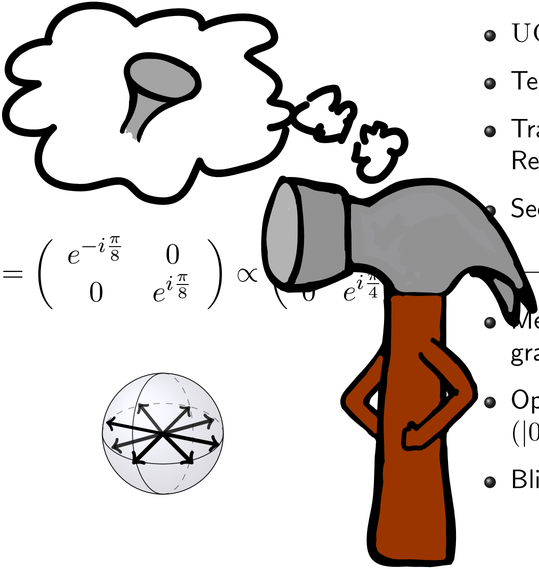
Applications?



- UQC= $\langle$Cliffords, $U_{\pi/8}\rangle$
- Teleportation-based UQC
- Transversal for some Reed-Muller codes
- Secure assisted UQC

$$U_{\pi/8} = \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix} \propto \begin{pmatrix} & \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

- Measurement-based UQC with graph states
- Optimal CHSH game with $(|00\rangle + |11\rangle)/\sqrt{2}$
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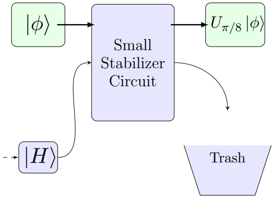
Application to Fault-tolerant QC

- We argued $|\psi_{U_v}\rangle$ was the qudit analogue of $|H\rangle = |\psi_{U_{\pi/8}}\rangle$.
- The key feature of $|H\rangle$ is that it is suitable for magic state distillation



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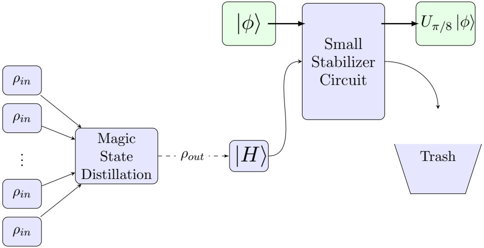

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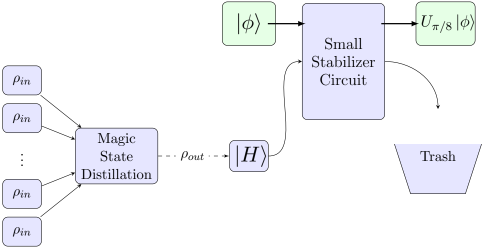

- Campbell, Anwar and Browne have shown $|\psi_{U_v}\rangle$ are qudit magic states.
- Nebe, Rains and Sloane have shown $\langle$Cliffords,$U_v\rangle$ enables UQC
- Geometrical features are good news!

E. T. Campbell, H. Anwar and D. E. Browne,
Magic state distillation in all prime dimensions using quantum Reed-Muller codes
arXiv:1205.3104v1, (2012).

UQC using perfect Cliffords + noisy U_υ gates

$$\mathcal{E}_{(|\psi_{U_\upsilon}\rangle,\varepsilon)}(\rho) \equiv (1-\varepsilon)U_\upsilon \rho U_\upsilon^\dagger + \varepsilon\frac{\mathbb{I}}{p} \qquad (\varepsilon \approx \text{depolarizing error rate})$$

- For what noise rates, ε, does $\mathcal{E}_{(|\psi_{U_\upsilon}\rangle,\varepsilon)}+$ Cliffords enable UQC?

$$|+\rangle \; — \boxed{\mathcal{E}_{(|\psi_{U_\upsilon}\rangle,\varepsilon)}} — \; (1-\varepsilon)\,|\psi_{U_\upsilon}\rangle\langle\psi_{U_\upsilon}| + \varepsilon\tfrac{\mathbb{I}}{p}$$

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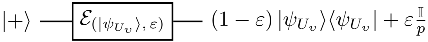
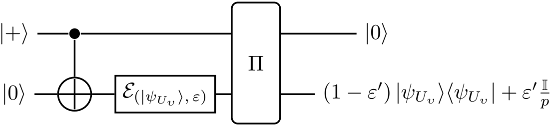




- Postselection "dilutes" the noise: $\varepsilon' < \varepsilon$.

  Exact relationship:
  $$\varepsilon' = \frac{\varepsilon}{p-(p-1)\varepsilon}$$

- Use this result, along with routines in [CAB'12], to find allowable ε

Noise thresholds for UQC

	Lower Bound	Upper Bound
$p = 2$	45.32%	45.32%
$p = 3$	58.15%	78.63%
$p = 5$	80.61%	95.20%
$p = 7$	72.24%	97.63%

Table: Bounds on threshold ε for UQC using noisy U_v + ideal Cliffords

How are these values found?

Lower bound:

- Postselction circuit & the best performing MSD routines given in [CAB'12].

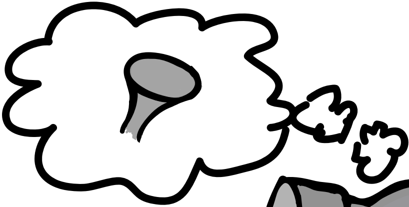
Upper bound:

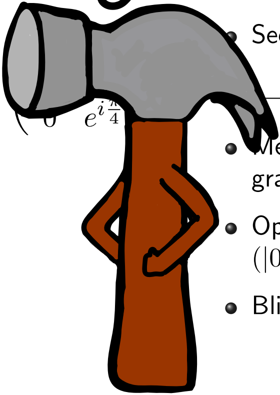
- Explicit facets of $\mathcal{CLIFF}$ for which $\mathrm{Tr}\left(W\left[(1-\varepsilon)|J_U\rangle\langle J_U| + \varepsilon\frac{\mathbb{I}}{p^2}\right]\right) = 0$

Note: U_v also maximally robust to phase damping noise $(p = 2, 3, 5, 7, ?)$

Open Question: Can the gaps be closed?

Applications?





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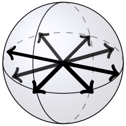

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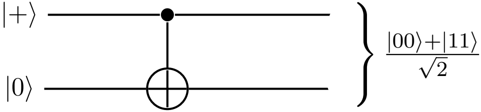
CHSH Bell Inequality

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2 \qquad (\lambda(A_j), \lambda(B_k) = \pm 1)$$



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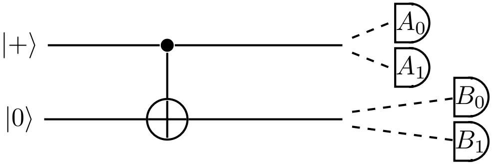

$$A_0 = X$$
$$A_1 = Y$$
$$B_0 = (X - Y)/\sqrt{2}$$
$$B_1 = (X + Y)/\sqrt{2}$$

- $2\sqrt{2} \not\leq 2$

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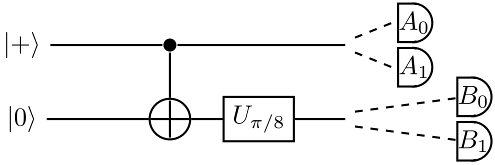

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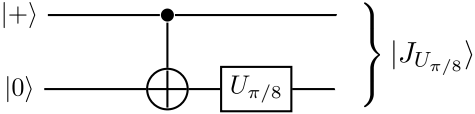

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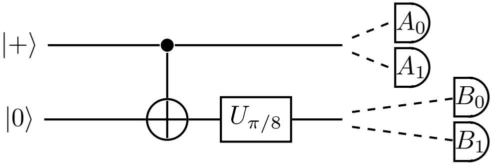

$A_0 = X$

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$B_0 = X$

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- $\langle \mathcal{B} \rangle = 2\sqrt{2} \not\leq 2$

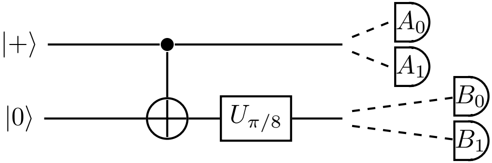
Maximizing quantity $\langle \mathcal{B} \rangle$ is related to maximizing

$$\sum_{\substack{a+b=st \bmod 2 \\ (a,b,s,t \in \mathbb{Z}_2)}} p(a,b|s,t)$$

where $p(\underbrace{a,b}_{\text{outcomes}}|\overbrace{s,t}^{\text{settings}})$ is a conditional prob.

CHSH Bell Inequality

$$\langle \mathcal{B} \rangle \leq 2 \qquad \mathcal{B} = \sum_{j,k \in \mathbb{Z}_2} (-1)^{jk} A_j B_k \qquad (\lambda(A_j), \lambda(B_k) = \pm 1)$$



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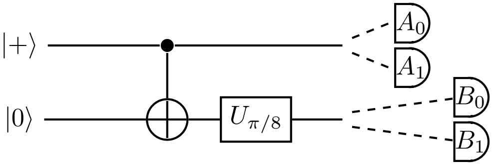
CHSH Bell Inequality

$$\langle \mathcal{B} \rangle \leq 2 \qquad \mathcal{B} = \sum_{j,k \in \mathbb{Z}_2} (e^{\frac{2\pi i}{2}})^{jk} A_j B_k \qquad (\lambda(A_j), \lambda(B_k) = \pm 1)$$



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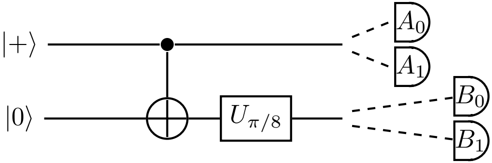
CHSH Bell Inequality

$$\langle \mathcal{B} \rangle \leq 2 \qquad \mathcal{B} = \sum_{j,k \in \mathbb{Z}_2} \omega^{jk} A_j B_k \qquad (\lambda(A_j), \lambda(B_k) = \pm 1)$$



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Generalized CHSH Bell Inequality for $p = 3$

$$\langle \mathcal{B} \rangle \leq 4.5 \qquad \mathcal{B} = \sum_{\substack{n \in \mathbb{Z}_3^* \\ j,k \in \mathbb{Z}_3}} \omega^{njk} A_j^n B_k^n \qquad (\lambda(A_j), \lambda(B_k) = \{\omega^0, \omega^1, \omega^2\})$$

Generalized CHSH Bell Inequality for $p = 3$

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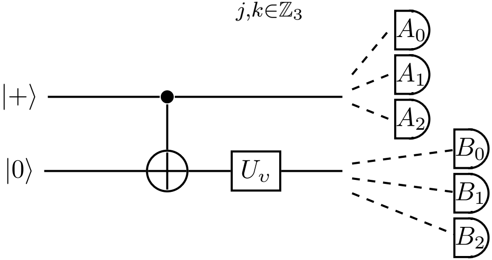

$$A_j = \omega^{j(j+1)} X Z^j$$

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- $\langle \mathcal{B} \rangle = 5.117 \nleq 4.5$

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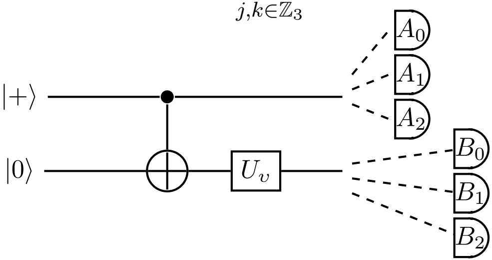

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Maximizing quantity $\langle \mathcal{B} \rangle$ is related to maximizing

$$\sum_{\substack{a+b+st=0 \bmod 3 \\ (a,b,s,t \in \mathbb{Z}_3)}} p(a, b | s, t)$$

where $p(\underbrace{a, b}_{\text{outcomes}} | \overbrace{s, t}^{\text{settings}})$ is a conditional prob.

Generalized CHSH Bell Inequalities

Originates with Ji *et al.*, Reexamined by Liang *et al.*

Se-Wan Ji, Jinhyoung Lee, James Lim, Koji Nagata, and Hai-Woong Lee
Multisetting Bell inequality for qudits,
Phys. Rev. A 78, 052103 (2008).

Yeong-Cherng Liang, Chu-Wee Lim, Dong-Ling Deng
Reexamination of a multisetting Bell inequality for qudits,
Phys. Rev. A 80, 052116 (2009).

Observation 1: For all the cases ($p \leq 17$) that we have checked, U_υ are optimal in the sense:

$$\mathrm{Tr}\left(|J_{U_\upsilon}\rangle\langle J_{U_\upsilon}|\mathcal{B}\right) = \lambda_{\max}(\mathcal{B})$$

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Observation 2: Suggests a natural generalization to $p \geq 3$ of the qubit result relating capability of operation $\mathcal{E}$ + stabilizer ops

Violation of CHSH inequality	$\Rightarrow$	Universal QC (via MSD)

M. Howard and J. Vala,
Nonlocality as a benchmark for universal quantum computation in Ising anyon topological quantum computers,
Phys. Rev. A **85**, 022304 (2012).

Open Questions & Thanks

- Are qudits better in any way?
  - How does one fairly compare qubits and qudits?
- Physical system that enables topologically protected qudit Cliffords? (cf. Ising anyons for $p = 2$)
- What does the rest of $\mathcal{C}_3$ look like?
- What does the diagonal subset of $\mathcal{C}_k$ look like?
- Can we close the gap between upper and lower bounds on noise thresholds?
- Are there applications in MUBs, SICs, Unitary designs?
- Can stronger statements be made relating nonlocality and UQC in the Clifford computer/magic state model of computation?

Thanks to Earl Campbell for many helpful discussions & comments