Explicit bounds on the entangled value of multiplayer XOR games

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CWI

Joint work with Thomas Vidick (MIT)

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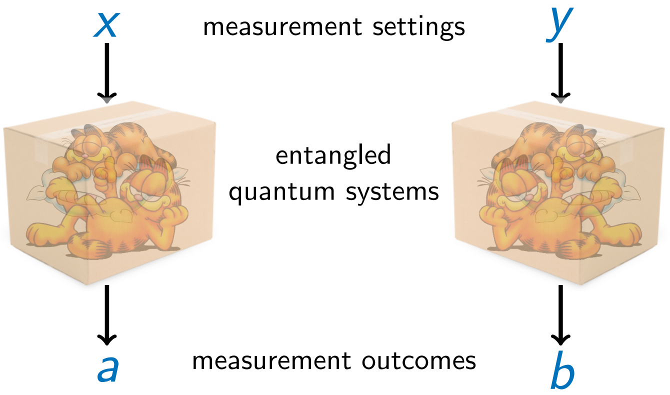
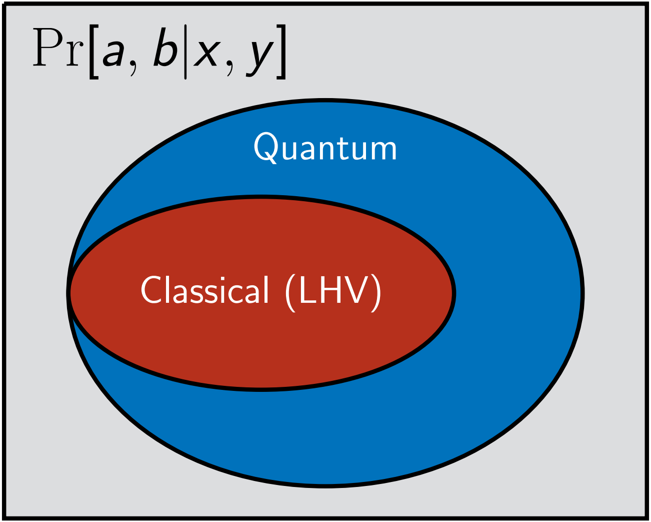
Entanglement and nonlocal correlations

- [Bell64] Measurements on entangled quantum systems can give outcomes that are *correlated* in a *non-classical* way



Qualitative picture of Bell's Theorem



Nonlocal games

- [CHTW04] Use *nonlocal games* to quantitatively study entanglement

x randomly selected questions y

noncommunicating players

a answers b

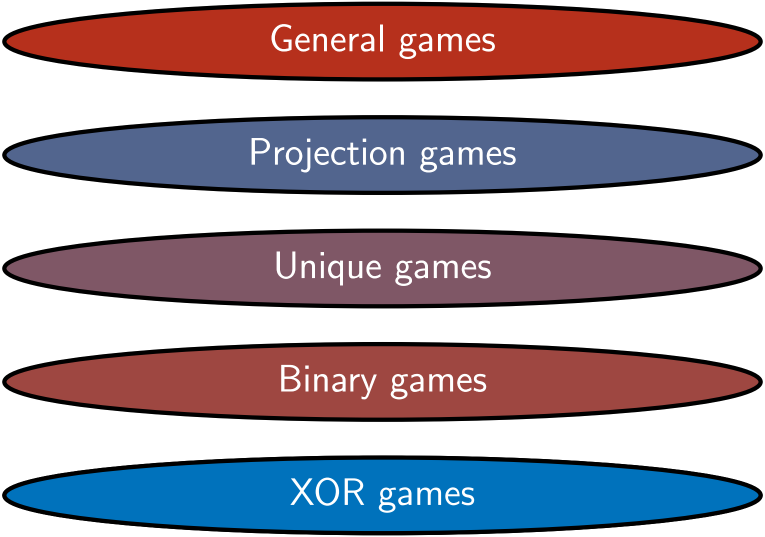
- Given predicate $f : (a, b, x, y) \mapsto \{\text{WIN, LOSE}\}$ known to both players
- compare the maximum winning probability *with entanglement* to the maximum winning probability *without entanglement*

Main question

How much advantage can entanglement give in nonlocal games?

. . . in terms of

- the number of possible questions
- the number of possible answers
- the local Hilbert space dimension of the shared state
- the number of players?


General games
Projection games
Unique games
Binary games
XOR games

XOR games

- $A \in \{-1, 1\}^{n \times n}$, probability distribution $\pi : [n] \times [n] \to [0, 1]$

$i \in [n]$ $\qquad$ $j \in [n]$

random questions $(i, j) \sim \pi$

answers

$x = \pm 1$ $\qquad$ $y = \pm 1$

- The players win iff $xy = A_{ij}$, equivalently iff $A_{ij}xy = +1$.

Example: The CHSH Game

i $\downarrow$

$$H_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

j $\downarrow$

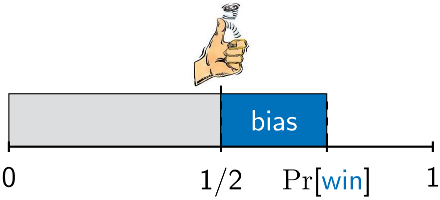
$\downarrow$ answers $\downarrow$

$x = \pm 1$ $y = \pm 1$

- Alice and Bob win iff $H_{ij}xy = +1$
- Classically, Alice and Bob win with prob. at most $3/4$
- but by sharing an EPR pair $(|00\rangle + |11\rangle)/\sqrt{2}$, with prob. 0.85!

The bias

- Note that XOR games can always be won with prob. $\geq 1/2$
- For random bits x, y, we have $\Pr[A_{ij}xy = +1] = 1/2$



- The bias equals $1/2$ times

$$\Pr[\text{win}] - \Pr[\text{lose}] = \mathbb{E}_{i,j}\Big[A_{ij}\, x_i y_j\Big]$$

where $x_i, y_j \in \{-1, 1\}$ are answers to questions i, j, resp.

- The advantage of entanglement is measured by the *ratio* of the entangled and classical biases

The classical bias

- For XOR game $G = (\pi, A)$ and $M_{ij} = \pi(i,j)A_{ij}$, the classical bias is given by

$$\beta(G) = \max \sum_{i,j} M_{ij}\, x_i y_j, \qquad \text{such that} \;\; x_i, y_j \in \{-1, 1\}$$

The entangled bias

- A $\{-1,1\}$*-valued observable* is a Hermitian matrix with ± 1 eigenvalues

- For XOR game $G = (\pi, A)$ and $M_{ij} = \pi(i,j)A_{ij}$, the entangled bias is given by

$$\begin{aligned} \beta^*(G) \quad = \quad & \max \sum_{i,j} M_{ij} \langle \psi | X_i \otimes Y_j | \psi \rangle, \\ & \text{such that} \quad |\psi\rangle \text{ is a state and } X_i, Y_j \text{ are} \\ & \qquad\qquad\quad \{-1,1\}\text{-valued observables} \end{aligned}$$

Thm. (Tsirelson). The following identity holds true:

$$\beta^*(G) = \max \sum_{i,j} M_{ij} \langle u_i, v_j \rangle, \qquad \text{such that} \quad \|u_i\|_2 = \|v_j\|_2 = 1$$

Grothendieck's inequality

Thm. (Grothendieck). There exists a universal constant K_G such that for any XOR game G, we have

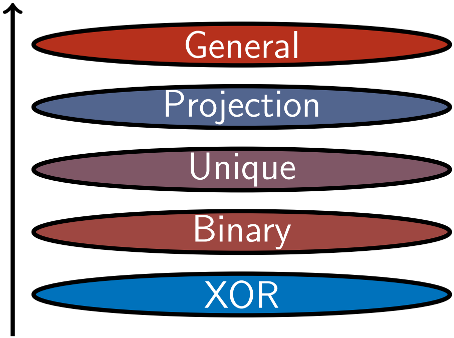
$$\beta^*(G) \leq K_G \, \beta(G)$$

- To this day, the exact value of K_G is unknown
- Davie (1984) and Reeds (1991) proved $K_G \gtrsim 1.68\ldots$
- Braverman, Makarychev, Makarychev and Naor (2011)

$$K_G < \frac{2}{\pi \ln(1+\sqrt{2})} = 1.78\ldots$$

- To observe larger contrast between classical and entangled systems, we need more general games than XOR games.
- There are two directions for generalization

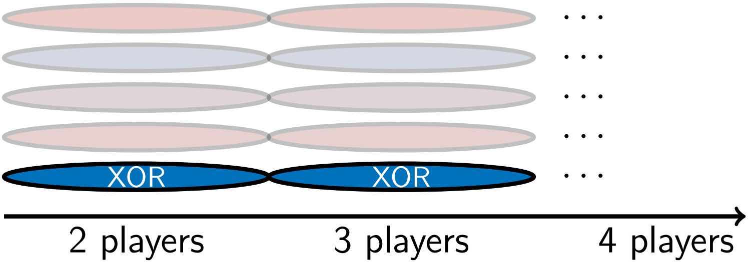
First direction: Moving *up*



- [JPP$^+$10, JP11] Upper bounds on the advantage in terms of the number of questions, answers and dimensions
- [BRSW11] Near-optimal lower bounds
- Disadvantage: complex measurements that may be hard to implement experimentally

Second direction: Moving *sideways*

- Stick with XOR games, but increase the number of *players*



- Involves *multipartite entanglement*, which is still poorly understood

- This is the direction we consider for the rest of this talk

Three-player XOR games

- Tensor $M : [n] \times [n] \times [n] \to [-1, 1]$ known to all players

$i \in [n]$ $\qquad$ $j \in [n]$ $\qquad$ $k \in [n]$



$x = \pm 1$ $\qquad$ $y = \pm 1$ $\qquad$ $z = \pm 1$

- Players win if $\mathrm{sgn}\left(M_{ijk}\right) xyz = +1$
- Can be won with prob. $1/2$ by flipping coins

The classical bias in three-player XOR games

- The classical bias is given by

$$\beta(G_M) = \max \sum_{i,j,k} M_{ijk}\, x_i y_j z_k, \quad \text{such that } x_i, y_j, z_k \in \{-1, 1\}$$

The entangled bias in three-player XOR games

- The entangled bias is given by

$$\beta^*(G) \;=\; \sup \sum_{i,j,k} M_{ijk} \langle\psi|X_i \otimes Y_j \otimes Z_k|\psi\rangle,$$

such that $|\psi\rangle$ is a state and X_i, Y_j, Z_k are $\{-1,1\}$-valued observables

Unbounded bias ratio in 3-player XOR games

Thm. [Pérez-García et al. 08] For any $R \geq 1$, there exists a 3-player XOR game with bias ratio $\beta^*/\beta \geq R$

- Striking in light of Tsirelson's bound for 2p XOR games. There is no "tripartite" Grothendieck inequality in this sense

- Uses local Hilbert space dimension $O(R^2)$ for Alice (optimal), but the dimensions for Bob and Charlie are *unbounded*

- The game is quite large: Alice gets $O(R^8)$ questions, but Bob and Charlie up to $\exp(R)$

- Highly non-explicit existence proof based on deep results from operator space theory

Our main result

Thm. [B, Vidick] For any integer $N = 2^n$ there exists a three-player XOR game G_N such that

$$\beta^*(G_N) \geq \widetilde{\Omega}(\sqrt{N})\, \beta(G_N)$$

- Moreover, the game G_N has N^2 questions per player (close to optimal)
- There is an entangled strategy achieving this gap with N dimensions per player (optimal) and only "Pauli observables"

Proof sketch of the construction

- The proof involves three steps:

  1. Construct a 3p XOR game from any *matrix* with appropriate size

  2. Relate classical and entangled biases to *spectral properties* of the matrix

  3. Use the probabilistic method to *prove existence* of a matrix with good spectral properties

Step 1: *The Pauli-Fourier expansion*

- The n-qubit Pauli matrices $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}^{\otimes n}$ form a basis for the space of N-by-N matrices

$$\underbrace{\boxed{M}}_{2^n \times 2^n} = \frac{1}{N} \sum_P \underbrace{\widehat{M}_P}_{\text{Pauli-Fourier coefficient}} \underbrace{\boxed{P}}_{n\text{-qubit Pauli}}$$

Step 1: *The Pauli-Fourier expansion*

- The n-qubit Pauli matrices $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}^{\otimes n}$ form a basis for the space of N-by-N matrices

N^3

$$\boxed{M} = \frac{1}{N^3} \sum_{P,Q,R} \widehat{M}_{PQR} \underbrace{\boxed{P \otimes Q \otimes R}}_{3n\text{-qubit Pauli}}$$

N^3

Pauli-Fourier coefficient

Step 1: *The game*

- Given a Hermitian matrix M of size 2^{3n}, suitably normalized
- pick n-qubit Paulis P, Q, R with prob. $|\widehat{M}_{PQR}|$

P | Q | R



$x = \pm 1$ | $y = \pm 1$ | $z = \pm 1$

- Players win iff $\mathrm{sgn}(\widehat{M}_{PQR})xyz = +1$
- The game tensor is given by $(\widehat{M}_{PQR})_{P,Q,R}$

Step 2: *Upper bound on the classical bias*

- We have $\widehat{M}_{PQR} = \langle M, P \otimes Q \otimes R \rangle$
- Plug this into the expression for the classical bias

$$
\begin{aligned}
\beta(G_M) &= \sum_{P,Q,R} \widehat{M}_{PQR}\, x_P y_Q z_R \\
&= \sum_{P,Q,R} \langle M, P \otimes Q \otimes R \rangle\, x_P y_Q z_R \\
&= \left\langle M, \underbrace{\left(\sum_P x_p P \right)}_{X} \otimes \underbrace{\left(\sum_Q y_Q Q \right)}_{Y} \otimes \underbrace{\left(\sum_R z_R R \right)}_{Z} \right\rangle
\end{aligned}
$$

- The matrices X, Y, Z are Hermitian and have Frobenius norm norm less than $N^{3/2}$

Step 2: *Upper bound on the classical bias*

- Define the 2-2-2 norm of M by

$$\|M\|_{2,2,2} = \max\langle M, X \otimes Y \otimes Z\rangle$$
$$\text{s.t. } X, Y, Z \text{ Hermitian, } \|X\|_F, \|Y\|_F, \|Z\|_F \leq 1$$

- The classical bias of the Pauli-Fourier game G_M is thus bounded by

$$\beta(G_M) \leq N^{9/2}\|M\|_{2,2,2}$$

Step 2: *Lower bound on the entangled bias*

- A lower bound on the entangled bias of G_M can also be obtained easily from the Pauli Fourier expansion of M
- Since the Pauli matrices are observables, we have for any $|\psi\rangle$

$$\begin{aligned}
\beta^*(G_M) &\geq \sum_{P,Q,R} \widehat{M}_{PQR} \langle\psi|P \otimes Q \otimes R|\psi\rangle \\
&= \langle\psi| \sum_{P,Q,R} \widehat{M}_{PQR} P \otimes Q \otimes R|\psi\rangle \\
&= N^3 \langle\psi|M|\psi\rangle
\end{aligned}$$

- This gives a lower bound in terms of the spectral norm

$$\beta^*(G_M) \geq N^3 \|M\|$$

Step 2: *Putting the spectral bounds together*

- Putting the bounds on the biases together gives for any Hermitian N^3-by-N^3 matrix M,

$$\frac{\beta^*(G_M)}{\beta(G_M)} \geq N^{-3/2} \frac{\|M\|}{\|M\|_{2,2,2}}$$

- Now, to find a good matrix M

Step 3: *Finding a good matrix*

- We use the probabilistic method

- Let $|g\rangle \sim \mathcal{N}(0,1)^{N^3}$ be a Gaussian vector and set

$$M_{(ijk),(i'j'k')} = \begin{cases} 0 & \text{if } i = i' \vee j = j' \vee k = k' \\ |g\rangle\langle g|_{(ijk),(i'j'k')} & \text{otherwise} \end{cases}$$

- Since M is "close" to the rank-1 matrix $|g\rangle\langle g|$, it is easy to lower bound $\|M\|$

- Set $|\psi\rangle = N^{-3/2}|g\rangle$. Then a χ^2-tail-bound gives that whp

$$\|M\| \geq \langle\psi|M|\psi\rangle \geq CN^3$$

Step 3: *Upper bounding the 2-2-2 norm*

- $\exists$ ***traceless*** Hermitian X, Y, Z in the Frobenius ball s.t.

$$\|M\|_{2,2,2} = \langle g|X \otimes Y \otimes Z|g\rangle - \underbrace{\mathrm{Tr}(X \otimes Y \otimes Z)}_{0}$$

- By a standard decomposition, we may restrict to normalized projectors (wrt Frobnenius norm) X', Y', Z', at a small loss:

$$\|M\|_{2,2,2} \lesssim \max \langle g|X' \otimes Y' \otimes Z'|g\rangle - \mathrm{Tr}(X' \otimes Y' \otimes Z')$$

Lem. (Latała). For Gaussian vector $|h\rangle$ and matrix A, $\langle h|A|h\rangle - \mathrm{Tr}(A)$ is concentrated around 0

- A delicate ε-net argument over normalized projectors gives that whp, $\|M\|_{2,2,2} \leq \widetilde{O}(N)$

Putting everything together

- We have shown:

  1. For any $N^3 \times N^3$ matrix M, there is a 3-player XOR game G_M such that
  $$\frac{\beta^*(G_M)}{\beta(G_M)} \geq N^{-3/2} \frac{\|M\|}{\|M\|_{2,2,2}}$$
  2. There is a distribution over matrices M such that whp, both
  $$\|M\| \geq CN^3 \qquad \text{and} \qquad \|M\|_{2,2,2} \leq \widetilde{O}(N)$$
  hold
  3. Hence, there is a matrix M s.t. $\beta^*(G_M) \geq \widetilde{\Omega}(\sqrt{N})\beta(G_M)$

Upper bounds on the maximum bias ratio

Upper bounds on the maximum bias ratio

Thm. [B, Vidick] For a three-player XOR game G with Q questions per player,

$$\beta^*(G) \leq K_G Q^{1/2} \beta(G)$$

- Our lower bound gave a factor of $\approx Q^{1/4}$
- Our gaps are nearly optimal (quadratically off) in terms of the size of the game

Proof outline of the upper bound

- The proof involves two steps:

  1. Upper bound the ratio if Charlie is classical

  2. Show that a quantum Charlie can be made classical at a loss of $1/\sqrt{Q}$ in the bias

Step 1: Suppose that Charlie is classical

Claim. In this case, the bias ratio is at most a constant

- To each question $k \in [Q]$, Charlie answers with a classically obtained bit $z_k \in \{-1, 1\}$
- After Charlie gets his question, Alice and Bob are left to play a two-player XOR game:
- to win, they have to answer bits x_i and y_j such that

$$M_{ijk} x_i y_j z_k = +1$$

- Since they don't know Charlie's question, they must use the same strategy for each k
- But by Tsirelson/Grothendieck, entanglement gives Alice and Bob at most a constant-factor advantage in the bias

Step 2: Making a quantum Charlie classical

Claim. A quantum Charlie gives at most a $\sqrt{Q}$-factor advantage over a classical Charlie

- For i.i.d. symmetric $\{-1, 1\}$-valued random variables $z_1, \ldots, z_Q$, we have
$$\mathbb{E}[z_i z_j] = \delta_{ij}$$
- Let $C_1, \ldots, C_Q$ be Charlie's observables. Then,
$$\mathbb{E}\, z_i \left(\sum_{j=1}^{Q} z_j C_j \right) = C_i$$
- On average over the $z_i s$, the matrix $z_1 C_1 + \cdots + z_Q C_Q$ squares to Q times the identity matrix

- Cauchy-Schwarz gives an instantiation of the z_is (a classical strategy) that drop by bias by at most a factor $\sqrt{Q}$

Open problems

- Find an explicit matrix satisfying the spectral properties proved possible by our probabilistic argument
- Close the gap between upper and lower bounds on the number of questions needed for a given bias ratio
- Remove the log factor in our upper bound on $\|M\|_{2,2,2}$
- Find out if unbounded ratios between the classical and entangled biases are possible for $\beta^* \approx 1/2$

Thank you!

arXiv:1108.5647 [quant-ph]