
(AN UNMATCHED LEFT PARENTHESIS CREATES AN UNRESOLVED TENSION THAT WILL STAY WITH YOU ALL DAY.

[xkcd.com/859]

Criticality without Frustration

IQC Waterloo, 9/2012

Sergey Bravyi
IBM Watson

Libor Caha
Slovak Academy of Sciences

Ramis Movassagh
Northeastern University

Peter Shor
MIT

Daniel Nagaj
universität wien

arXiv: 1203.5801

Thanks: IQC, QESSENCE, LPP QWAC

Criticality without Frustration

IQC Waterloo, 9/2012

Sergey Bravyi
IBM Watson

Libor Caha
Slovak Academy of Sciences

Ramis Movassagh
Northeastern University

Peter Shor
MIT

Daniel Nagaj
universität wien

arXiv: 1203.5801

Thanks: IQC, QESSENCE, LPP QWAC

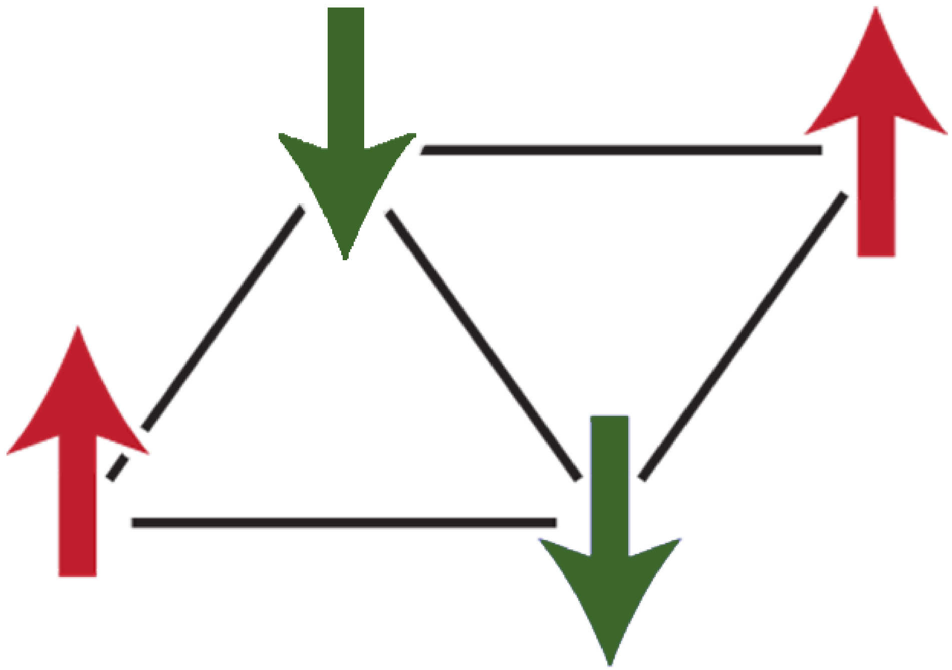
Criticality without Frustration

IQC Waterloo, 9/2012

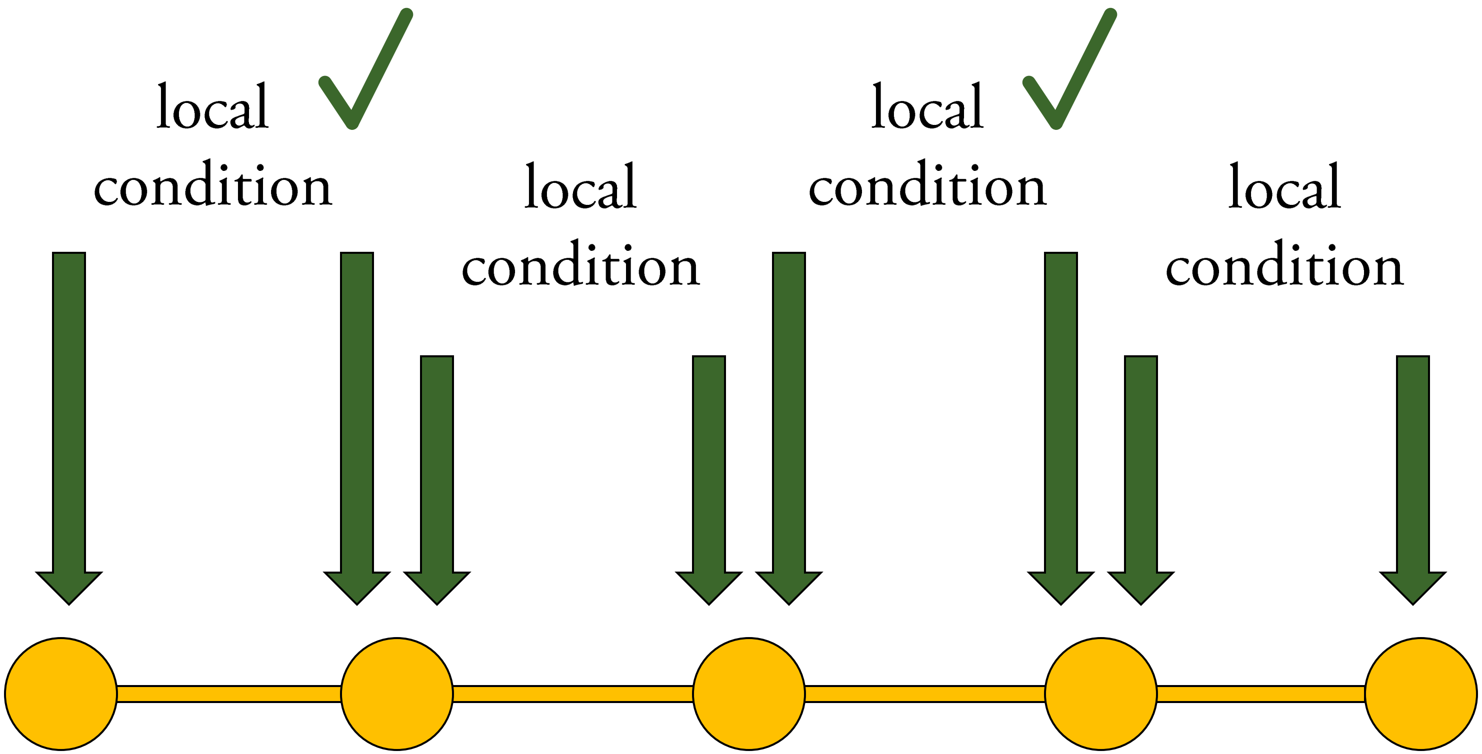
1. finding ground states (1D)
2. fun with qutrits
3. formidable calculations
4. frustration free future

1 Frustrated systems

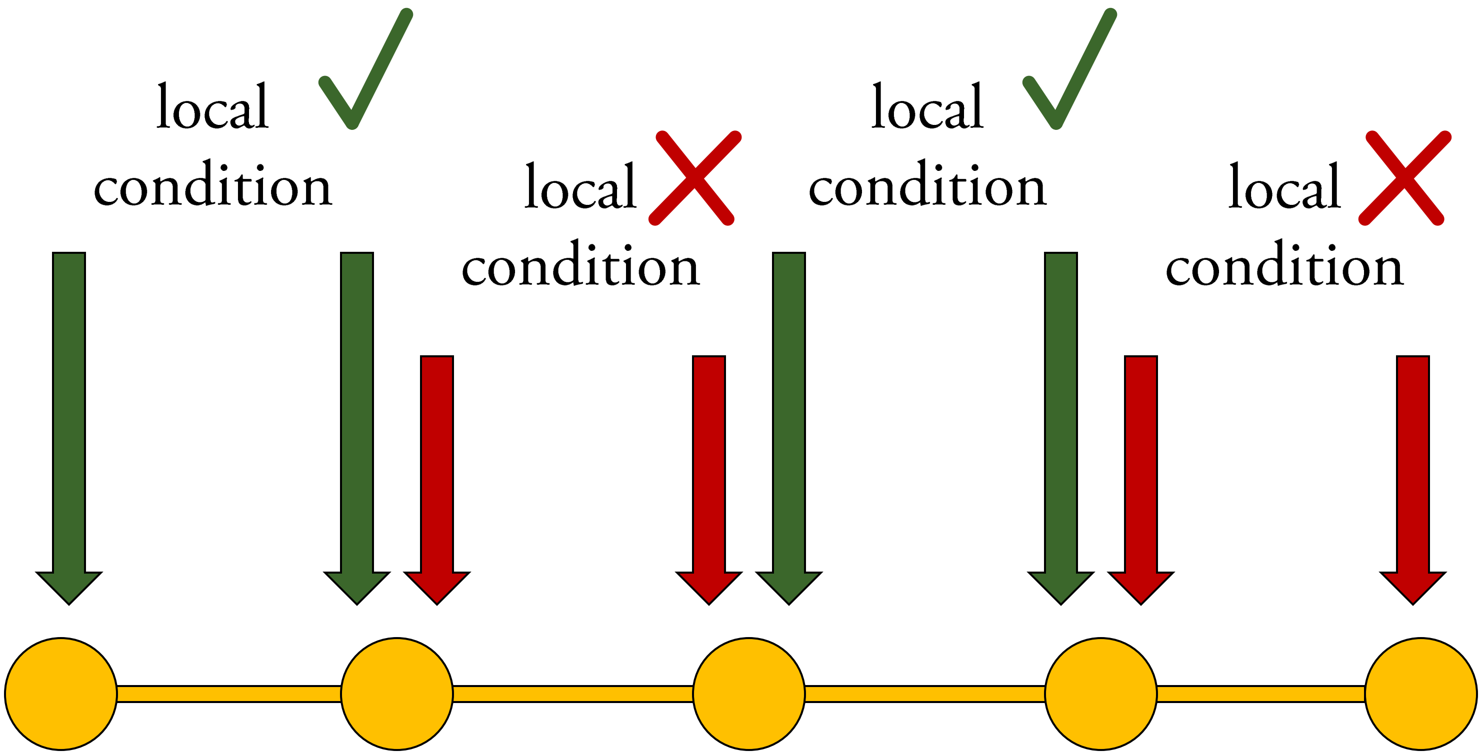
Sometimes you can't make everybody happy.



1 Frustrated ground states in 1D



1 Frustrated ground states in 1D



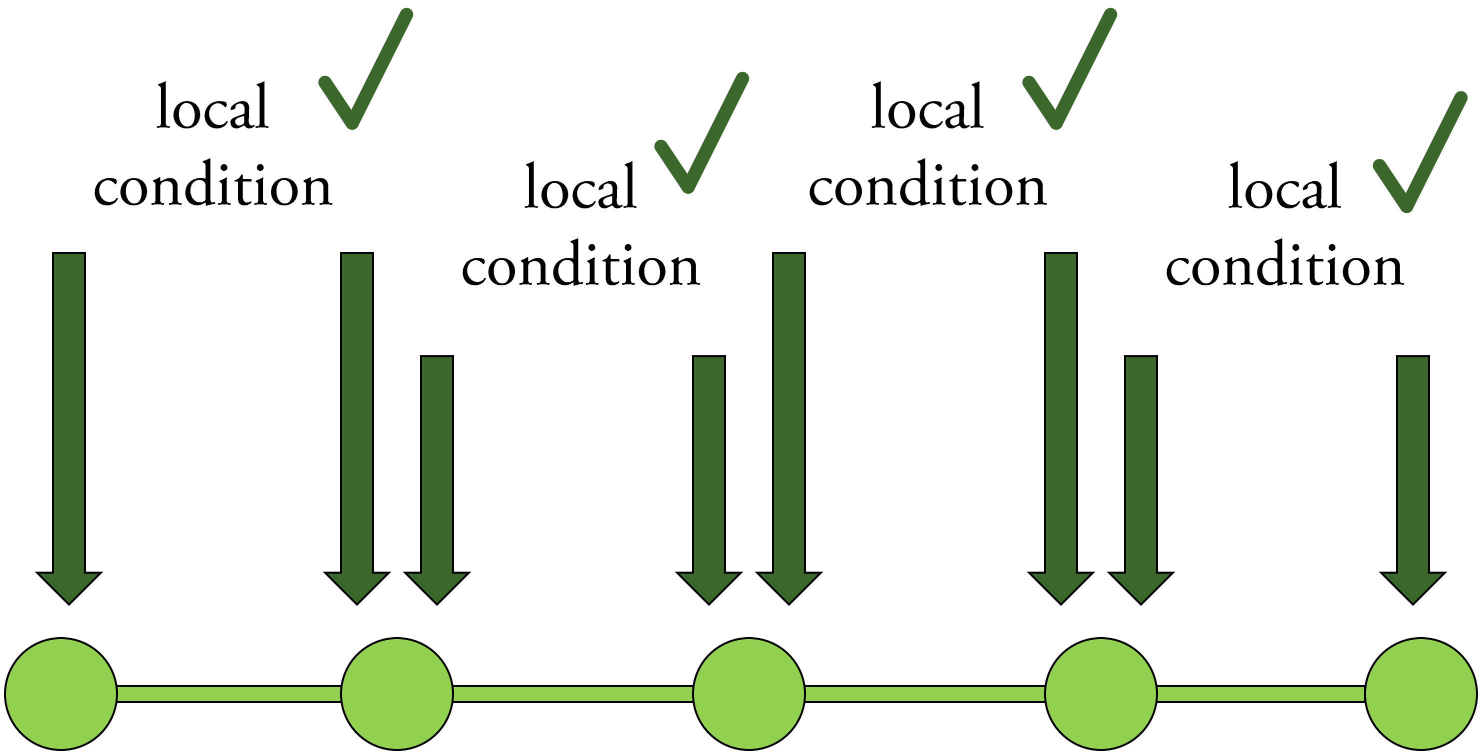
a global ground state

HARD?

find & describe it?
is it entangled?


frustrated
FRUST
RATED

Unfrustrated ground states in 1D



a global & local
ground state

EASY?

find & describe it?
is it entangled?

1 Classical 1D: simple

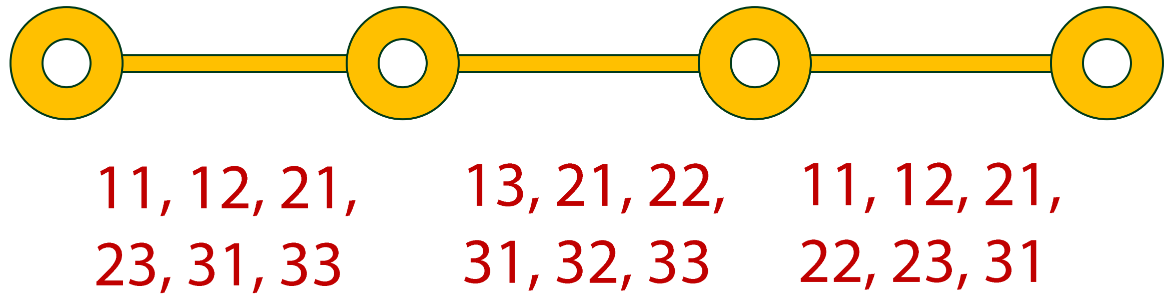
Having higher dits doesn't add anything.

- unfrustrated ground states on a line?

1 Classical 1D: simple

Having higher dits doesn't add anything.

- unfrustrated ground states on a line?



- classical 2-SAT in 1D: a list of forbidden states

allowed pairs: 13, 22, 32 | 11, 12, 23 | 13, 32, 33

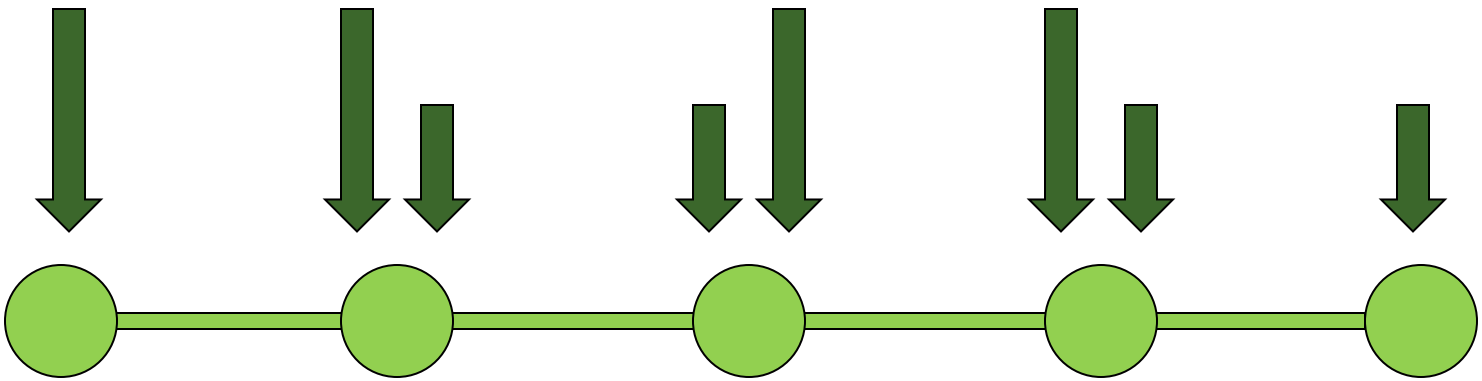
allowed states: 1, 2, 3 | 2, 3 | 3 | 2,3

the MAX-version is also solvable (dynamical programming)

1 Unfrustrated qubit chains in 1D

A quantum example.

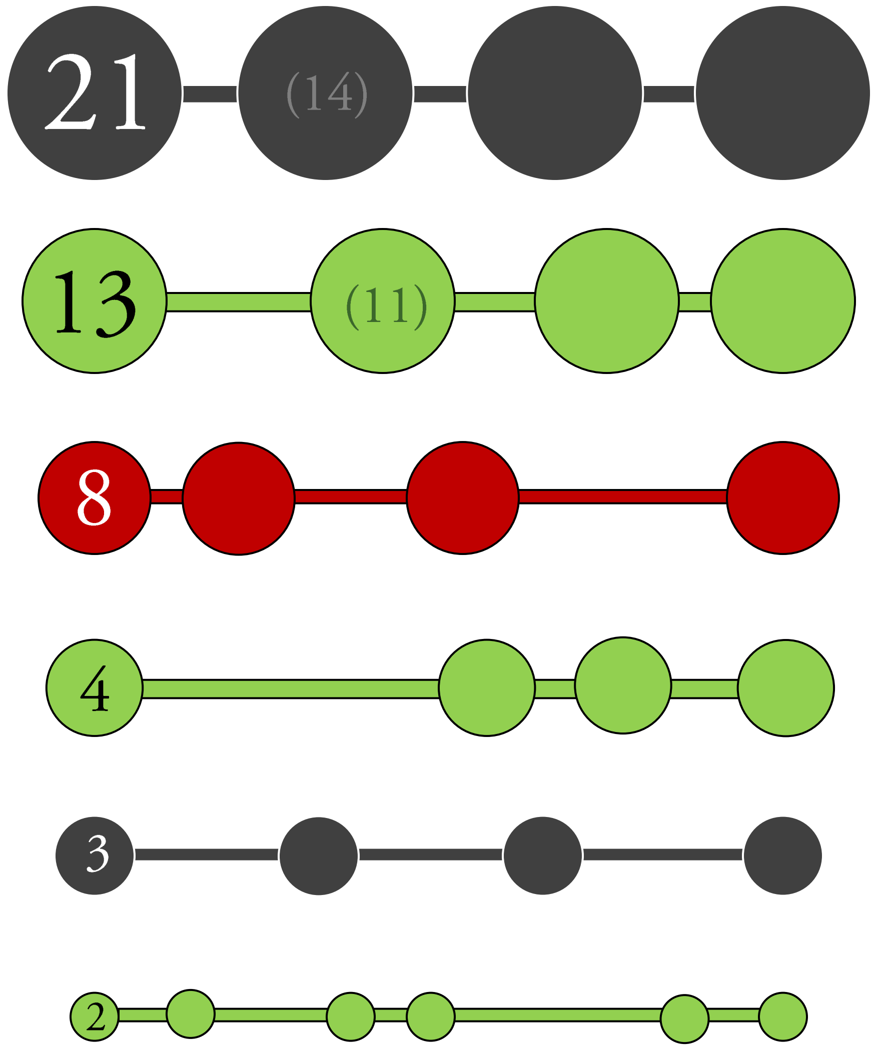
$$H = \frac{1}{2}\sum_{k=1}^{L-1} |01 - 10\rangle\langle 01 - 10|_{k,k+1} \otimes \mathbb{I}$$



a simple product ground state $|11111\rangle$

1 Ground states in 1D

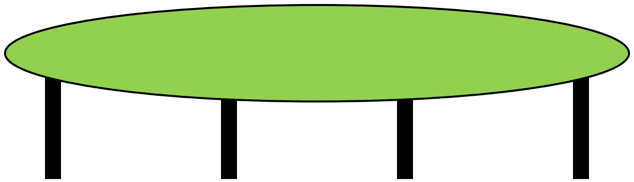
How hard is it to find/describe them?



entropy ~ L	trans. invariant [Irani]
QMA$_1$-comp.	history state [A+'06]
QMA-comp.	frustrated [N+'12]
very entangled	random projectors [M+'10]
const. entropy	trans. invariant AKLT model
product states	Q 2-SAT in P [Bravyi, C+]

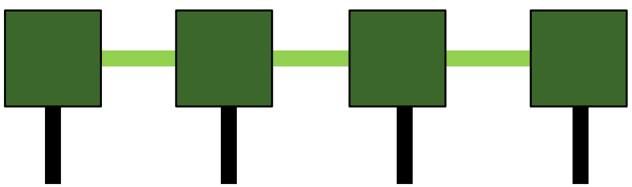
1 Finding 1D ground states

How to describe or approximate them?



$$|\psi\rangle = \sum_{s,t,u,v=0}^{1} c_{stuv} |stuv\rangle$$

Schmidt decompositions ➡ a local description



$$c_{stuv} = \sum_{a,b,c=1}^{\chi} A^s_a B^t_{ab} C^u_{bc} D^v_c$$

- Matrix Product States & DMRG

… low Schmidt rank ansatz
… local optimization
… matrix size ~ Schmidt rank

- Are they hard to find?
- Can they be very entangled?

entanglement

qubits: NO
[Chen et al.]

qudits (high d): YES
[AGIK, Irani]

qubits

How hard is it to find/describe them?

- Are they hard to find?
- Can they be very entangled?

entanglement

qubits: NO
[Chen et al.]

qudits (high *d*): YES
[AGIK, Irani]

qu3its?

1 FF ground states in 1D

How hard is it to find/describe them?

- Are they hard to find?

entanglement

- Can they be very entangled?

qubits: NO
[Chen et al.]

qudits (high *d*): YES
[AGIK, Irani]

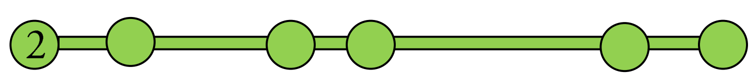
- Nice & surprising properties?

unique? gapped?
translationally invariant?

const. entropy

trans. invariant
AKLT model

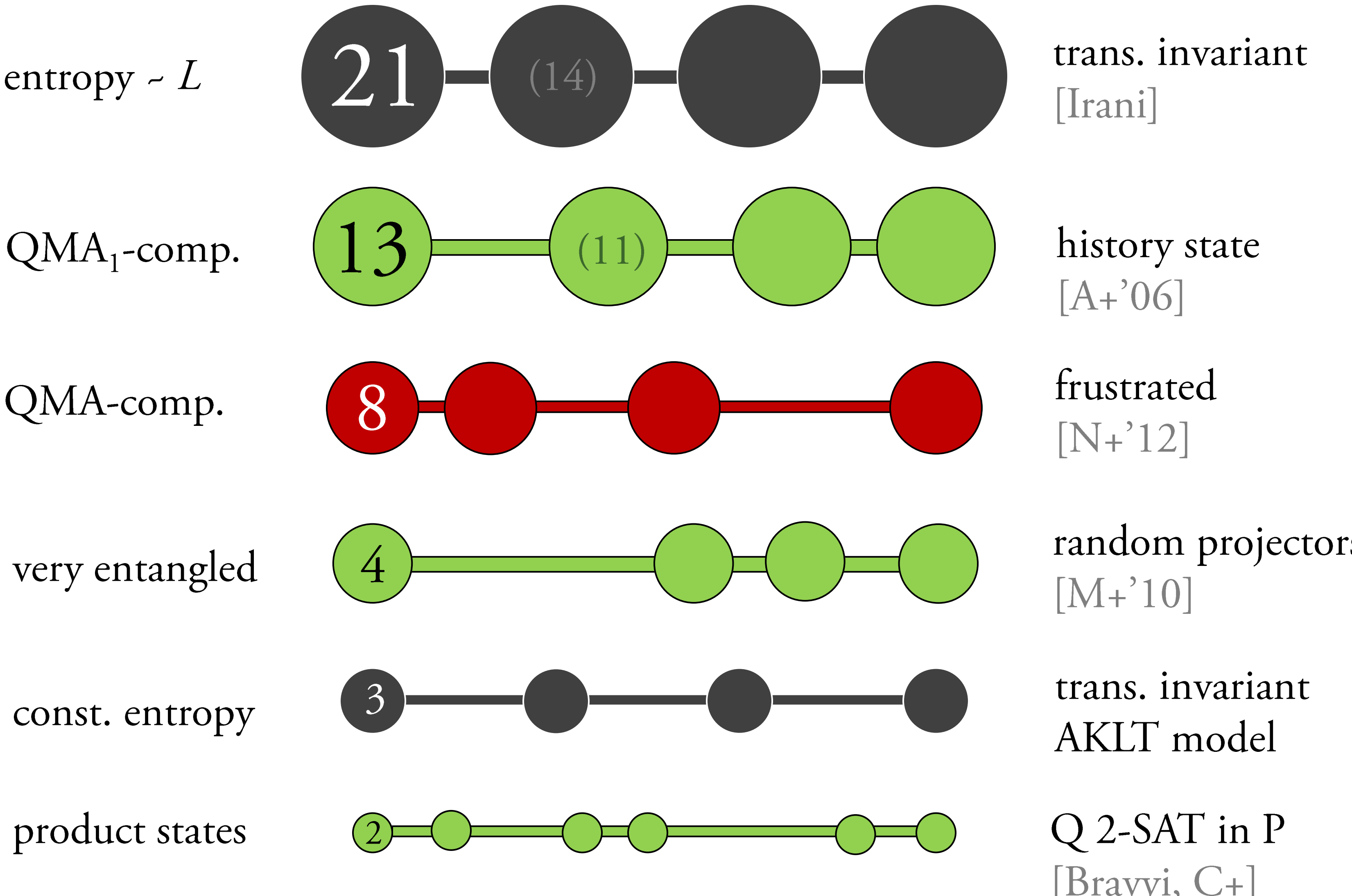
product states



Q 2-SAT in P
[Bravyi, Chen]

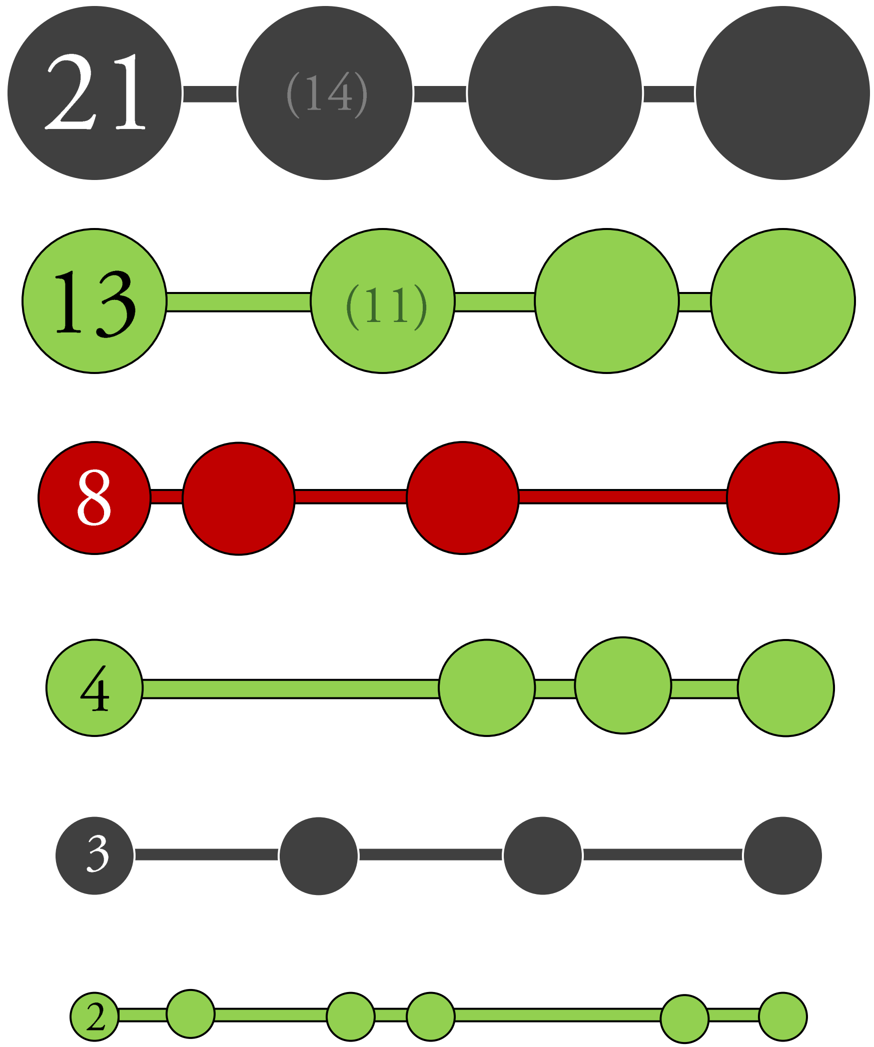
1 Ground states in 1D

How hard is it to find/describe them?



1 Ground states in 1D

How hard is it to find/describe them?



entropy ~ L	trans. invariant	[Irani]
QMA$_1$-comp.	history state	[A+'06]
QMA-comp.	frustrated	[N+'12]
very entangled	random projectors	[M+'10]
■ entropy ~ log L	trans. invariant	[B+'12]
product states	Q 2-SAT in P	[Bravyi, C+]

2 Quantum + more dimensions: encode a computation

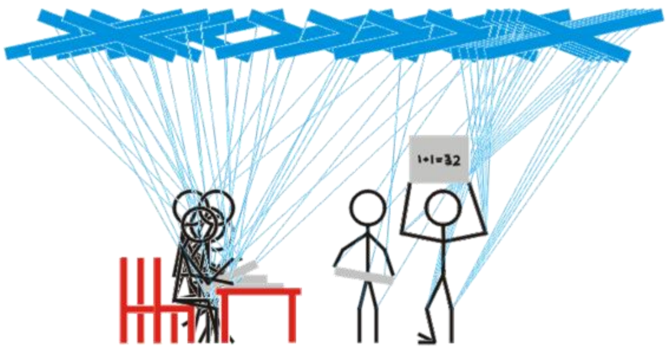
- Kitaev's Hamiltonian written in a local way

$$H_K = \frac{1}{2}\sum_{t=1}^{T}\left(|\psi_t\rangle - |\psi_{t-1}\rangle\right)\left(\langle\psi_t| - \langle\psi_{t-1}|\right)$$

- the history state

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}}\sum_{t=0}^{T}|\psi_t\rangle$$

a global & local ground state



2 Quantum + more dimensions: encode a computation

- 2-locally checkable transitions

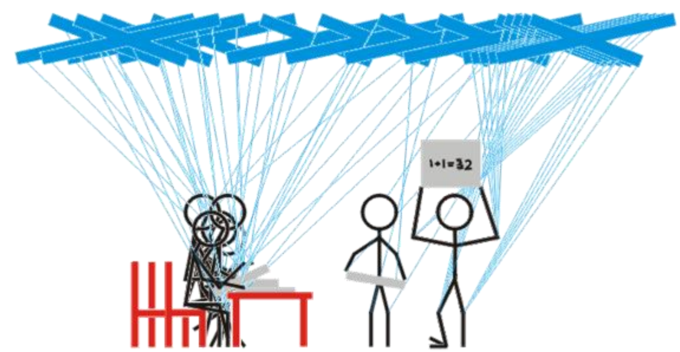
$$\tfrac{1}{2}\big(|ab\rangle - |cd\rangle\big)\big(\langle ab| - \langle cd|\big)$$

0-energy states

$$\cdots|ab\rangle\cdots$$
$$+\cdots|cd\rangle\cdots$$

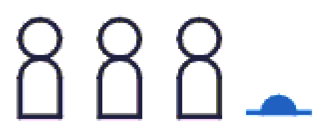
- the history state

$$|\psi_{hist}\rangle = \frac{1}{\sqrt{T+1}}\sum_{t=0}^{T}|\psi_t\rangle$$



a global & local ground state

high-*d* qubits
the data “moves”

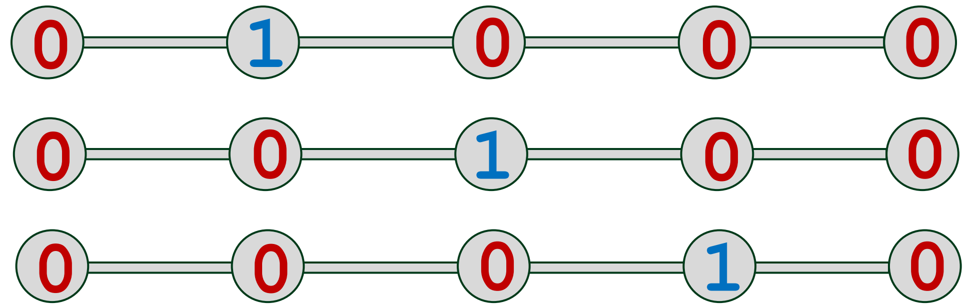


2 A less complicated task: moving qubits

Hopping on a line.

- allowed transitions

  `01` ↔ `10`



- Hamiltonian: 2-local projectors

  $\frac{1}{2}\left(|01\rangle - |10\rangle\right)\left(\langle 01| - \langle 10|\right)$

  ground states: uniform superpositions

- invariant subspaces with different # of `1`'s

  degenerate ground state (1 in each subspace)

  cut down the deneneracy: rule out `11`

  still a "dead" (product) ground state

`1100`
`1010`
`1001`
`0110`
`0101`
`0011`

`0000`

2 Surfing with qutrits

Moving a domain wall.

- the surfer construction
  W, @, w … a surfer riding a wave

  transition rule   @w ↔ W@

  2-local projector to check the transition

  $$\tfrac{1}{2}\big(|\text{@w}\rangle - |\text{W@}\rangle\big)\big(\langle\text{@w}| - \langle\text{W@}|\big)$$

  ensure a single wave … rule out

  wW, Ww, @@, w@, @W
  + endpoint w/W

- a unique, entangled ground state
  quantum computation on a single qubit

@wwwwwwwwww
W@wwwwwwwww
WW@wwwwwwww
WWW@wwwwwww
WWWW@wwwwww
WWWWW@wwwww
WWWWWW@wwww
WWWWWWW@www
WWWWWWWW@ww
WWWWWWWWW@w
WWWWWWWWWW@

2 Surfing with qutrits

- is the surfer state entangled?

  Schmidt decomposition

  $$\sum_{j=1}^{\chi} \lambda_j |\phi_j\rangle_A \otimes |\psi_j\rangle_B$$

- constant MPS dimension χ = 2

  same as in the AKLT model

```
@wwwwwwwwww
W@wwwwwwwww
WW@wwwwwwww
WWW@wwwwwww
WWWW@wwwwww
WWWWW@wwwww
WWWWWW@wwww
WWWWWWW@www
WWWWWWWW@ww
WWWWWWWWW@w
WWWWWWWWWW@
```

(@wwww+W@www+...) ⊗ wwwwww

+WWWWWW ⊗ (@wwww+W@www+...)

2 Entanglement and superpositions

Does one imply the other?

- a lot of terms in a superposition

  not necessarily entangled

00|00	01|00	10|00	11|00
00|01	01|01	10|01	11|01
00|10	01|10	10|10	11|10
00|11	01|11	10|11	11|11

$$(0+1)\,(0+1) \;\otimes\; (0+1)\,(0+1)$$

2 Unique ground states

Invariant subspaces & killing all but one.

- interesting: lots of entanglement

  large Schmidt number, ent. entropy

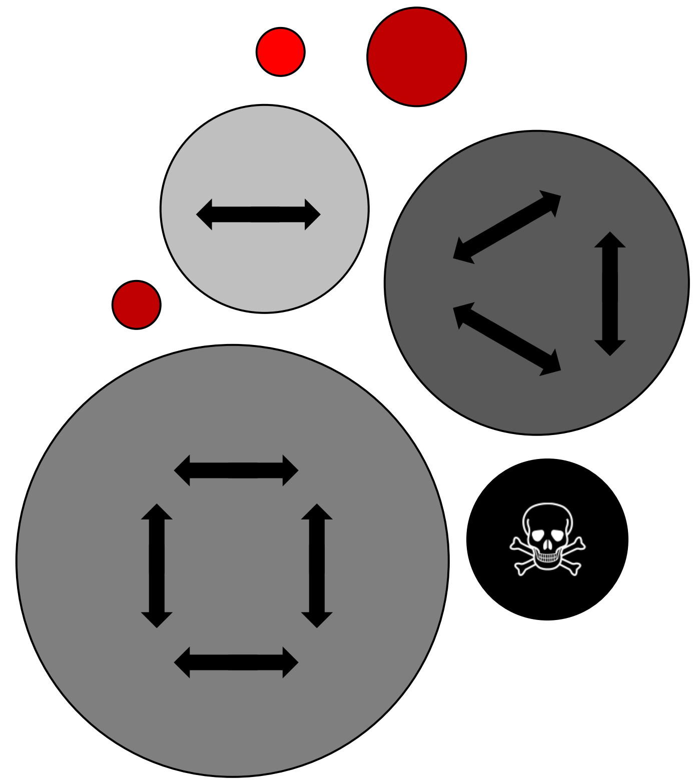
- build a superposition

  history state-like?

  transition rules

  connected components

$$\frac{1}{\sqrt{C_N}} \sum_s |s\rangle$$



- make it unique

  local rules (+ endpoints)

wwwwwww W@wwwww WW@@www WWWWWWW

2 The bracket construction

[[[Yes, this is it.]]]

- strings of 3 letters: [ ] –

  transition rules
  – – ↔ [ ]
  [– ↔ – [
  –] ↔] –

  checkable by 2-local projectors

  $$\tfrac{1}{2}\big(|ab\rangle - |cd\rangle\big)\big(\langle ab| - \langle cd|\big)$$

  dislike bad brackets at the ends

  ] … or … [

- a unique ground state

  well bracketed

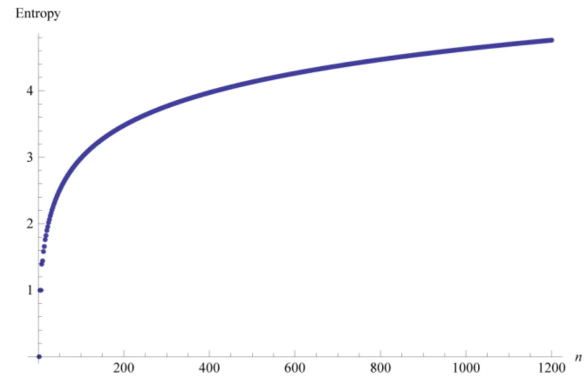
  $$\frac{1}{\sqrt{M_n}} \sum_{s\in S^n_{0,0}} |s\rangle$$

– – – –
[ ] – –
[–] –
[– –]
– [ ] –
– [–]
– – [ ]
[ ] [ ]
[[ ]]

3 Fantastic properties

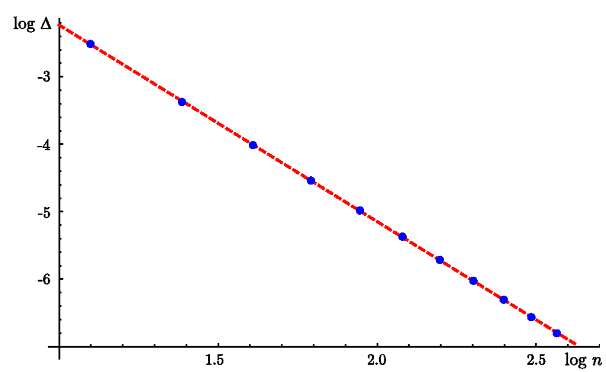
We will soon see this.

- logarithmic scaling of entanglement entropy



- a unique ground state of a frustration-free qutrit chain

- a polynomial Hamiltonian gap



3 Fantastic properties

We will soon see this.

- logarithmic scaling of entanglement entropy

CRITICALITY

- a unique ground state of a frustration-free qutrit chain

without frustration

- a polynomial Hamiltonian gap

3 Hilbert space structure

Making the ground state unique.

- allowed transitions

– – ↔ [ ]

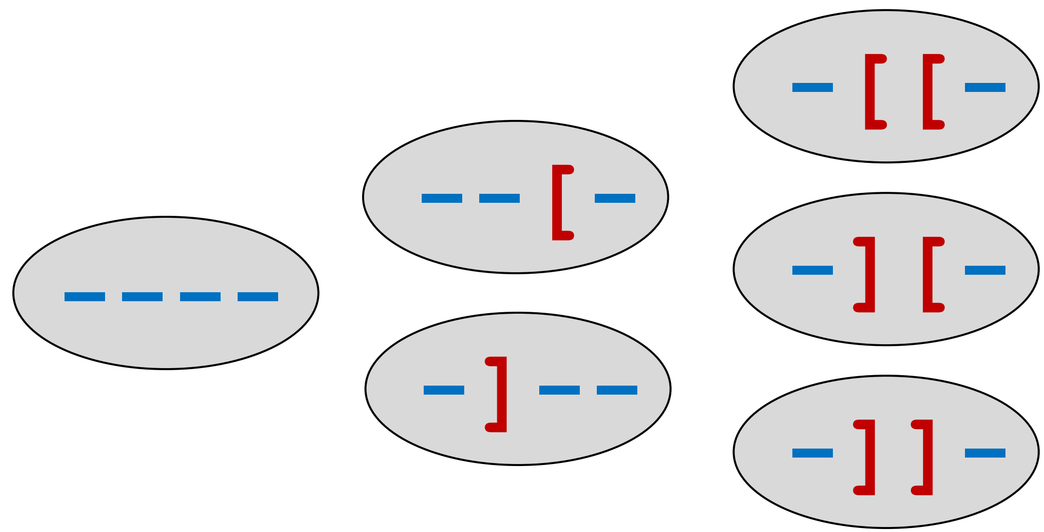
[– ↔ – [

–] ↔] –

$H \geq 0$, a sum of projectors

$\frac{1}{2}\left(|ab\rangle - |cd\rangle\right)\left(\langle ab| - \langle cd|\right)$

- invariant subspaces: (L,R) "bad" brackets



balanced

– [ ] – [– [–]]

[[–] [[–]]]

unbalanced

[– – [ ]] – –] –

– [ ]] – [ ] [[ ]

3 Hilbert space structure

Making the ground state unique.

- allowed transitions

  -- ↔ [ ]

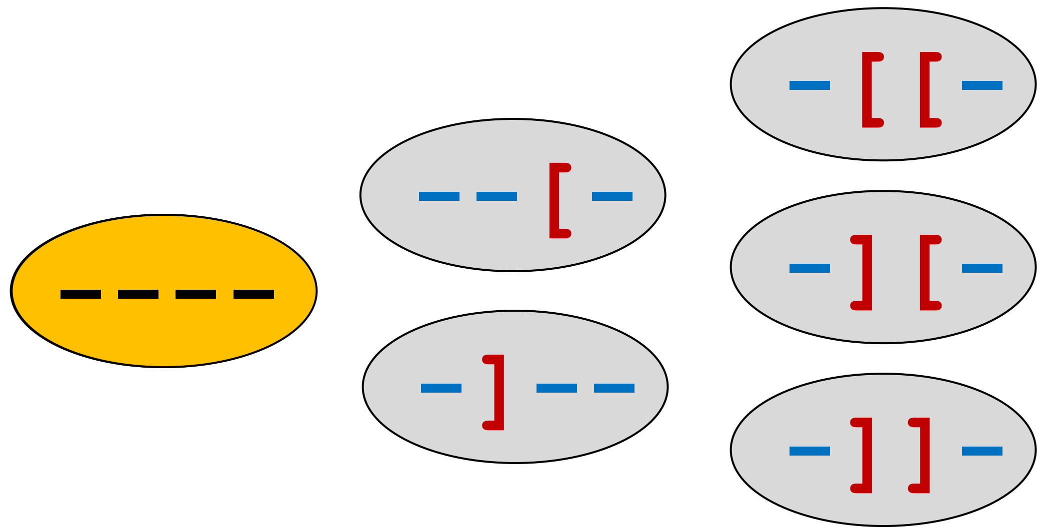
  [- ↔ - [

  -] ↔] -

  $H \geq 0$, a sum of projectors

  $\frac{1}{2}\left(|ab\rangle - |cd\rangle\right)\left(\langle ab| - \langle cd|\right)$

- invariant subspaces:
  (L,R) "bad" brackets



- uniform superpositions within subspaces
- add endpoint projectors

  a unique, unfrustrated ground state

3 The well-bracketed superposition

- the Motzkin state
  uniform superposition

$$|\mathcal{M}_n\rangle = \frac{1}{\sqrt{M_n}} \sum_{s \in S^n_{0,0}} |s\rangle$$

[[ ]] – [ ] – – [– [–]] [–]

– [– [ ] – – [–] – [–]] [–]

– – – –

[ ] – –

[–] –

[– –]

– [ ] –

– [–]

– – [ ]

[ ] [ ]

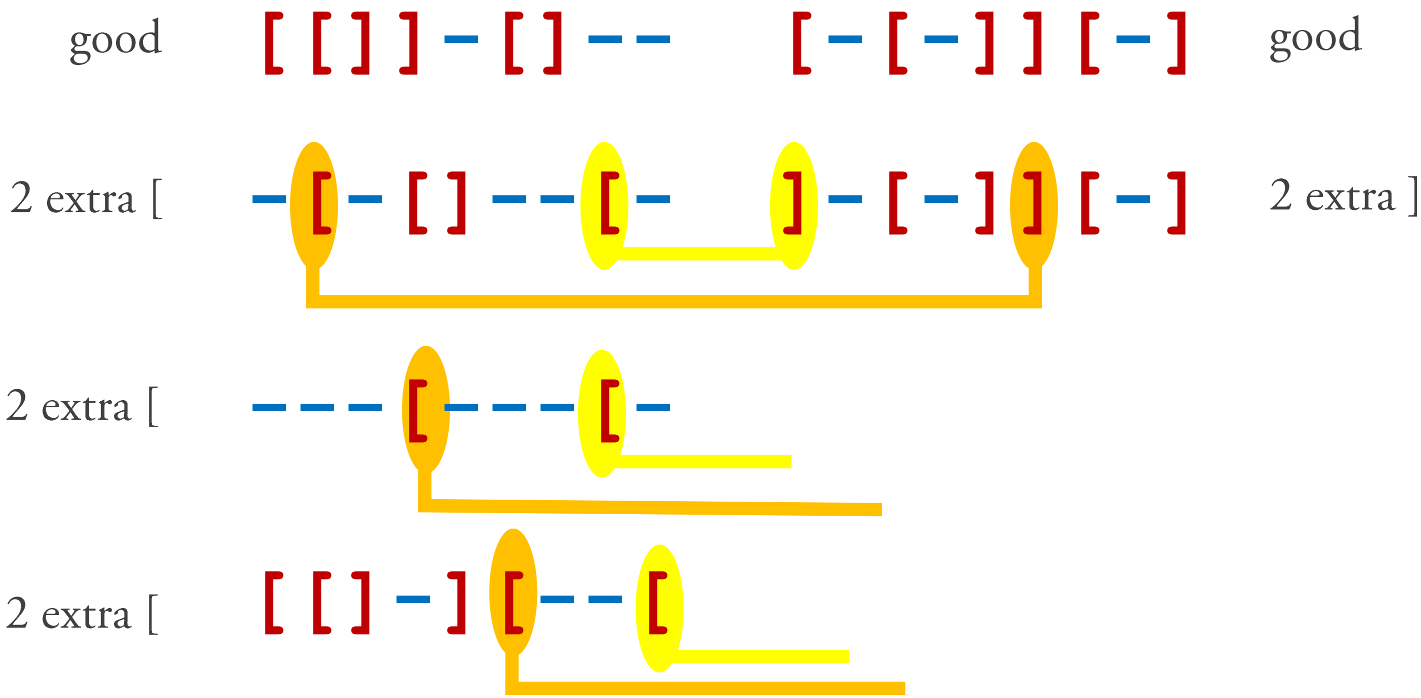
[[ ]]

3 Is the ground state entangled?

Cutting the ground state in half.

- the Motzkin state
  uniform superposition

$$|\mathcal{M}_n\rangle = \frac{1}{\sqrt{M_n}} \sum_{s \in S_{0,0}^n} |s\rangle$$

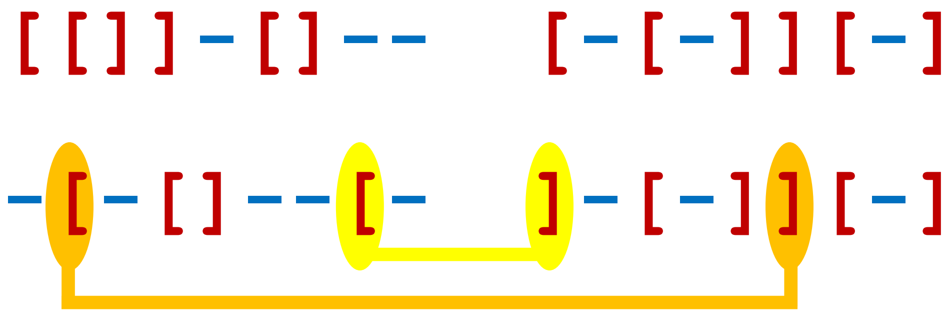


3 Is the ground state entangled?

Cutting the ground state in half.

- the Motzkin state

$$|\mathcal{M}_n\rangle = \frac{1}{\sqrt{M_n}} \sum_{s \in S_{0,0}^n} |s\rangle$$

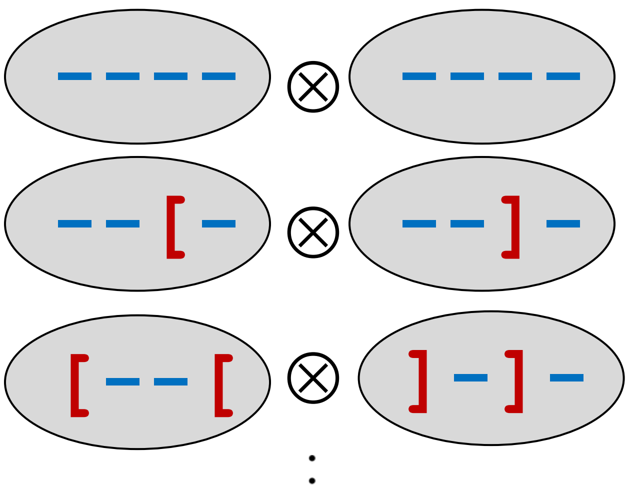


- Schmidt decomposition

subspaces of length $n/2$

m matching bad brackets

# of terms: $\chi = n/2 + 1$



3 Is the ground state entangled?

Cutting the ground state in half.

- the Motzkin state

$$|\mathcal{M}_n\rangle = \frac{1}{\sqrt{M_n}} \sum_{s \in S^n_{0,0}} |s\rangle$$

$$\sum_{m=0}^{n/2} \sqrt{p_m}\, |\hat{C}_{0,m}\rangle_A \otimes |\hat{C}_{m,0}\rangle_B$$

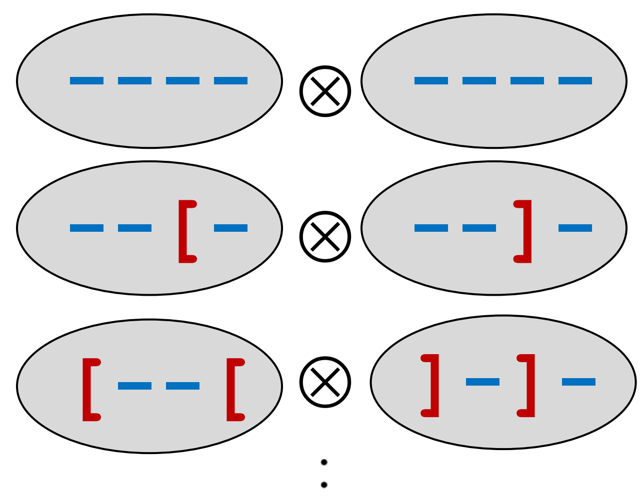
uniform superposition
length $n/2$, m extra left brackets

m extra
right brackets

- Schmidt decomposition

subspaces of length $n/2$
m matching bad brackets

# of terms: $\chi = n/2 + 1$



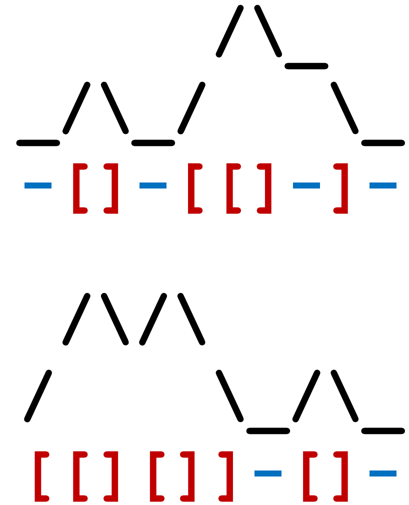
3 Quantifying entanglement

Required to compute the coefficients.

- Motzkin number M_n

  \# of mountains of height ≤ n/2

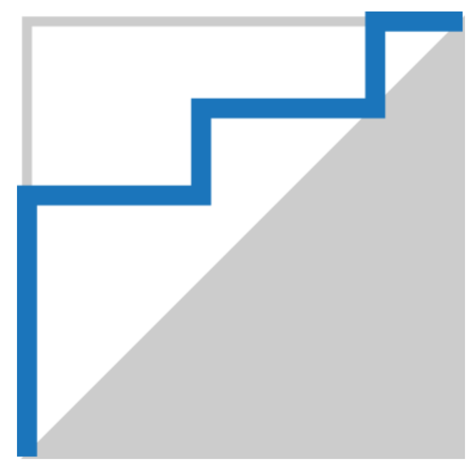
$$M_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} C_k \binom{n}{2k}$$



- Catalan number C_k

  \# of up/right paths above
  the diagonal on a $k \times k$ grid

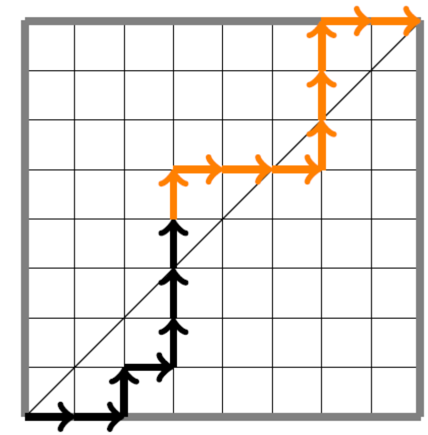
$$C_k \propto \frac{4^n}{n^{3/2}\sqrt{\pi}}$$



3 Calculating the Catalan numbers

- André's reflection method

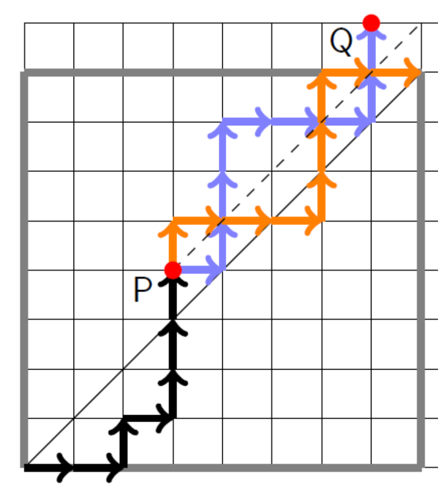
count all paths



subtract paths
crossing the diagonal

reflect after the first crossing

… paths on a $(k-1) \times (k+1)$ grid



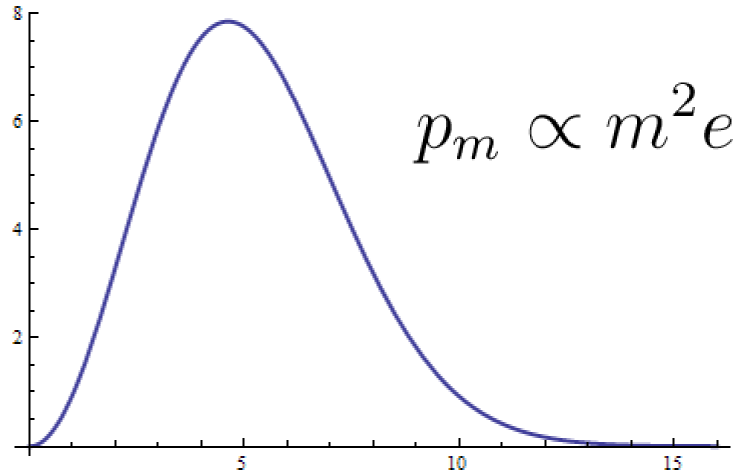
$$C_k = \binom{2k}{k} - \binom{2k}{k+1}$$

3 Entanglement Entropy

A cut through the middle.

- many significant Schmidt coefficients

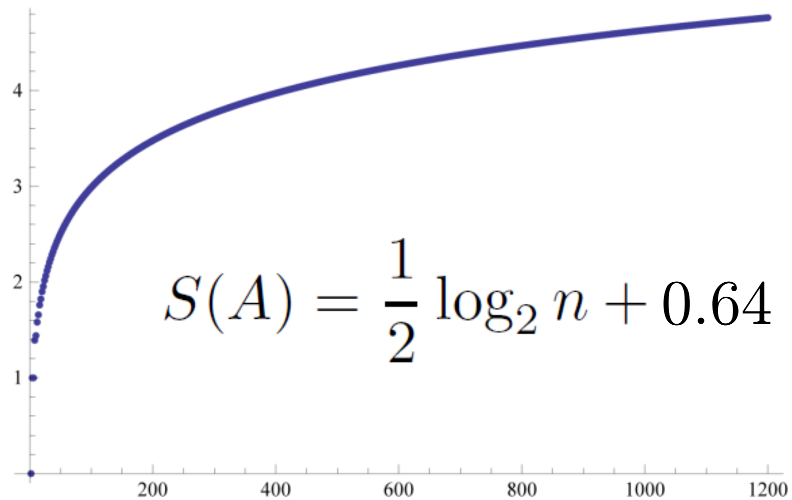
$$\sum_{m=0}^{n/2} \sqrt{p_m}\, |\hat{C}_{0,m}\rangle_A \otimes |\hat{C}_{m,0}\rangle_B$$



$$p_m \propto m^2 e^{-\frac{3m^2}{n}}$$

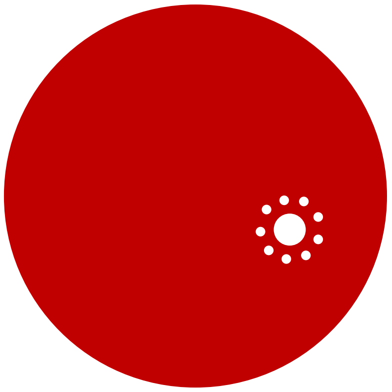
- logarithmic entanglement entropy

$$S(A) = -\operatorname{Tr} \rho_A \log_2 \rho_A$$



$$S(A) = \frac{1}{2} \log_2 n + 0.64$$

a unique entangled ground state



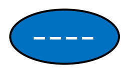
3 Bounding the energy gap

A prelude to a long story.

- frustration-free ground state: E=0

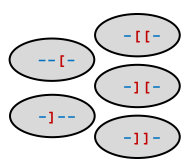
$$H = H_{move} + H_{create} + H_{end}$$

- well-bracketed subspace

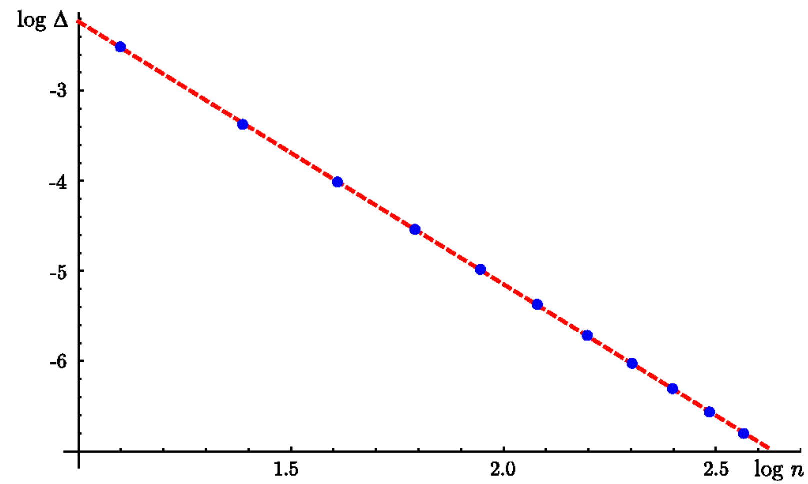


… second eigenvalue?

- unbalanced subspaces



… lowest energy?



$$\log \Delta = -0.68 - 2.91 \cdot \log n$$

3 Is the gap small at all?

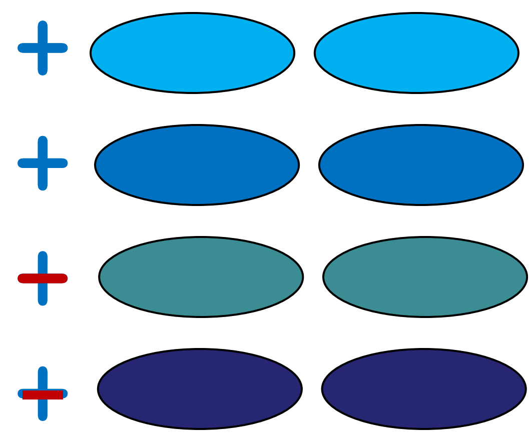
Test: twist the ground state.

- ground state: uniform

$$\sum_{m=0}^{n/2} \sqrt{p_m}\, |\hat{C}_{0,m}\rangle_A \otimes |\hat{C}_{m,0}\rangle_B$$

- an almost orthogonal state

  from some k on use $-\sqrt{p_m}$



- caught by only a few terms

  an upper bound on the gap

$$\Delta \leq O\left(\frac{1}{\sqrt{n}}\right)$$

3 Lower bounding the good subspace gap

A long story.

-[]-[[]]- Hamiltonian on Motzkin paths

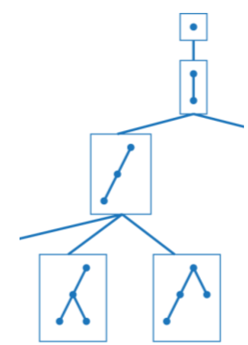
effective Hamiltonian on Dyck paths

$P_{x,y}$ Markov chain on Dyck paths

graph congestion bounds the gap Δ

congestion from canonical paths

canonical paths from fractional matchings



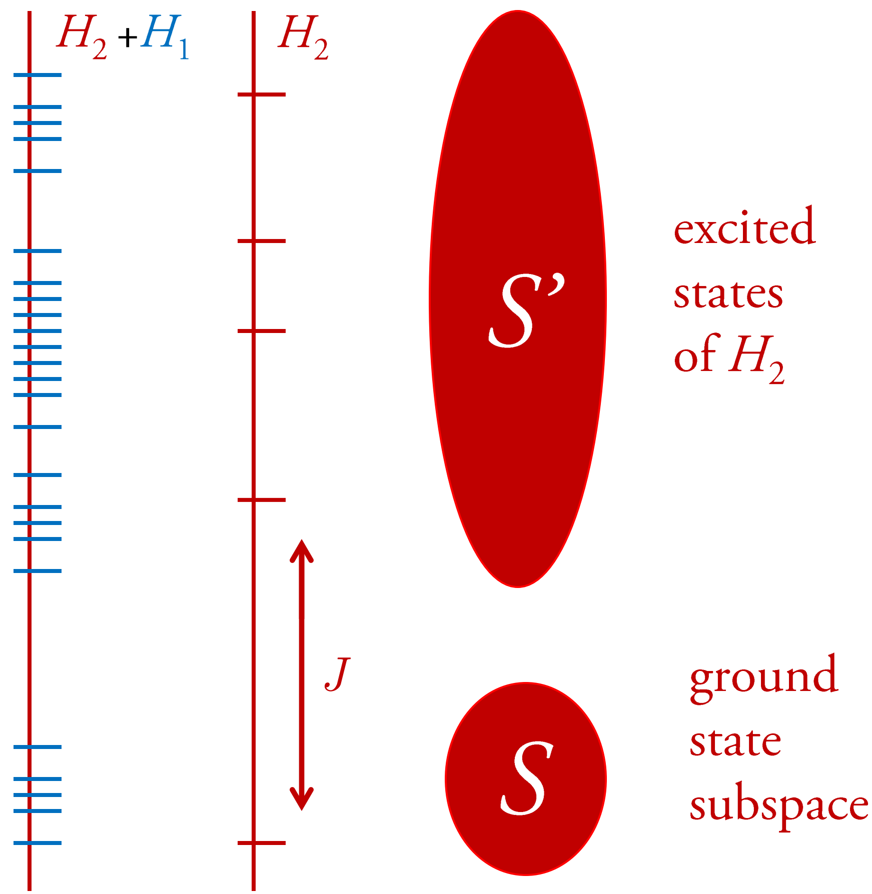
3 The projection lemma

Combining HUGE and tiny Hamiltonians.

$$H_2 + H_1$$

Lemma 1 *Let $H = H_1 + H_2$ be the sum of two Hamiltonians operating on some Hilbert space $\mathcal{H} = \mathcal{S} + \mathcal{S}^{\perp}$. The Hamiltonian H_2 is such that $\mathcal{S}$ is a zero eigenspace and the eigenvectors in $\mathcal{S}^{\perp}$ have eigenvalue at least $J > 2\|H_1\|$. Then,*

$$\lambda(H_1|_{\mathcal{S}}) - \frac{\|H_1\|^2}{J - 2\|H_1\|} \leq \lambda(H) \leq \lambda(H_1|_{\mathcal{S}}).$$



- estimate the small eigenvalues of $H_2 + H_1$ by an effective Hamiltonian

$$H_1|_S$$

3 Projection lemma, "good" subspace

- full Hamiltonian

-- ↔ []

$$H = H_{move} + H_{create} + H_{end}$$

[- ↔ -[

-] ↔]-

3 Projection lemma, "good" subspace

- full Hamiltonian

$$-- \leftrightarrow [\,]$$

$$H|_{good} = H_{move} + H_{create}$$

- pretend the "create" part is small

$$H_\epsilon = H_{move} + \epsilon H_{create}$$

ferromagnetic
Heisenberg
spin ½ chain
gap $O(n^{-2})$

S' — excited states of H_{move}

- low spectrum

$$H_{create}|S$$

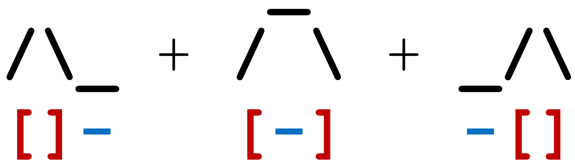
well-bracketed words
uniformly spread

S — ground states of H_{move}

3 From Motzkin paths to Dyck Paths

Good-bye, spaces!

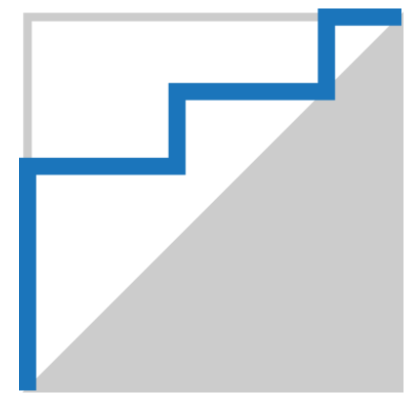
- a new basis



superpositions of Motzkin paths with the same Dyck path

[ ]

labeled by Dyck paths



[[[ ]] [ ]] [ ]

- low spectrum

$$H_{create}|S$$

well-bracketed words uniformly spread

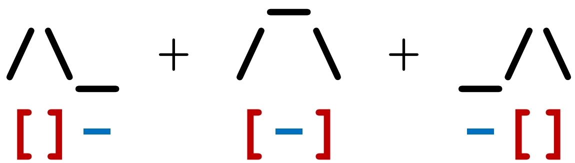
S

ground states of H_{move}

From Motzkin paths to Dyck Paths

Good-bye, spaces!

- a new basis



superpositions of Motzkin paths with the same Dyck path

[ ]

labeled by Dyck paths

- low spectrum

$H_{create}|S$

well-bracketed words uniformly spread

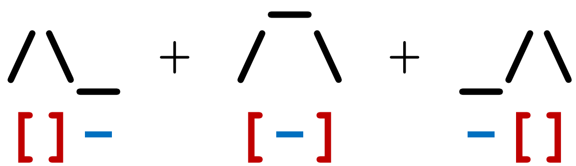
S

ground states of H_{move}

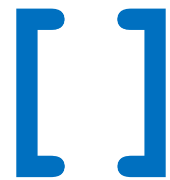
3 From Motzkin paths to Dyck Paths

Good-bye, spaces!

- a new basis



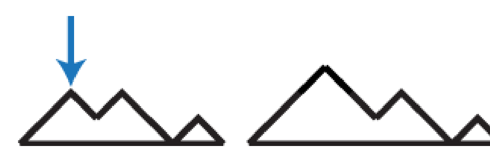
superpositions of Motzkin paths
with the same Dyck path



labeled by
Dyck paths

- effective Hamiltonian: a random walk on Dyck paths

- connections: erosion / eruption



3 Many types of paths

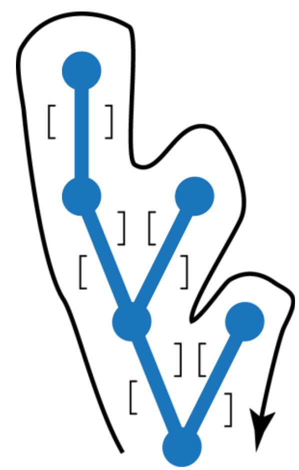
Mapping Dyck paths to trees.

- walking on Dyck paths

well-bracketed words [[[ ]] [ ]] [ ]

mountains

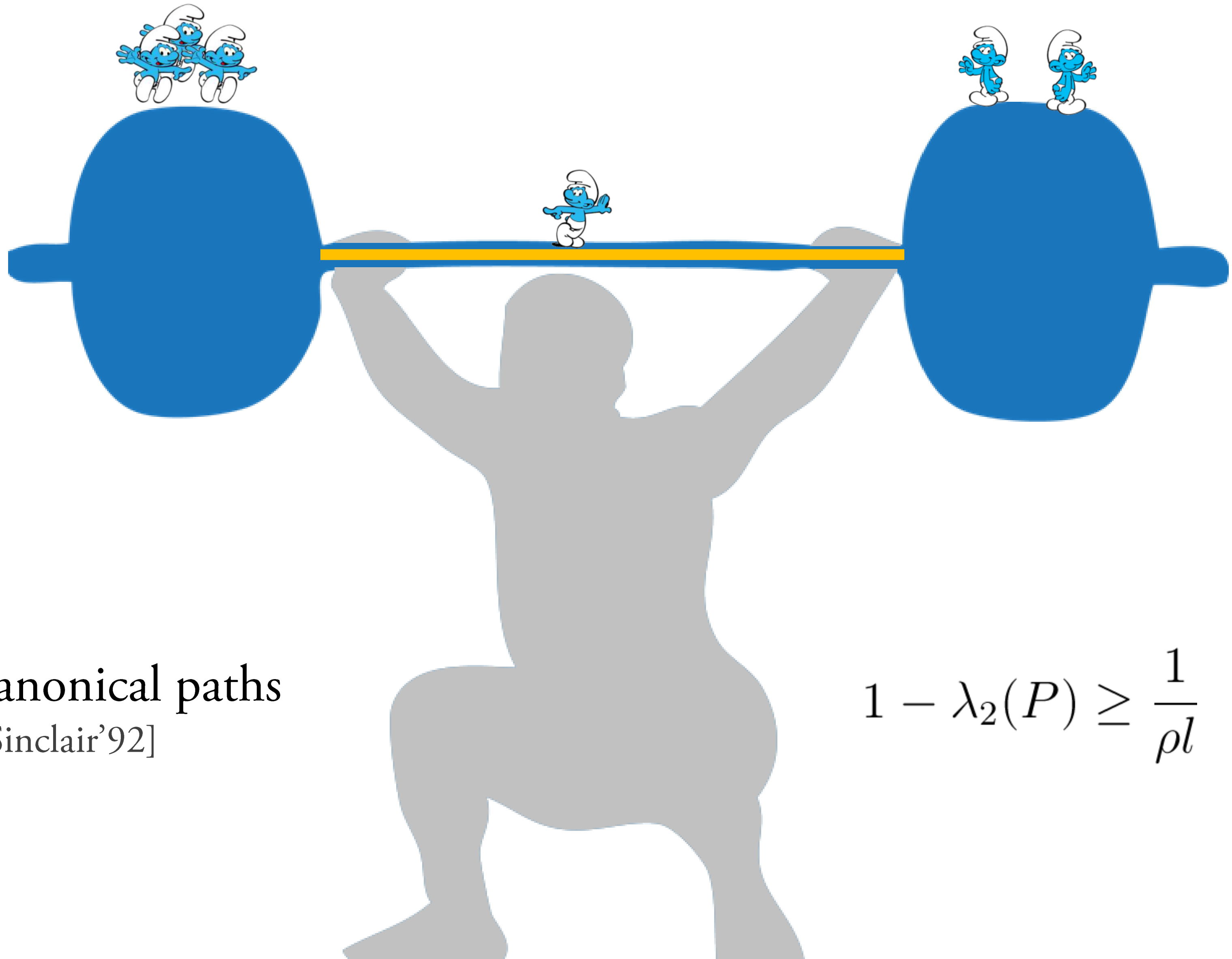
trees



- connections:
  cut/grow leaves

3 Congestion in a graph

Is any edge overworked?



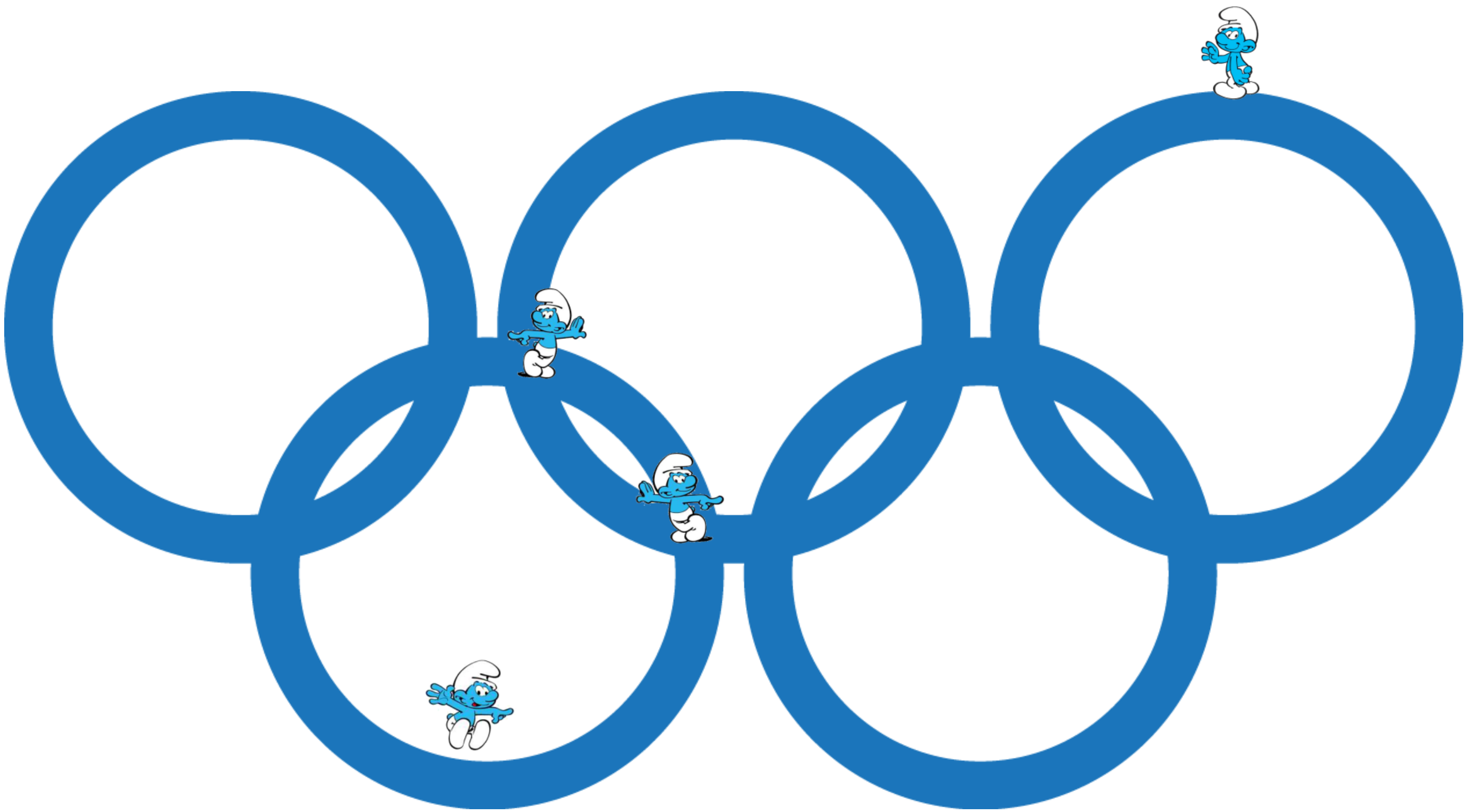
canonical paths

[Sinclair'92]

$$1 - \lambda_2(P) \geq \frac{1}{\rho l}$$

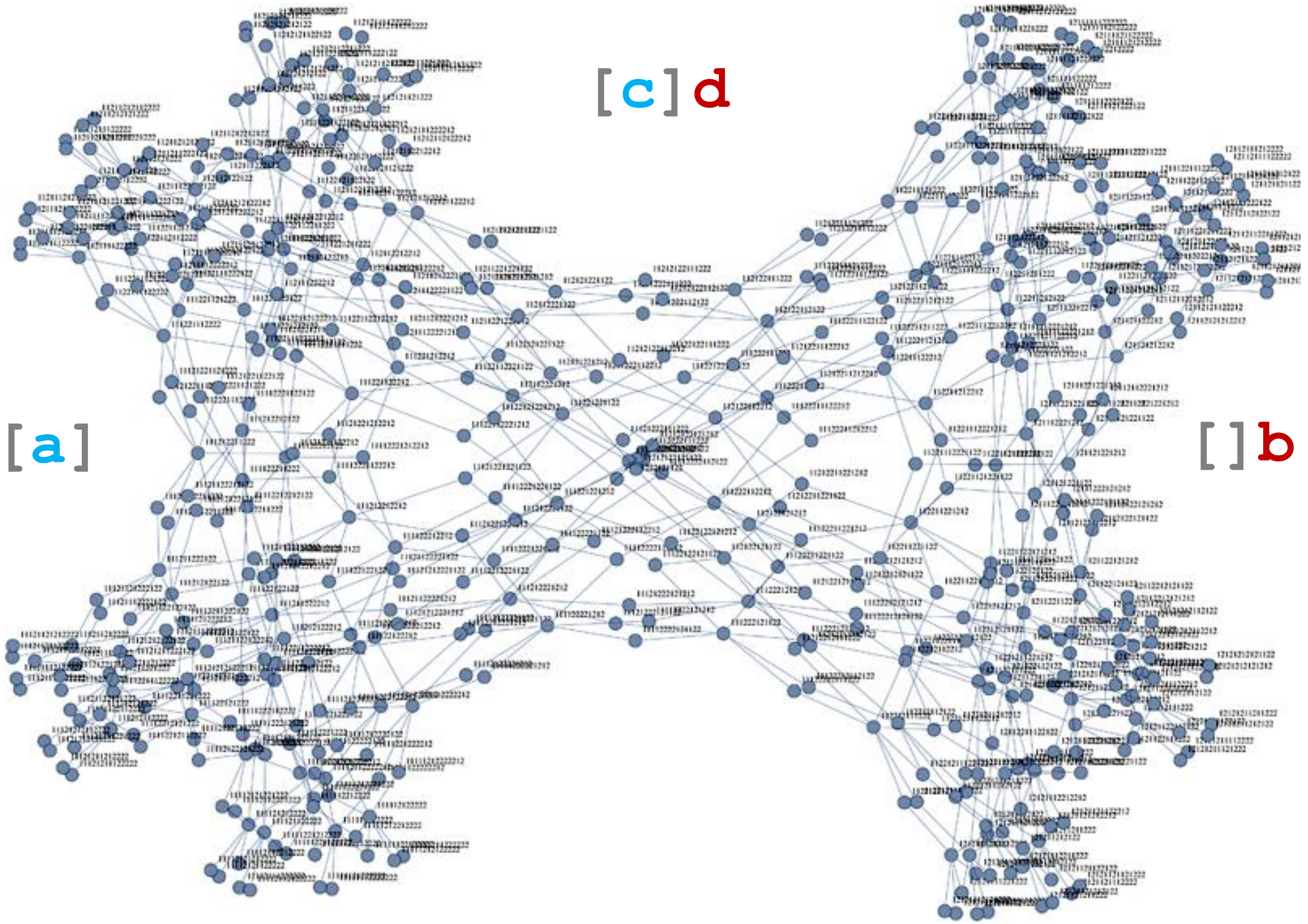
3 Congestion in a graph

Is any vertex overworked?



3 Connecting Dyck paths

It looks well connected. How to prove it?

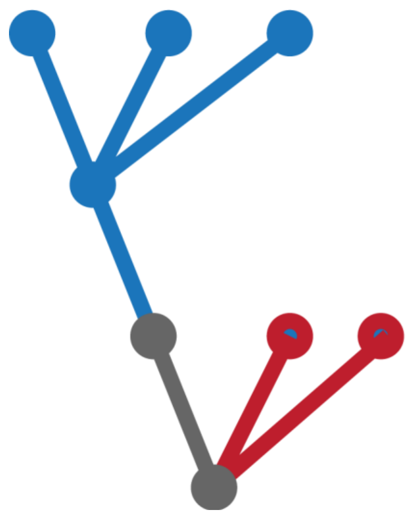


3 Pruning and growing subtrees

The lesson: be careful!

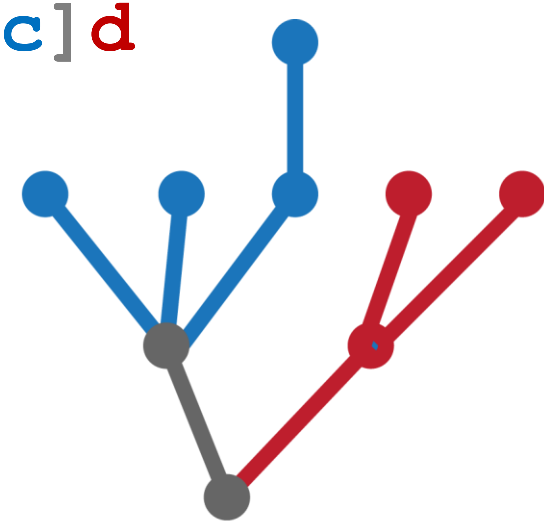
[[[][][]]][][]

[a]b

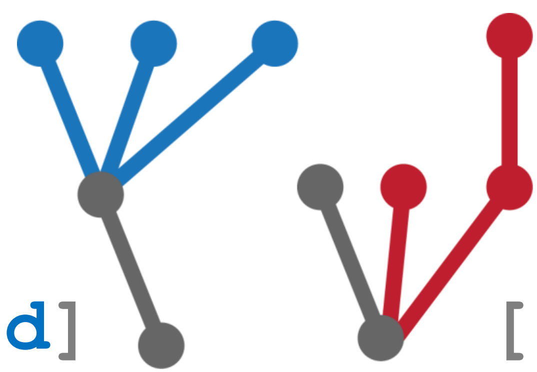


[[][][[]]][[][]]

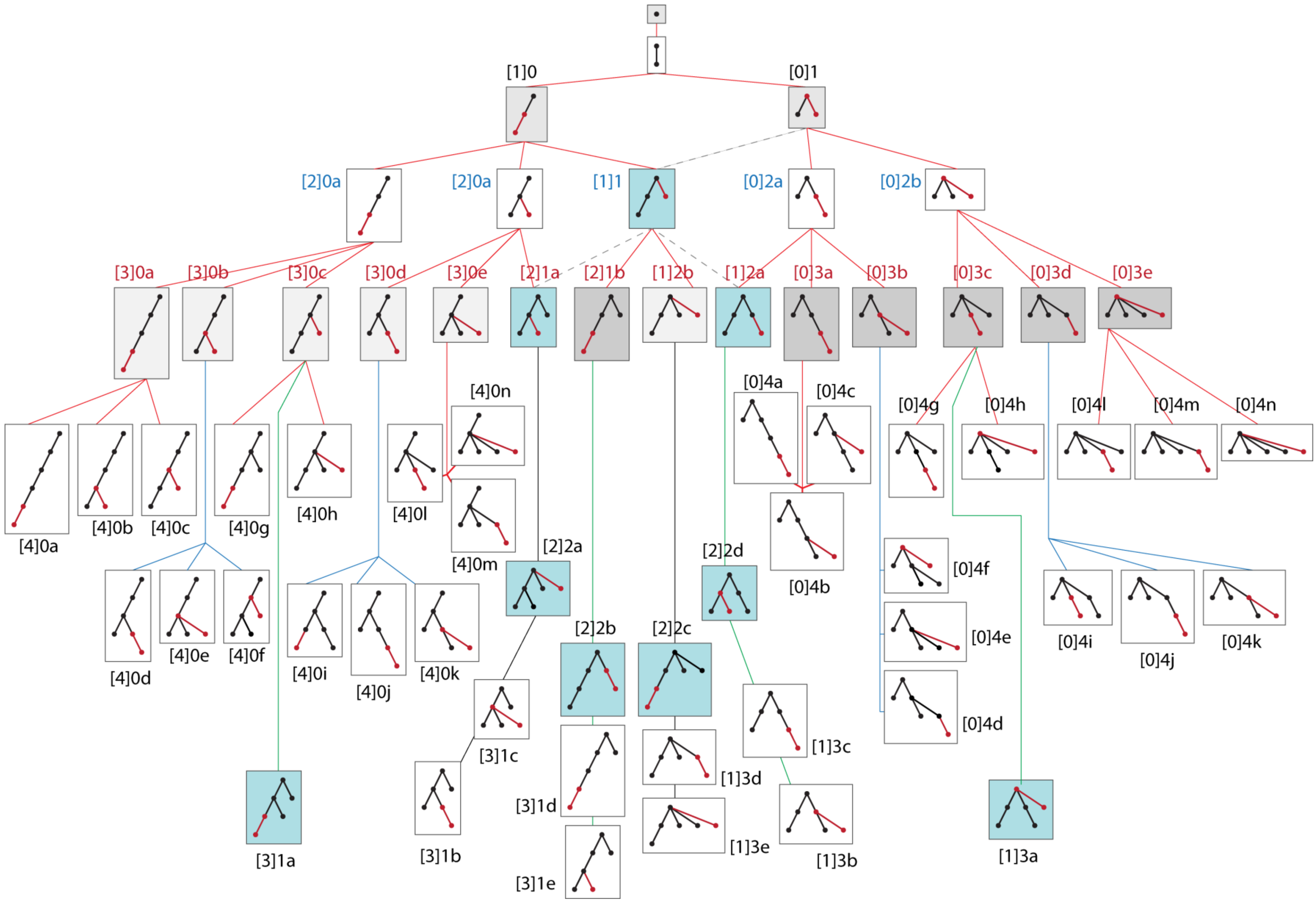
[c]d



- transform subtrees
- cut first, grow later: NO
  cut/grow/cut/grow: YES



[d] []e



[1]0
[0]1
[2]0a
[2]0a
[1]1
[0]2a
[0]2b
[3]0a
[3]0b
[3]0c
[3]0d
[3]0e
[2]1a
[2]1b
[1]2b
[1]2a
[0]3a
[0]3b
[0]3c
[0]3d
[0]3e
[4]0a
[4]0b
[4]0c
[4]0g
[4]0h
[4]0l
[4]0n
[4]0m
[0]4a
[0]4c
[0]4b
[0]4g
[0]4h
[0]4l
[0]4m
[0]4n
[4]0d
[4]0e
[4]0f
[4]0i
[4]0j
[4]0k
[2]2a
[2]2d
[0]4f
[0]4e
[0]4d
[0]4i
[0]4j
[0]4k
[2]2b
[2]2c
[3]1c
[1]3c
[1]3d
[3]1d
[1]3e
[1]3b
[3]1a
[3]1b
[3]1e
[1]3a

3 Growing trees: at most 4 children

Balance as well as possible.

- canonical growing

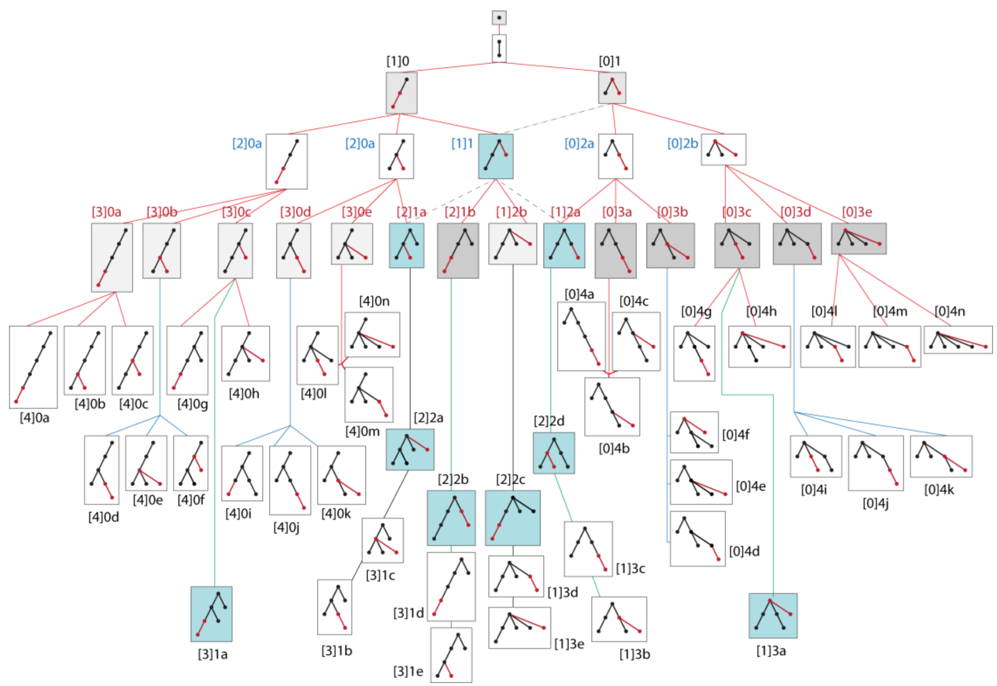
  initial steps

- cut & grow

  a left/right balanced randomized strategy

- proof of existence

  fractional matchings



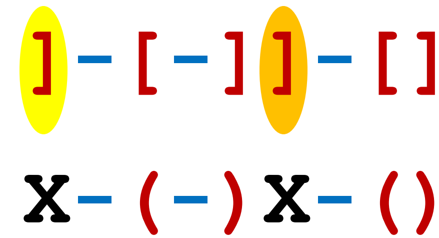
$$\lambda_{2,good} = O(n^{-c})$$

3 Eigenvalues from bad subspaces

We'll catch them at the end.

- unbalanced brackets don't mix

$$H' = H_{rest} + H_{moveX}$$

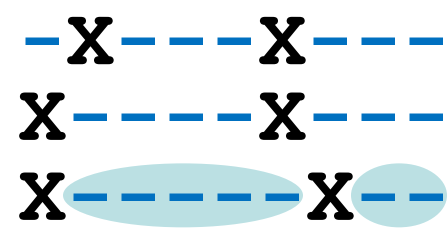


- projection lemma 1: move the **X**'s

$$H'_\epsilon = H_{rest} + \epsilon H_{moveX}$$

a new effective subspace

calculate the coefficients



a well-bracketed superposition

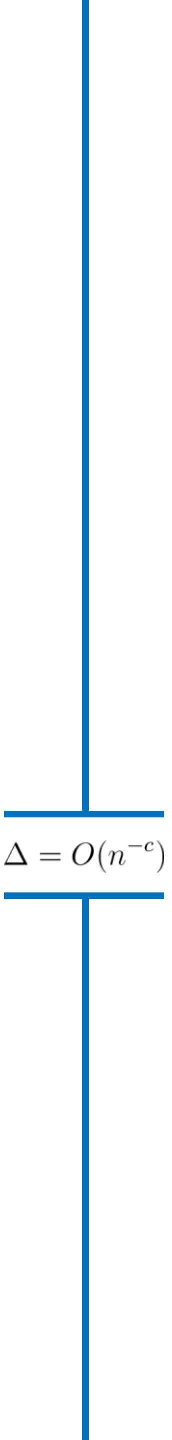
- projection lemma 2: watch the ends

$$H'' = H_{moveX} + \delta H_{end}$$

in the g. subspace of $H_{move\ X}$

$$\lambda_{1,bad} = O(n^{-c})$$

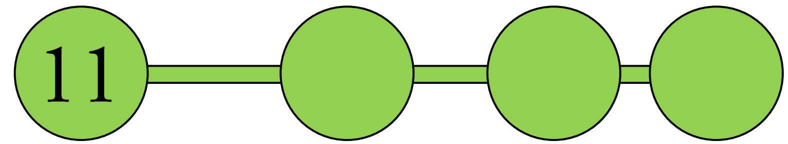
a polynomially small gap



4 Criticality without frustration in 1D

Conclusions.

- even happy chains can be difficult and fun

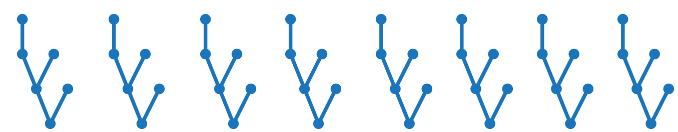


- brackets are way cooler than surfers

-[[]-[]-[]]-

A unique, entangled, frustration-free ground state of a translationally-invariant qutrit chain with a polynomial gap.

- the tools we have used



Schmidt decomposition, Catalan numbers, Motzkin & Dyck paths, random walks, projection lemma, canonical paths, fractional matchings

- optimal entropy scaling?
- a class of problems?
- pushdown automata?

```
WWWWWW@wwwwww
WWWWWWW@wwwww
WWWWWWWW@wwww
```

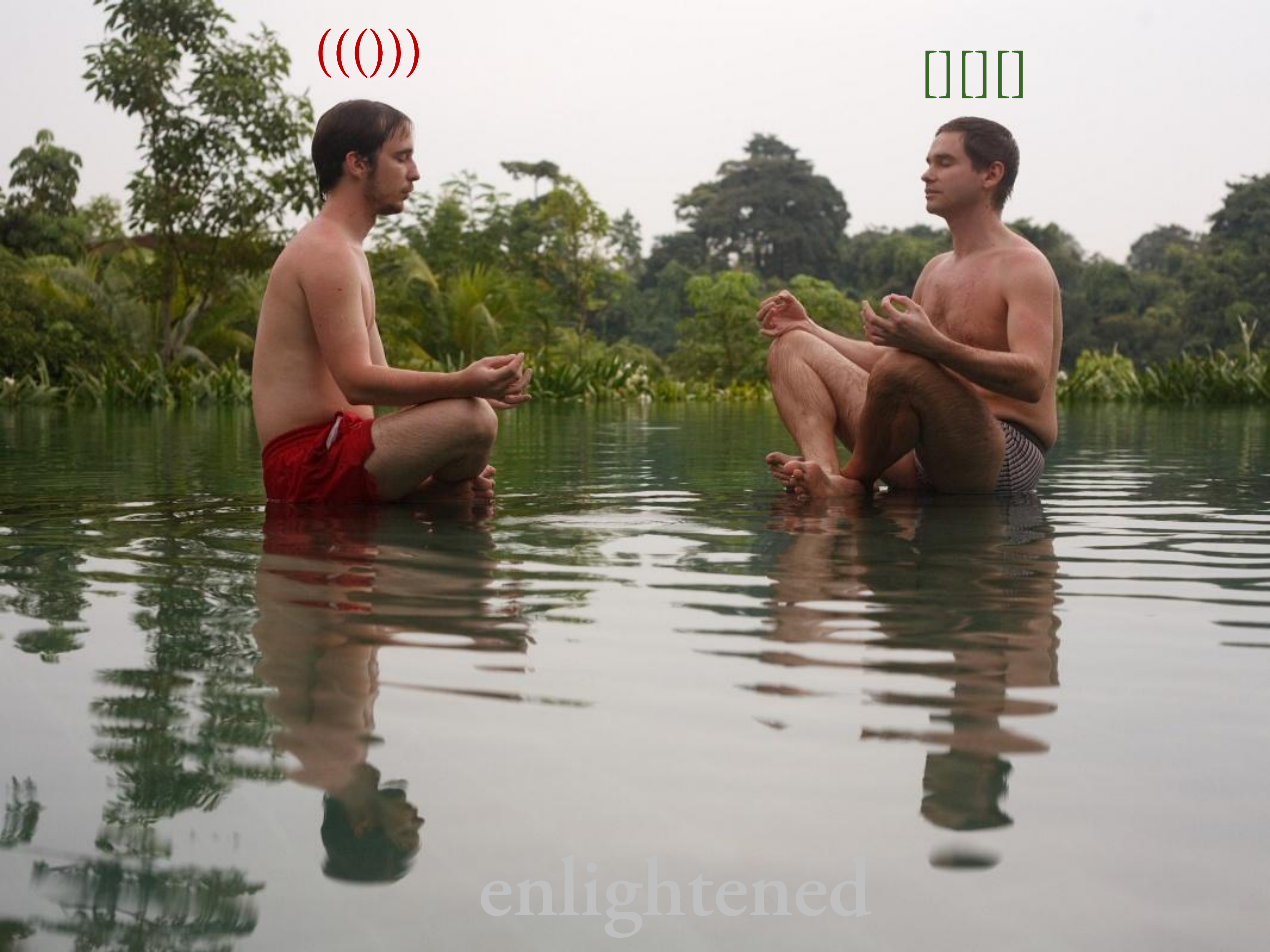
- more bracket types [M'12]

```
-AB-AB-AB-AB
```

- thermodynamic limit?
- do we need boundaries?
- connections to CFT's?

```
------------
-[--[]--][]-
-[-[--]-]---
-[-]--[-]---
-[--[][--]]-
------------
```


frustrated
FRUST
RATED


((()))
[][][]
enlightened

Criticality without Frustration

IQC Waterloo, 9/2012

Sergey Bravyi
IBM Watson

Libor Caha
Slovak Academy of Sciences

Ramis Movassagh
Northeastern University

Peter Shor
MIT

Daniel Nagaj
universität wien

arXiv: 1203.5801

Thanks: IQC, QESSENCE, LPP QWAC

Criticality without Frustration

IQC Waterloo, 9/2012

Sergey Bravyi
IBM Watson

Libor Caha
Slovak Academy of Sciences

Ramis Movassagh
Northeastern University

Peter Shor
MIT

Daniel Nagaj
universität wien

arXiv: 1203.5801

Thanks: IQC, QESSENCE, LPP QWAC