Quantum Theory in a New Frame

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QM is about
Information.

$H(X)$

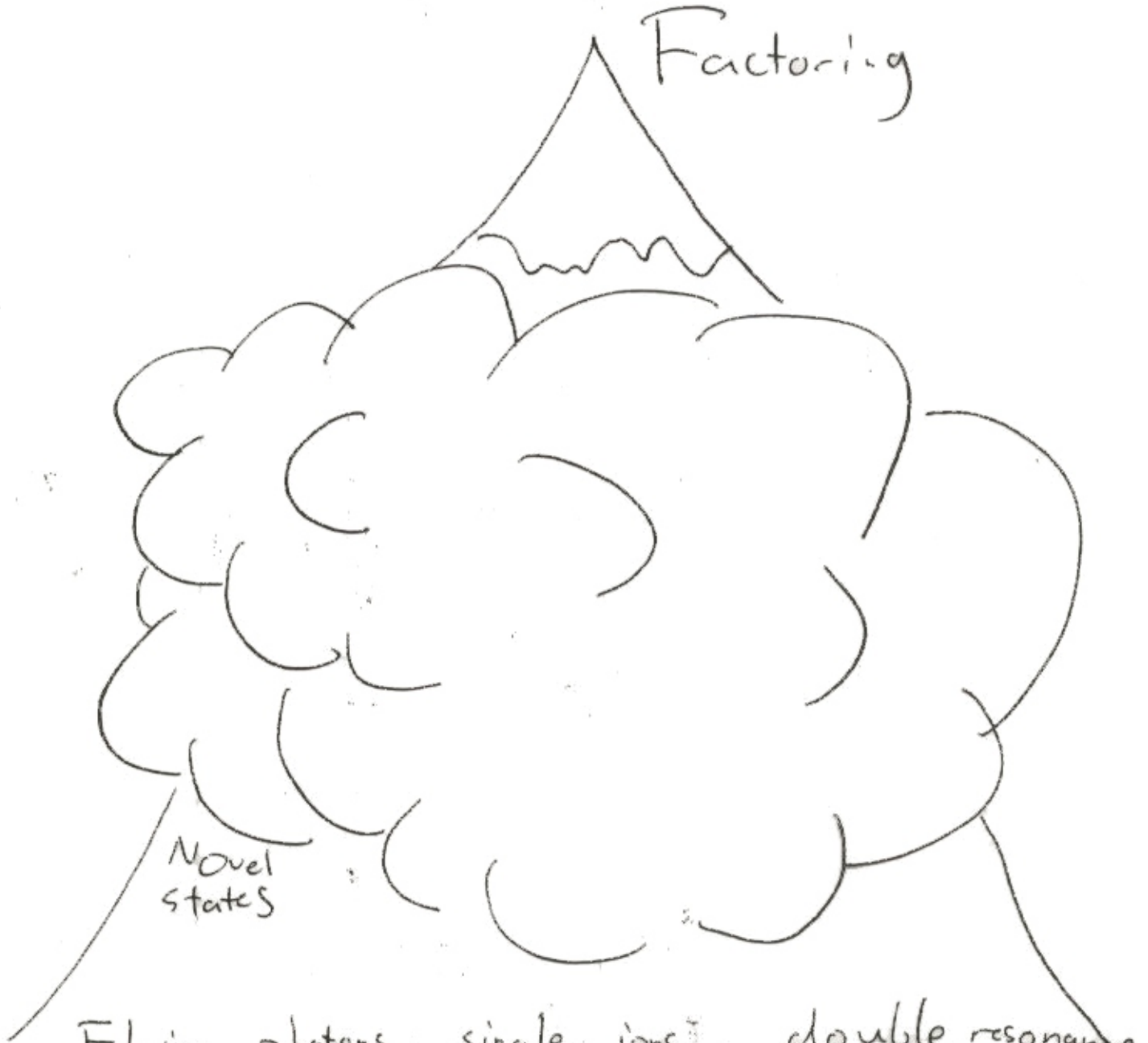
Plain old ordinary Shannon information:

ignorance

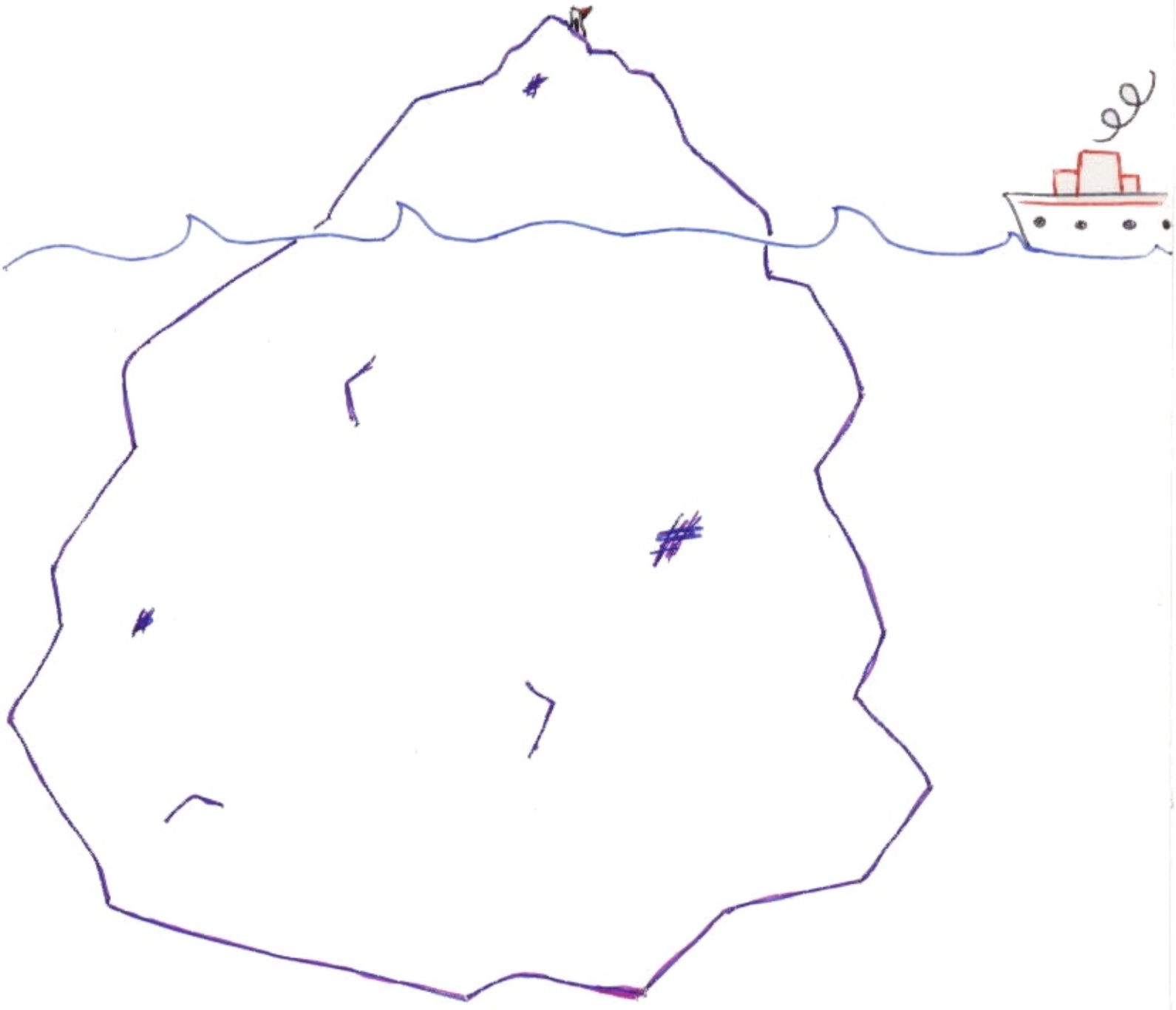
lack of predictability

QUIC Kickoff, Seth Lloyd

The view of quantum computation in 1994:



– Looks steep



Quantum Mechanics:
The Axioms and Our Imperative!

States correspond to density operators ρ over a Hilbert space $\mathcal{H}$.	*Give an information theoretic reason if possible!*
Measurements correspond to positive operator-valued measures (POVMs) $\{E_d\}$ on $\mathcal{H}$.	*Give an information theoretic reason if possible!*
$\mathcal{H}$ is a complex vector space, not a real vector space, not a quaternionic module.	*Give an information theoretic reason if possible!*
Systems combine according to the tensor product of their separate vector spaces, $\mathcal{H}_{\text{AB}} = \mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$.	*Give an information theoretic reason if possible!*
Between measurements, states evolve according to trace-preserving completely positive linear maps.	*Give an information theoretic reason if possible!*
By way of measurement, states evolve (up to normalization) via outcome-dependent completely positive linear maps.	*Give an information theoretic reason if possible!*
Probabilities for the outcomes of a measurement obey the Born rule for POVMs $\text{tr}(\rho E_d)$.	*Give an information theoretic reason if possible!*

The distillate that remains—the piece of quantum theory with no information theoretic significance—will be our first unadorned glimpse of "quantum reality." Far from being the end of the journey, placing this conception of nature in open view will be the start of a great adventure.

Hardy's New Axioms

1. **Definiteness**. Associated with any given pure state is a unique maximal effect giving probability equal to one. This maximal effect does not give probability equal to one for any other pure state.
2. **Information Locality**. A maximal measurement on a composite system is effected if we perform maximal measurements on each of the components.
3. **Tomographic Locality**. The state of a composite system can be determined from the statistics collected by making measurements on the components.
4. **Compound Permutatability**. There exists a compound reversible transformation on any system effecting any given permutation of any given maximal set of distinguishable states for that system.
5. **Preparability**. Filters are non-mixing and non-flattening.

Chiribella, D'Ariano, and Perinotti's "5 Axioms and 1 Postulate"

1. **Causality**. The probability of a measurement outcome at a certain time does not depend on the choice of measurements that will be performed later.
2. **Perfect Distinguishability**. If a state is not completely mixed, then there exists at least one state that can be perfectly distinguished from it.
3. **Ideal Compression**. Every source of information can be encoded in a suitable physical system in a lossless and maximally efficient fashion. Here lossless means that the information can be decoded without errors and maximally efficient means that every state of the encoding system represents a state in the information source.
4. **Local Distinguishability**. If two states of a composite system are different, then we can distinguish between them from the statistics of local measurements on the component systems.
5. **Pure Conditioning**. If a pure state of system AB undergoes an atomic measurement on system A, then each outcome of the measurement induces a pure state on system B.
6. **Purification**. Every state has a purification. For fixed purifying system, every two purifications of the same state are connected by a reversible transformation on the purifying system.

Wilce's "Four and a Half Axioms"

Let a physical system be modeled by a pair $(\mathcal{U}, \Omega)$, where $\mathcal{U}$ is a test space with outcome-space X and Ω is a closed, convex, outcome-separating set of continuous states thereon.

1. **Symmetry**. There is a compact group G acting continuously on $(\mathcal{U}, \Omega)$, in such a way that (i) G acts fully symmetrically on $\mathcal{U}$, and (ii) G acts transitively on Ω_{ext}.
2. **Minimization**. There exists a minimizing G-invariant, positive inner product on V^*.
3. **Sharpness**. To every outcome $x \in X$, there corresponds a unique state $\epsilon_x \in \Omega$ with $\epsilon_x(x) = 1$.
4. **Correlation**. Every state is the marginal of a correlating non-signaling state.
5. **Filtering**. For every test E and every $f : E \to (0, 1]$, there exists an order-isomorphism $\phi : V^* \to V^*$ with $\phi(x) = f(x)x$.

Special Relativity

c is constant.

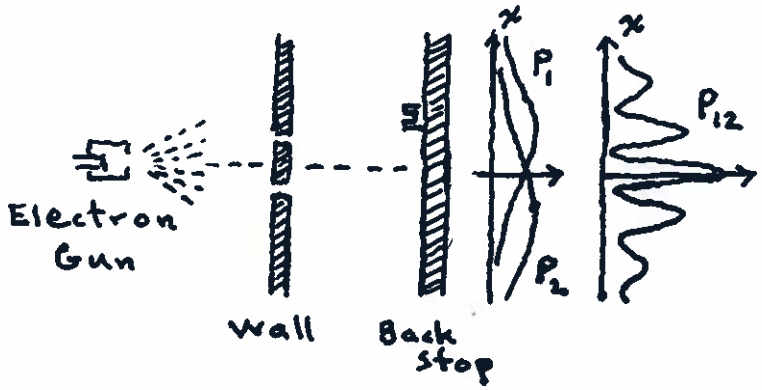
Physics is constant.

If one really understood the central point [of quantum theory] and its necessity in the construction of the world, one ought to be able to state it in one clear, simple sentence. Until we see the quantum principle with this simplicity we can well believe that we do not know the first thing about the universe ... and ... our place in it.

— John Archibald Wheeler

QBies Friday Group Meeting

Feynman 1



"The result P_{12} obtained with both holes is clearly not the sum of P_1 and P_2, the probabilities for each hole alone. In analogy with our water-wave experiment, we say: 'There is interference.'

For electrons: $P_{12} \neq P_1 + P_2$."

Instead $P_1 = |\varphi_1|^2$, $P_2 = |\varphi_2|^2$, $P_{12} = |\varphi_1 + \varphi_2|^2$.

"We shall tackle immediately the basic element of the mysterious behavior in its most strange form. We choose to examine a phenomenon which is impossible, absolutely impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery."

— R. P. Feynman, 1964

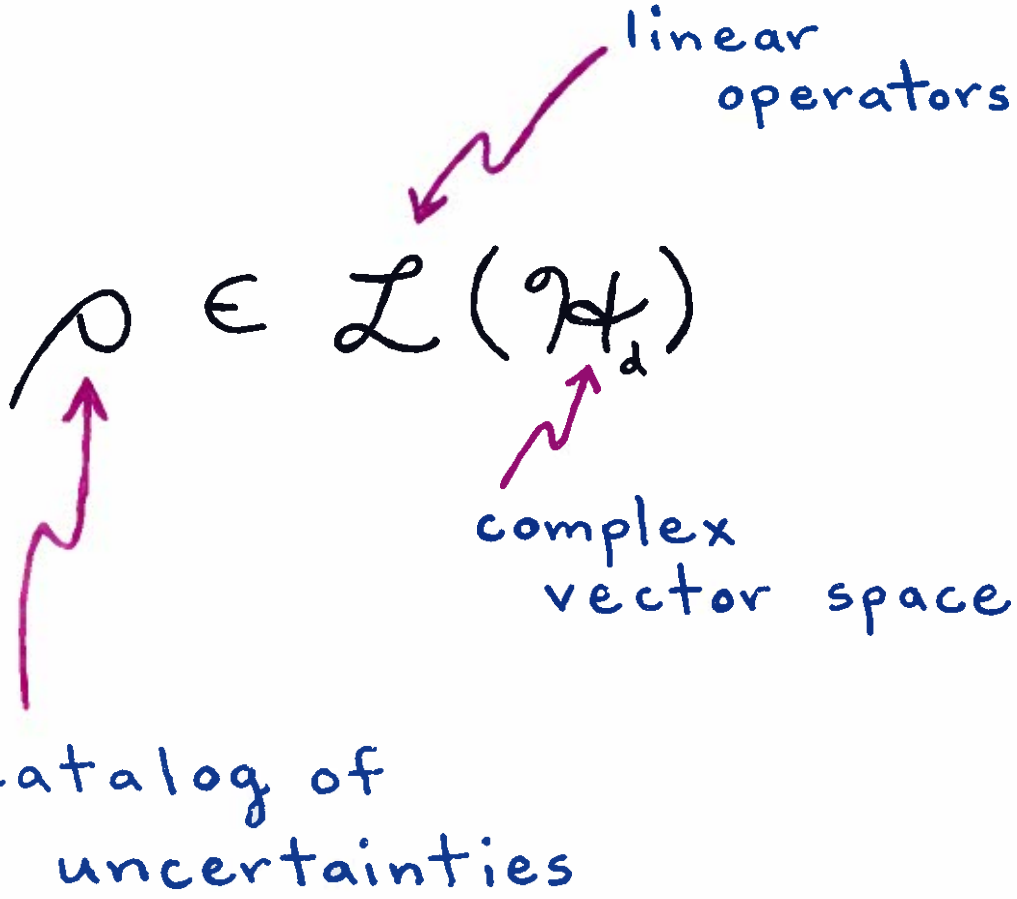
R. P. Feynman, "The Concept of Probability in Quantum Mechanics," 1951

The new theory asserts that there are experiments for which the exact outcome is fundamentally unpredictable, and that in these cases one has to be satisfied with computing probabilities of various outcomes. But far more fundamental was the discovery that in nature the laws of combining probabilities were **not** those of the classical probability theory ...

I should say, that in spite of the implication of the title of this talk the concept of probability is not altered in quantum mechanics. When I say the probability of a certain outcome of an experiment is p, I mean the conventional thing ...

What is changed, and changed radically, is the method of calculating probabilities.

Density Operators



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1) $\rho^\dagger = \rho$

2) $\mathrm{tr}\,\rho = 1$

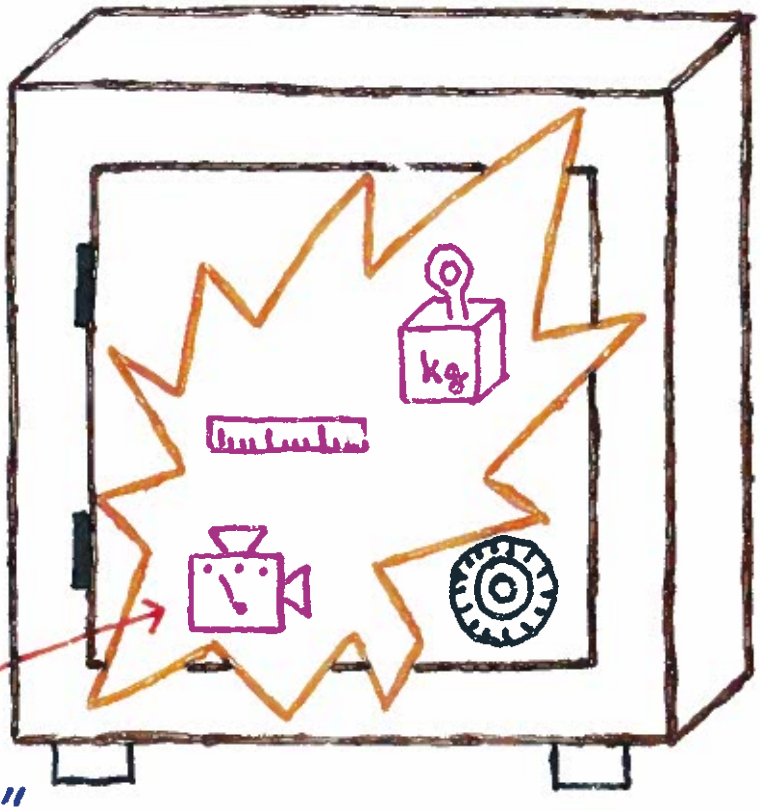
3) $\lambda_i(\rho) \geq 0$

($\lambda_i(\rho)$: eigenvalues)

> convex hull of the set $\{ |\psi\rangle\langle\psi| : |\psi\rangle \in \mathcal{H}_d \}$


p(h)

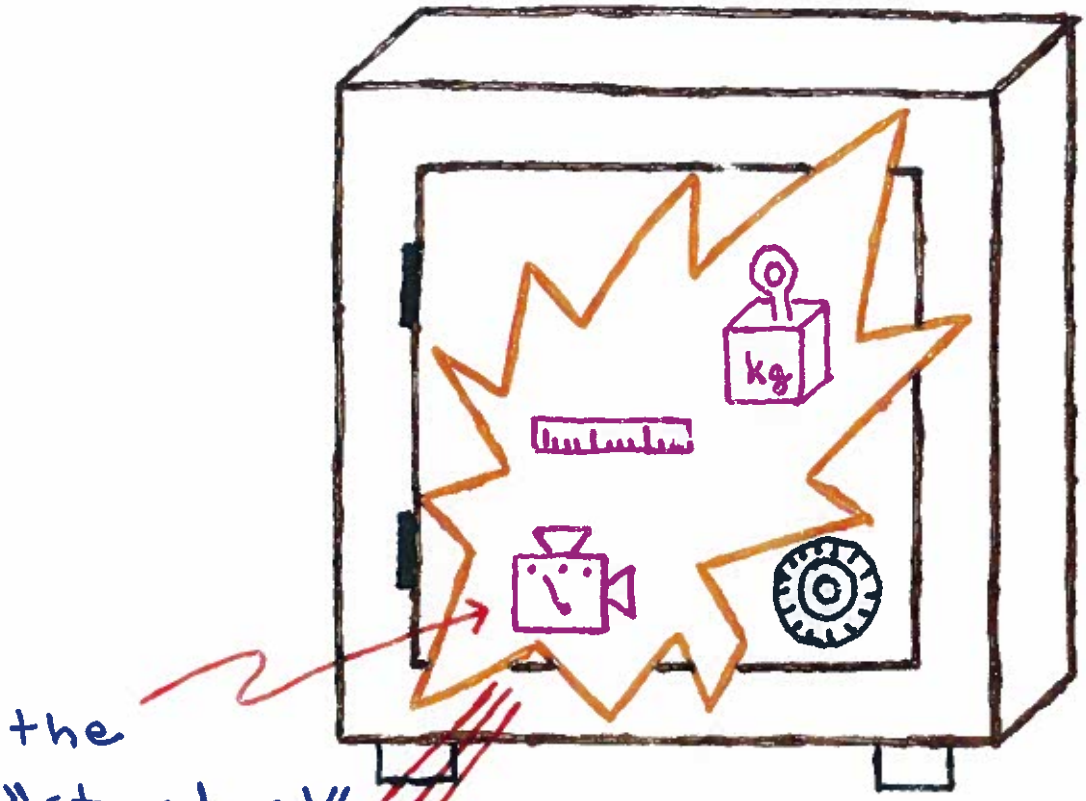
Bureau of Standards



the "standard" quantum measurement

ρ

Bureau of Standards



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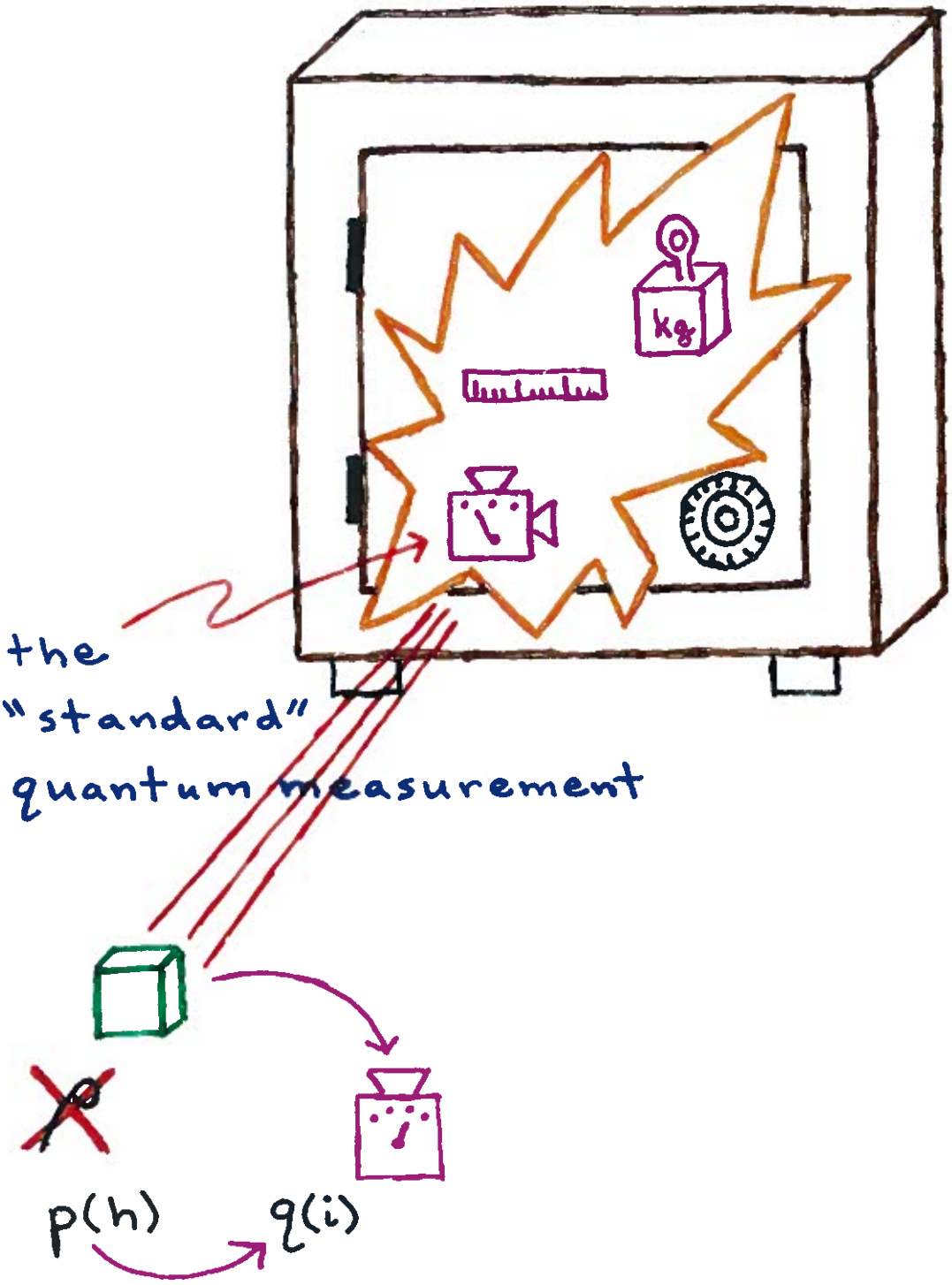
Bureau of Standards



von Neumann

~~Standard~~ measurements

not good enough for

the bureau.

$$H = \sum_i \alpha_i \Pi_i \quad , \quad \Pi_i = |i\rangle\langle i|$$

$$p(i) = \mathrm{tr}\,\rho\Pi_i = \langle i|\rho|i\rangle$$

$$\Rightarrow \begin{pmatrix} \rho_{11} & & \\ & \rho_{22} & \\ & & \ddots \end{pmatrix}$$

Standard Measurements	Generalized Measurements
$\{\Pi_i\}$	$\{E_b\}$
$\langle\psi\|\Pi_i\|\psi\rangle \geq 0, \forall\|\psi\rangle$	$\langle\psi\|E_b\|\psi\rangle \geq 0, \forall\|\psi\rangle$
$\sum_i \Pi_i = I$	$\sum_b E_b = I$
$p(i) = \mathrm{tr}\,\rho\Pi_i$	$p(b) = \mathrm{tr}\,\rho E_b$
$\Pi_i \Pi_j = \delta_{ij}\Pi_i$	—

Informational Completeness

quantum states

$\rho \in \mathcal{L}(\mathcal{H}_D)$ — D^2-dimensional vector space

Choose POVM $\{E_h\}$, $h = 1, \ldots, D^2$, with E_h all linearly independent. (Can be done.)

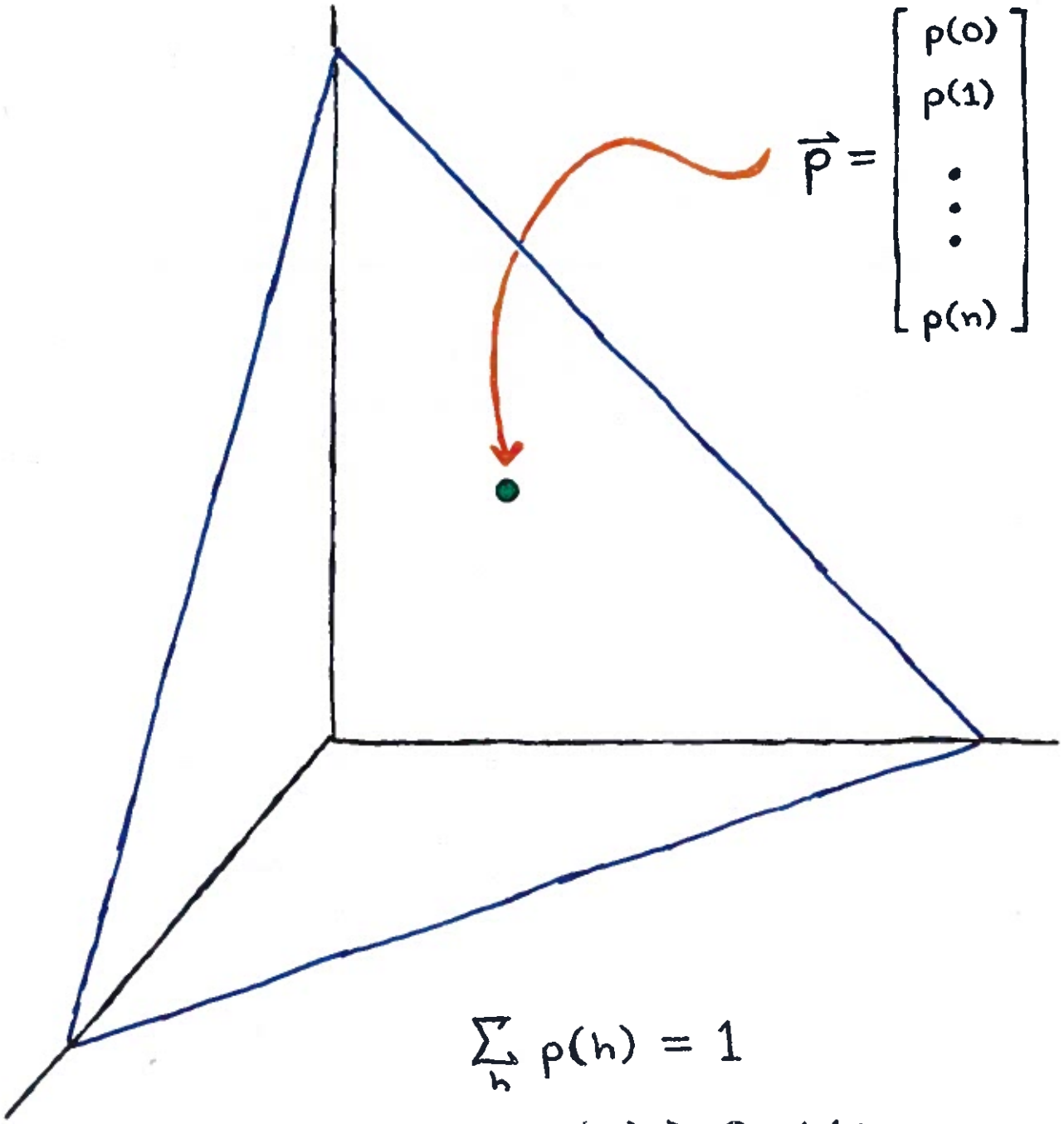
D^2 numbers $p(h) = \mathrm{tr}\,\rho E_h$ determine ρ.

Because $(A,B) = \mathrm{tr}\,A^\dagger B$ is an inner product.

↑ projection of ρ onto E_h

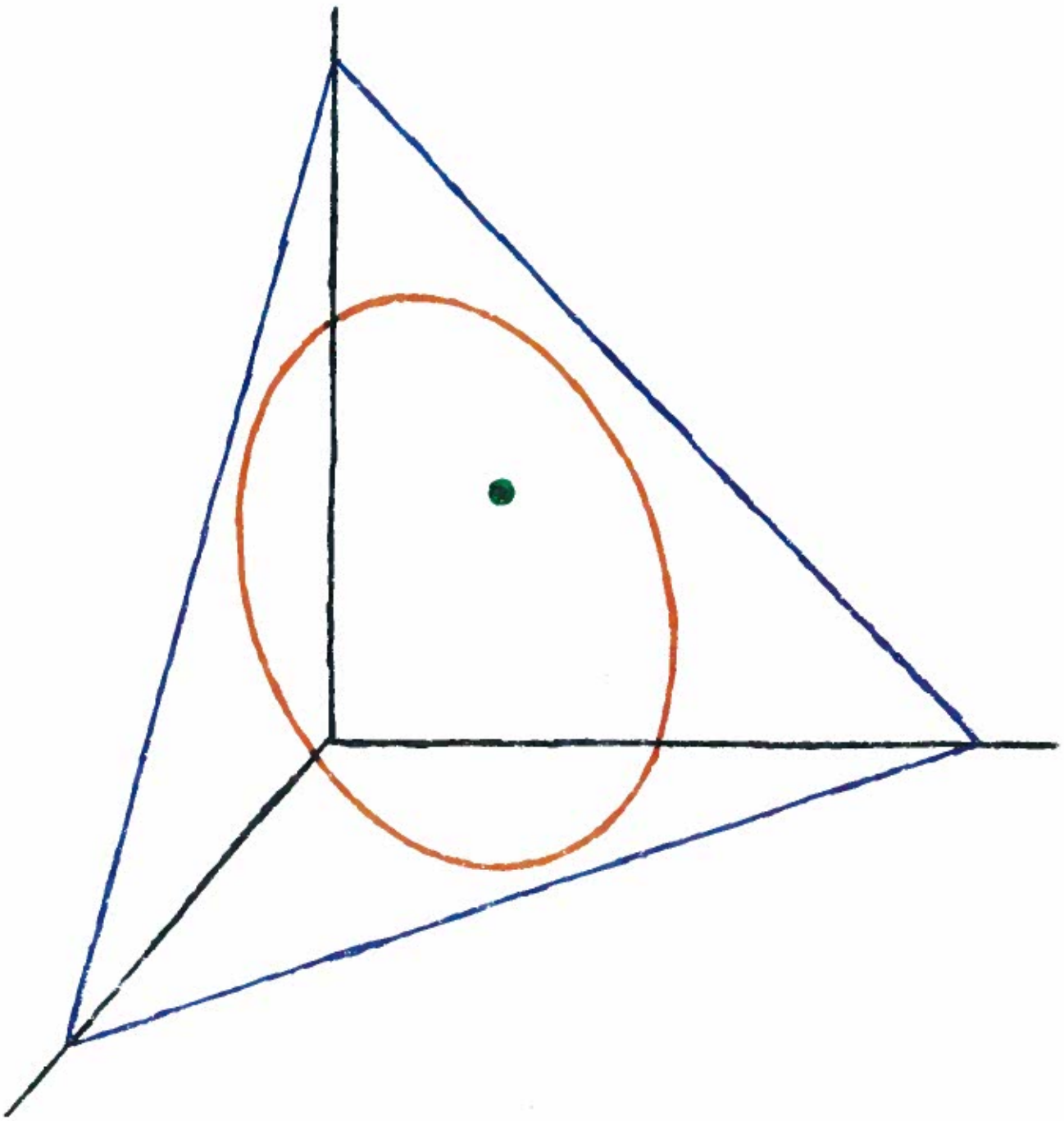
Any such $\{E_h\}$ can be the standard quantum measurement.

Probability Simplex



$$\sum_h p(h) = 1$$

$$p(h) \geq 0 \quad \forall\, h$$



Path Back to Density Ops

Suppose $\{E_j\}$, $j = 1, \ldots, d^2$, is ICP.

Then $p(j)$ determines ρ.

But also $\rho = \sum_j \alpha_j E_j$ for some α_j's.

Thus

$$p(j) = \mathrm{tr}\, \rho E_j = \sum_k \alpha_k \mathrm{tr}\, E_j E_k$$

i.e.

$$\vec{p} = M\vec{\alpha} \quad \text{where} \quad M = [\mathrm{tr}\, E_j E_k]$$

M: nonnegative matrix

and so

$$\boxed{\vec{\alpha} = M^{-1}\vec{p}}$$

Analogy

$$|\psi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$$

$$\rho = \sum_{i=1}^{d^2} \alpha_i A_i$$

operators

What properties should we demand of these?

Measure of Orthonormality

ADF, 0707.2071

Suppose A_i, $i = 1, \ldots, d^2$

positive semi-definite.

And $\mathrm{tr}\, A_i^2 = 1$.

"Orthonormality"

$$K = \sum_{i \neq j} (\mathrm{tr}\, A_i A_j)^2$$

smaller the better

Can prove

$$K \geq \frac{d^2(d-1)}{d+1} \quad \text{with} = \text{iff}$$

$$\mathrm{tr}\, A_i A_j = \frac{1}{d+1} \quad \forall\, i \neq j$$

A_i — rank-1

A Very Fundamental Mmt ?

Suppose d^2 projectors $\Pi_i = |\psi_i\rangle\langle\psi_i|$ satisfying

$$\mathrm{tr}\,\Pi_i \Pi_j = \frac{1}{d+1} \quad , \quad i \neq j$$

exist. ← Called SIC.

Can prove:

1) the Π_i linearly independent

2) $\sum_i \frac{1}{d} \Pi_i = I$

So good for Bureau of Standards.

Also

$$p(i) = \frac{1}{d}\,\mathrm{tr}\,\rho\Pi_i$$

$$\boxed{\rho = \sum_i \left[(d+1)p(i) - \frac{1}{d}\right]\Pi_i}$$

CAF, QIC 4, 467 (2002)

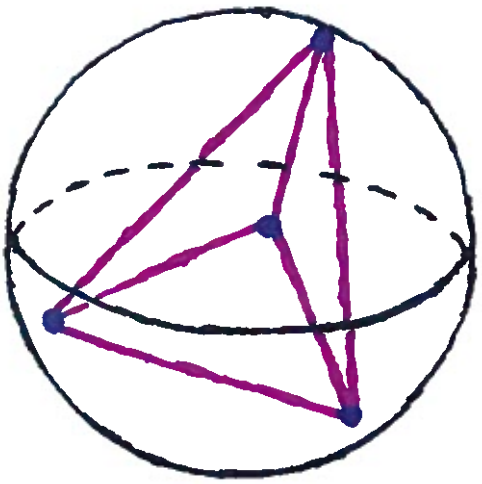
But do they

EXIST

?

SIC Sets

dimension 2



any
regular
tetrahedron

dimension 3

Let $\omega = e^{\frac{2\pi i}{3}}$.

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \omega \\ -\bar{\omega} \end{bmatrix} \quad \begin{bmatrix} 0 \\ \bar{\omega} \\ -\omega \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ \bar{\omega} \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -\omega \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -\bar{\omega} \\ 0 \end{bmatrix}$$

Dimension 6
Appleby's "pencil and paper" Solution

$$|\psi\rangle = \frac{\alpha}{3\sqrt{2}} \begin{pmatrix} f_+ \\ \sigma^5 f_- \\ \sigma^8 f_+ \\ \sigma^{-3} f_- \\ \sigma^8 f_+ \\ \sigma^5 f_- \end{pmatrix} + \frac{\beta_- e^{i\theta_+}}{3\sqrt{2}} \begin{pmatrix} \sigma^8 f_- \\ \sigma^{-7} f_+ \\ f_- \\ \sigma^{-7} f_+ \\ \sigma^8 f_- \\ \sigma^9 f_+ \end{pmatrix} + \frac{\beta_+ e^{i\theta_-}}{3\sqrt{2}} \begin{pmatrix} \sigma^8 f_- \\ \sigma^9 f_+ \\ \sigma^8 f_- \\ \sigma^{-7} f_+ \\ f_- \\ \sigma^{-7} f_+ \end{pmatrix}$$

where

$$\sigma = e^{i\pi/12}$$

$$f_\pm = \sqrt{3 \pm \sqrt{3}}$$

$$g = \sqrt{6\sqrt{21} - 18}$$

$$\alpha = \sqrt{\frac{7 - \sqrt{21}}{14}}$$

$$\beta_\pm = \sqrt{\frac{7 + \sqrt{21} \pm \sqrt{14\sqrt{21} - 42}}{28}}$$

$$e^{i\theta_\pm} = \frac{1}{2}\left(\sqrt{46 - 6\sqrt{21} \mp 6g} \pm i\sqrt{18 + 6\sqrt{21} \pm 6g} \right)^{\frac{1}{3}}$$

Evidence for Existence

Exact Solutions

$d = 2 - 16, 19, 24, 28, 31, 35, 37, 43, 48$

(Appleby, Grassl, Zauner, ...)

Numerical Approximations

$d = 2 - 76, \_\_, 78 - 81, \_\_, \_\_, \_\_, 85, \_\_, 87$

(E. Schnetter, A.J. Scott)

Where there is symmetry,

there are groups?

Might it be possible to generate $|\psi_j\rangle$ as the orbit of some group action?

I.e. might there be some group G and some unitary representation $\{U_g : g \in G\}$ such that

$$|\psi_g\rangle = U_g |\psi_0\rangle$$

is SIC?

Toolbox

Generalized Pauli Operators

$$X: \ |j\rangle \longrightarrow |j+1\rangle$$

$$Z: \ |j\rangle \longrightarrow \omega^j |j\rangle$$

where $\omega = e^{\frac{2\pi i}{d}}$

"Error Operators"

~~"Displacement Operators"~~

$$D_{jk} = \tau^{jk} X^j Z^k$$

where $\tau = -e^{\pi i/d}$

$$\mathrm{tr}\, D_{jk}^{\dagger} D_{\ell m} = d\, \delta_{j\ell} \delta_{km}$$

$$D_{jk} D_{\ell m} = \tau^{k\ell - jm} D_{j+\ell, k+m}$$

JMP 45 (2004)

Renes et al. Conjecture

Let $|\psi_{jk}\rangle = D_{jk}|\psi\rangle$ ← "fiducial state"

$$\Pi_{jk} = |\psi_{jk}\rangle\langle\psi_{jk}| \, .$$

Conjecture:

There exists a $|\psi\rangle$ such that

$$\mathrm{tr}\, \Pi_{jk}\Pi_{\ell m} = \frac{1}{d+1} \quad \text{when } (j,k) \neq (\ell,m).$$

"Π_{jk} is an SIC"

Evidence:

SIC POVMs and Clifford groups in prime dimensions

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Abstract. We show that in prime dimensions not equal to three, each group covariant symmetric informationally complete positive operator valued measure (SIC POVM) is covariant with respect to a unique Heisenberg–Weyl (HW) group. Moreover, the symmetry group of the SIC POVM is a subgroup of the Clifford group. Hence, two SIC POVMs covariant with respect to the HW group are unitarily or antiunitarily equivalent if and only if they are on the same orbit of the extended Clifford group. In dimension three, each group covariant SIC POVM may be covariant with respect to three or nine HW groups, and the symmetry group of the SIC POVM is a subgroup of at least one of the Clifford groups of these HW groups respectively. There may exist two or three orbits of equivalent SIC POVMs for each group covariant SIC POVM, depending on the order of its symmetry group. We then establish a complete equivalence relation among group covariant SIC POVMs in dimension three, and classify inequivalent ones according to the geometric phases associated with fiducial vectors. Finally, we uncover additional SIC POVMs by regrouping of the fiducial vectors from different SIC POVMs which may or may not be on the same orbit of the extended Clifford group.

PACS numbers: 03.65.-w, 03.65.Wj, 02.10.De, 03.67.-a

Galois Automorphisms of a Symmetric Measurement

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Abstract

SIC-POVMs (Symmetric Informationally Complete Positive Operator Valued Measures) have been constructed in every dimension ≤ 67. However, a proof that they exist in every finite dimension has yet to be constructed. In this paper we examine the Galois group of SIC-POVMs covariant with respect to the Weyl-Heisenberg group, or WH SICs (the great majority, though not all of the known examples are of this type). Scott and Grassl have noted that every known exact WH SIC is expressible in radicals, which means that the corresponding Galois group is solvable. They have also calculated the Galois group for most known exact examples. The purpose of this paper is to take the analysis of Scott and Grassl further. We first prove a number of theorems regarding the structure of the Galois group and the relation between it and the extended Clifford group. We then examine the Galois group for the known exact fiducials and on the basis of this we propose a list of 8 conjectures concerning its structure. These conjectures represent a considerable strengthening of the theorems we have actually been able to prove. Finally we generalize the concept of an anti-unitary to the concept of a g-unitary, and show that every WH SIC fiducial is an eigenvector of a family of g-unitaries.

We conclude this introductory section by drawing the reader's attention to two particularly striking points to emerge from our analysis. Let Π be a fiducial projector. Let $\mathbb{E}$ be the smallest normal extension of $\mathbb{Q}$ containing the standard basis matrix elements of Π and $\tau = -e^{\frac{i\pi}{d}}$. $\mathbb{E}$ only depends on the extended Clifford group orbit to which Π belongs. It turns out that if $d > 3$ then $\mathbb{E}$ is an Abelian extension of the real quadratic field

$$\mathbb{Q}\left(\sqrt{(d-3)(d+1)}\right) \tag{2}$$

for all 27 extended Clifford group orbits on which an exact fiducial is known. The

Tight informationally complete quantum measurements

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We introduce a class of informationally complete positive-operator-valued measures which are, in analogy with a tight frame, "as close as possible" to orthonormal bases for the space of quantum states. These measures are distinguished by an exceptionally simple state-reconstruction formula which allows "painless" quantum state tomography. Complete sets of mutually unbiased bases and symmetric informationally complete positive-operator-valued measures are both members of this class, the latter being the unique minimal rank-one members. Recast as ensembles of pure quantum states, the rank-one members are in fact equivalent to weighted 2-designs in complex projective space. These measures are shown to be optimal for quantum cloning and linear quantum state tomography.

PACS numbers: 03.65.Wj,03.67.-a,02.10.Ud
Keywords: quantum measurement, informational completeness, frame theory, combinatorial design

I. INTRODUCTION

The retrieval of classical data from quantum systems, a task described by quantum measurement theory, is an overlooked – though important – component of quantum information processing [1]. The ability to precisely determine a quantum state is paramount to tests of quantum information processing devices such as quantum teleporters, key

A Kochen-Specker inequality from a SIC

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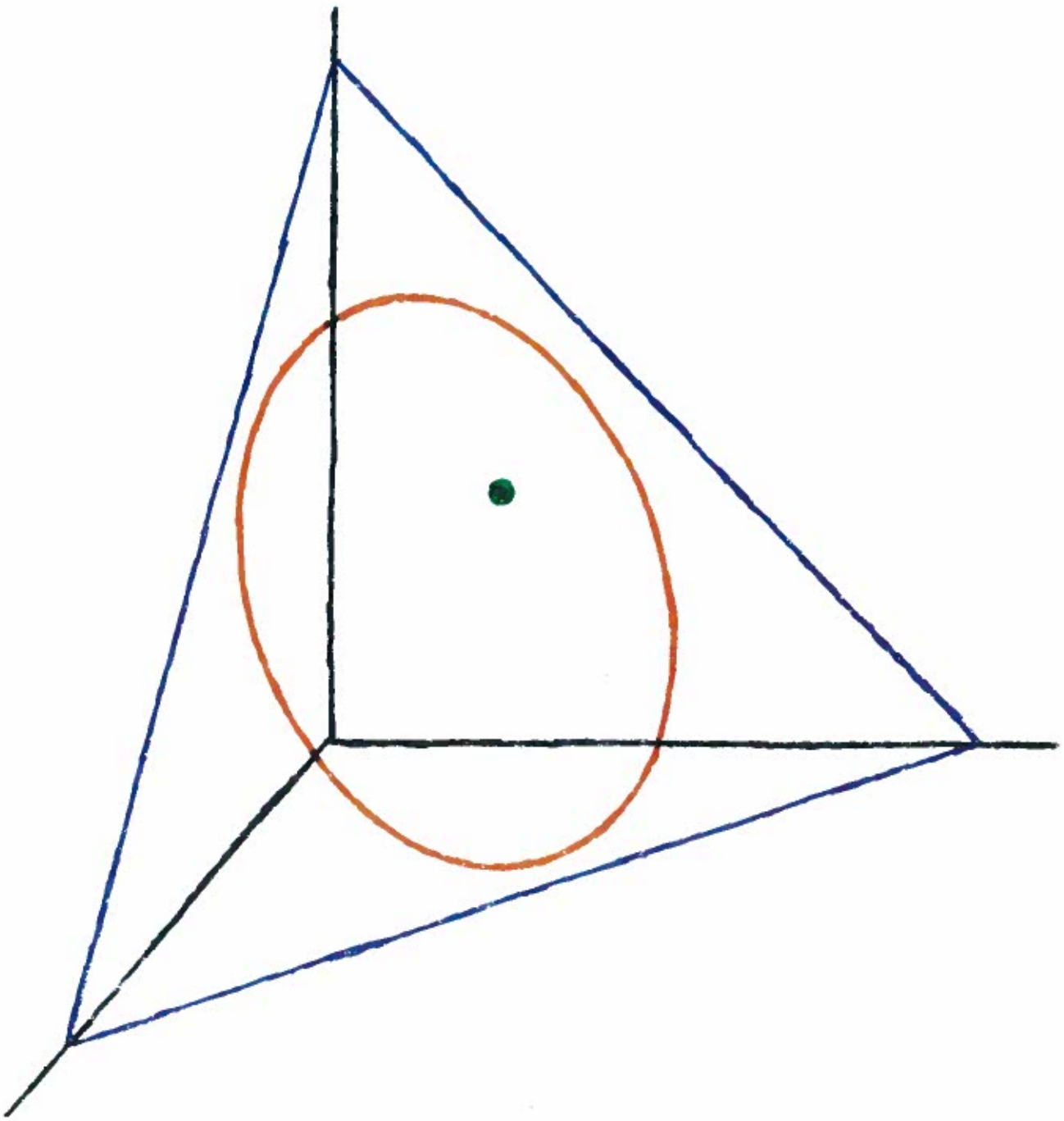
(Dated: October 7, 2011)

Yu and Oh [1] have given a state-independent proof of the Kochen-Specker theorem in three dimensions using only 13 rays. The proof consists of showing that a non-contextual hidden variable theory necessarily leads to an inequality that is violated by quantum mechanics. We give a similar proof making use of 21 rays that constitute a SIC and four Mutually Unbiased Bases.

PACS numbers: 03.65.Ta, 03.65.Ud

Introduction.—The Kochen-Specker theorem states that a certain kind of hidden variable theory cannot be consistent with quantum mechanics. The idea is to assign truth values (1 for true, 0 for false) to a finite set of measurements represented by projectors onto rays in Hilbert space. These assignments must obey the Kochen-Specker rules, namely no two orthogonal projectors can both be true, and one member of each complete orthonormal basis must be true. Since two orthogonal projectors commute they represent compatible measurements. Note that the assignment made for a particular projector is independent of which particular set of mutually compatible measurements it belongs to—even though it may belong logically impossible. On the other hand the reformulation of the Kochen-Specker theorem in terms of inequalities has led to a number of recent experimental tests [9–13]. Using inequalities also has the incidental advantage that the Kochen-Specker theorem can be proved over the rational numbers [14].

Our purpose is to give a state-independent proof along the same lines as Yu and Oh, but starting from a configuration of rays in three dimensions that is of independent interest: a symmetric informationally-complete POVM (SIC) and a complete set of mutually unbiased bases (MUB). The resulting configuration of 21 rays is highly symmetric, and we believe that it has some ad-



Pure States in SIC Language

Conditions

$$\rho^+ = \rho \quad , \quad \mathrm{tr}\,\rho^2 = \mathrm{tr}\,\rho^3 = 1$$

translate to

$$\sum_i p(i)^2 = \frac{2}{d(d+1)}$$

and

$$\sum_{jkl} c_{jkl}\, p(j)\, p(k)\, p(l) = \frac{d+7}{(d+1)^3}$$

where

$$c_{jkl} = \mathrm{Re}\,\mathrm{tr}\,\Pi_j \Pi_k \Pi_l$$

Could these be independently motivatable physical constants?

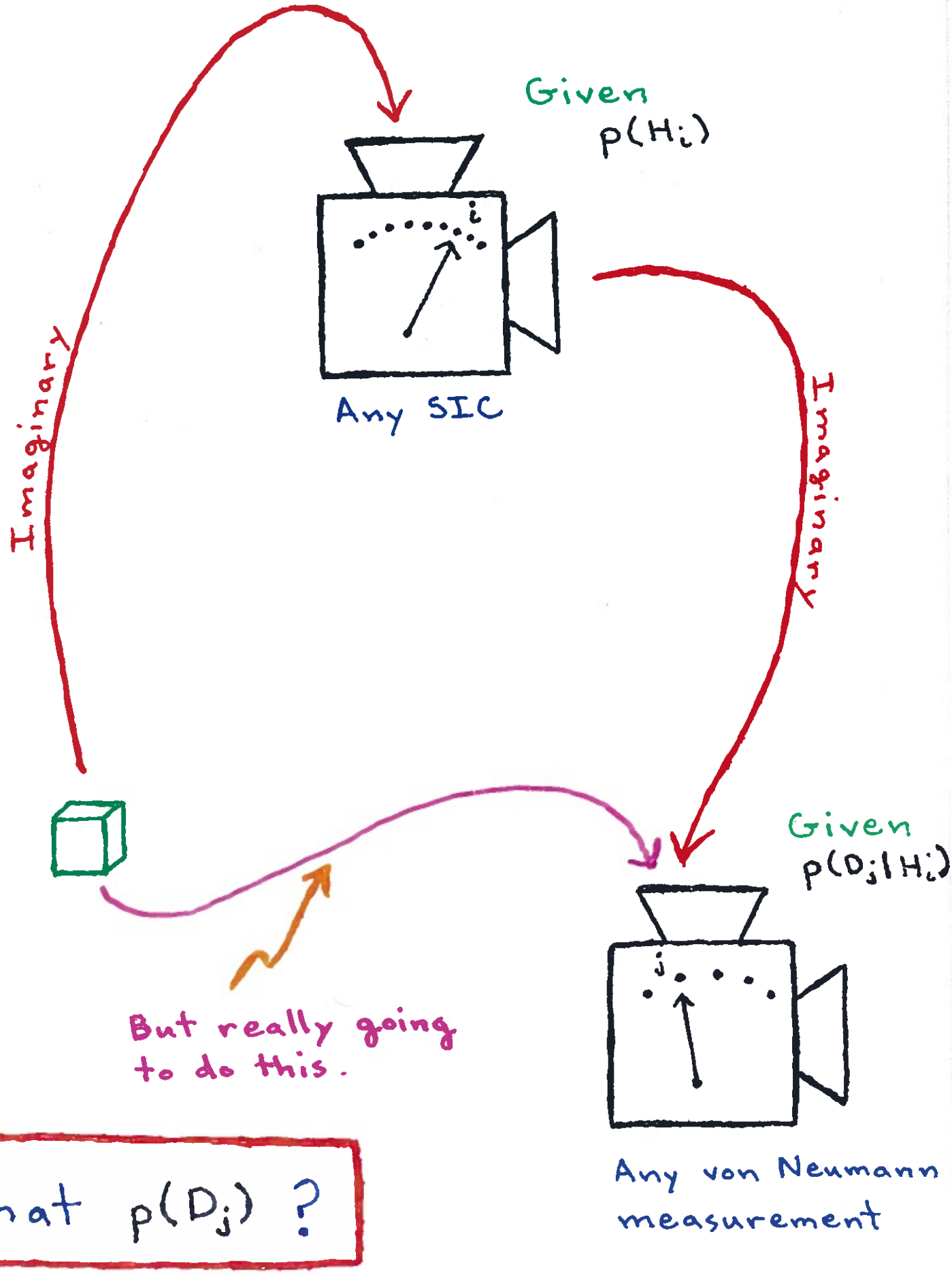
Gelo Tabia's "QBic Equation" for a Qutrit

arXiv:1207.6035v1

$$\sum_i p(i)^3 \;-\; 3 \sum_{(ijk)\in Q} p(i)p(j)p(k) \;=\; 0$$

where Q consists of all lines in the 3×3 affine plane:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \qquad \text{I.e.,} \qquad Q \;=\; \left\{ \begin{array}{ccc} (123) & (456) & (789) \\ (147) & (258) & (369) \\ (159) & (267) & (348) \\ (168) & (249) & (357) \end{array} \right\}$$


Given
$p(H_i)$
i
Any SIC
Imaginary
Imaginary
Given
$p(D_j|H_i)$
But really going to do this.
j
Any von Neumann measurement
What $p(D_j)$?

In this case,

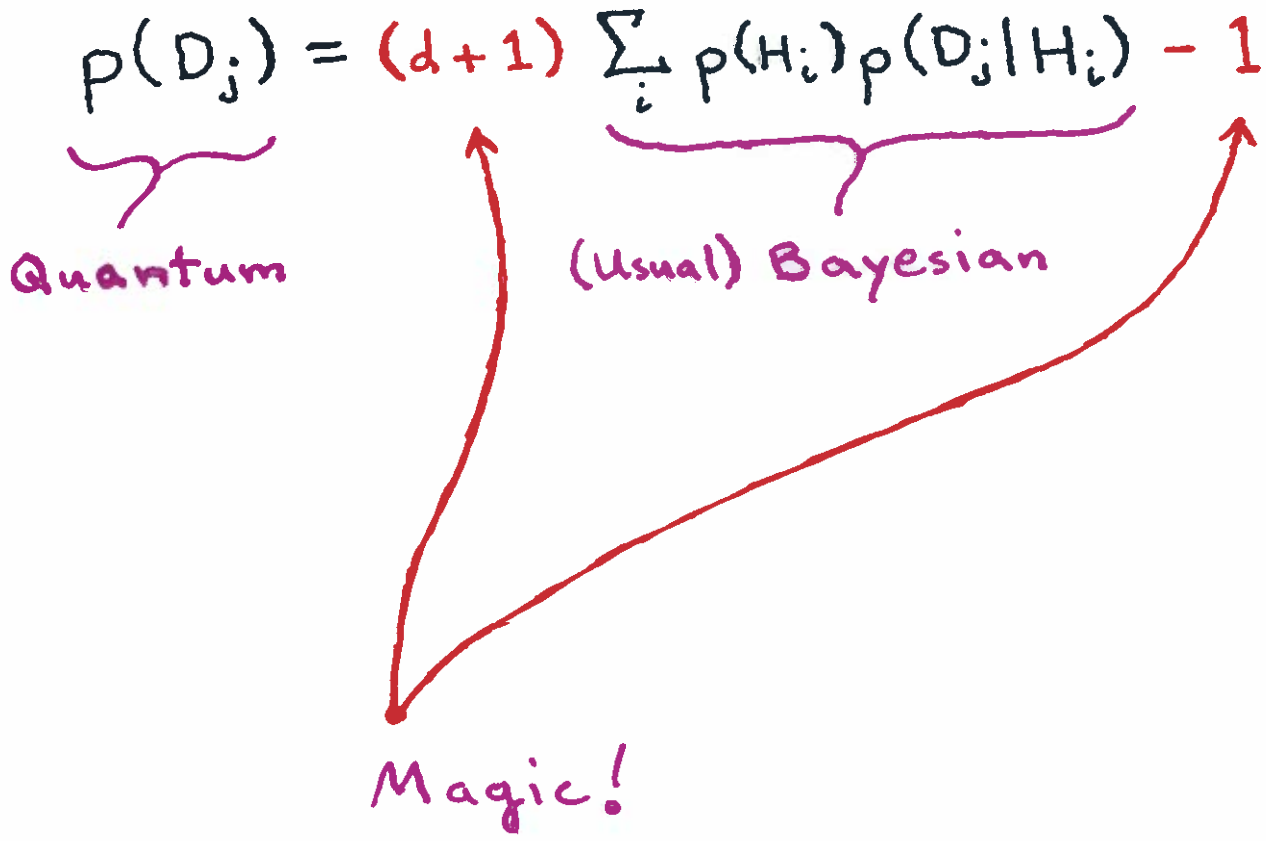
$$p(D_j) \neq \sum_i p(H_i)\, p(D_j \mid H_i) .$$

As Ballentine (1986) points out, there are hidden conditionals

$p(D_j)$ really $p(D_j \mid C_1)$

$p(H_i)$ really $p(H_i \mid C_2)$

$p(D_j \mid H_i)$ really $p(D_j \mid H_i, C_2)$


$p(D_j) = (d+1) \sum_i p(H_i)\, p(D_j|H_i) - 1$
Quantum
(Usual) Bayesian
Magic!

Law of Total Probability:

$$p(D_j) = \sum_i p(H_i)\, p(D_j|H_i)$$

The Born Rule:

$$q(D_j) = \mathrm{tr}\, \hat{\rho} \hat{D}_j$$

$$= (d+1)\, p(D_j) - 1$$

dimensionality of the system

Could we take diagram and modified Law of Total Probability

$$p(D_j) = (d+1)\sum_i p(H_i)\,p(D_j|H_i) - 1$$

as a fundamental postulate of quantum mechanics?

Nearly the consistency
of this equation alone
implies a significant,
nontrivial convex structure.

Homework

containing the $\vec{e}_k$

Call a set $\mathcal{S} \subseteq \Delta_{d^2}$ within the probability simplex

a) consistent if for any $\vec{p}, \vec{q} \in \mathcal{S}$

$$\frac{1}{d(d+1)} \leq \vec{p} \cdot \vec{q} \leq \frac{2}{d(d+1)} ,$$

b) maximal if adding any further $\vec{p} \in \Delta_{d^2}$ makes it inconsistent

Example: If $\mathcal{S}$ is set of quantum states, it is consistent & maximal.

Problem: Characterize all such $\mathcal{S}$; compare to quantum.

Sphere Too Big for Simplex

Re-reference points to center $\vec{c}$ of probability simplex:

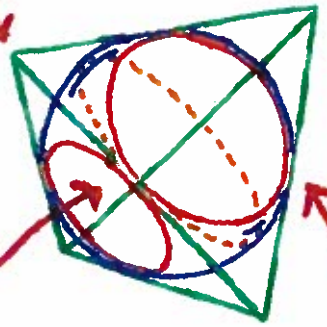
$$\vec{p}' = \vec{p} - \vec{c} \ .$$

Consistency condition becomes:

$$\frac{-1}{d^2(d+1)} \leq \vec{p}' \cdot \vec{q}' \leq \frac{d-1}{d^2(d+1)} \ .$$

Sphere too big for simplex!

face of low dimension not reached



face of high dimension cuts off sphere

sphere touches face of dimension $\frac{1}{2}d(d+1) - 1$

Examples

1) Take $\vec{q} = \vec{p}$. Consequently must have

$$\vec{p} \cdot \vec{p} \leq \frac{2}{d(d+1)}$$

Same as quantum.

2) Consider a subset $\{\vec{p}_k\} \subseteq \mathcal{S}$ with $k = 1, \ldots, m$ such that

$$\vec{p}_k \cdot \vec{p}_k = \frac{2}{d(d+1)}$$

$$\vec{p}_k \cdot \vec{p}_\ell = \frac{1}{d(d+1)} \qquad k \neq \ell .$$

How large can m be?

Answer: d, same as quantum

If one really understood the central point [of quantum theory] and its necessity in the construction of the world, one ought to be able to state it in one clear, simple sentence. Until we see the quantum principle with this simplicity we can well believe that we do not know the first thing about the universe ... and ... our place in it.

— John Archibald Wheeler