CHARACTERIZING THE NUMBER OF COLOURED $m-$ARY PARTITIONS MODULO m, WITH AND WITHOUT GAPS

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ABSTRACT. In a pair of recent papers, Andrews, Fraenkel and Sellers provide a complete characterization for the number of m-ary partitions modulo m, with and without gaps. In this paper we extend these results to the case of coloured m-ary partitions, with and without gaps. Our method of proof is different, giving explicit expansions for the generating functions modulo m.

1. INTRODUCTION

An m-ary partition is an integer partition in which each part is a nonnegative integer power of a fixed integer $m \geq 2$. An m-ary partition *without gaps* is an m-ary partition in which m^j must occur as a part whenever m^{j+1} occurs as a part, for every nonnegative integer j.

Recently, Andrews, Fraenkl and Sellers [AFS15] found an explicit expression that characterizes the number of m-ary partitions of a nonnegative integer n modulo m; remarkably, this expression depended only on the coefficients in the base m representation of n. Subsequently Andrews, Fraenkel and Sellers [AFS16] followed this up with a similar result for the number of m-ary partitions without gaps, of a nonnegative integer n modulo m; again, they were able to obtain a (more complicated) explicit expression, and again this expression depended only on the coefficients in the base m representation of n. See also Edgar [E16] and Ekhad and Zeilberger [EZ15] for more on these results.

The study of congruences for integer partition numbers has a long history, starting with the work of Ramanujan (see, e.g., [R19]). For the special case of m-ary partitions, a number of authors have studied congruence properties, including Churchhouse [C69] for $m = 2$, Rødseth [R70] for m a prime, and Andrews [A71] for arbitrary positive integers $m \geq 2$. The numbers of m-ary partitions without gaps had been previously considered by Bessenrodt, Olsson and Sellers [BOS13] for $m = 2$.

In this note, we consider m-ary partitions, with and without gaps, in which the parts are *coloured*. To specify the number of colours for parts of each size, we let $\mathbf{k} = (k_0, k_1, \ldots)$ for positive integers $k_0, k_1, \ldots$, and say that an m-ary partition is $\mathbf{k}$-*coloured* when there are k_j colours for the part m^j, for $j \geq 0$. This means that there are k_j different kinds of parts of the same size m^j. Let $b_m^{(\mathbf{k})}(n)$ denote the number of $\mathbf{k}$-coloured m-ary partitions of n, and let $c_m^{(\mathbf{k})}(n)$ denote the number of $\mathbf{k}$-coloured m-ary partitions of n without gaps. For the latter, some part m^j of any colour must occur as a part whenever some part m^{j+1} of any colour (not necessarily the same colour) occurs as a part, for every nonnegative integer j.

We extend the results of Andrews, Fraenkel and Sellers in [AFS15] and [AFS16] to the case of $\mathbf{k}$-coloured m-ary partitions, where m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$. Our method of proof is different, giving explicit expansions for the generating functions modulo m. These expansions depend on the following simple result.

Proposition 1.1. *For positive integers m, a with m relatively prime to $(a-1)!$, we have*

$$(1-q)^{-a} \equiv (1-q^m)^{-1} \sum_{\ell=0}^{m-1} \binom{a-1+\ell}{a-1} q^\ell \pmod{m}.$$

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Proof. From the binomial theorem we have

$$(1-q)^{-a} = \sum_{\ell=0}^{\infty} \binom{a-1+\ell}{a-1} q^{\ell}.$$

Now using the falling factorial notation $(a-1+\ell)_{a-1} = (a-1+\ell)(a-2+\ell)\cdots(1+\ell)$ we have

$$\binom{a-1+\ell}{a-1} = ((a-1)!)^{-1}\,(a-1+\ell)_{a-1}.$$

But

$$(a-1+\ell+m)_{a-1} \equiv (a-1+\ell)_{a-1} \pmod{m},$$

for any integer ℓ, and $((a-1)!)^{-1}$ exists in $\mathbb{Z}_m$ since m is relatively prime to $(a-1)!$, which gives

$$\binom{a-1+\ell+m}{a-1} \equiv \binom{a-1+\ell}{a-1} \pmod{m}, \tag{1}$$

and the result follows. $\square$

2. Coloured m-ary partitions

In this section we consider the following generating function for the numbers $b_m^{(\mathbf{k})}(n)$ of $\mathbf{k}$-coloured m-ary partitions:

$$B_m^{(\mathbf{k})}(q) = \sum_{n=0}^{\infty} b_m^{(\mathbf{k})}(n) q^n = \prod_{j=0}^{\infty} \left(1-q^{m^j}\right)^{-k_j}.$$

The following result gives an explicit expansion for $B_m^{(\mathbf{k})}(q)$ modulo m.

Theorem 2.1. *If m is relatively prime to $(k_0-1)!$ and to $k_j!$ for $j \geq 1$, then we have*

$$B_m^{(\mathbf{k})}(q) \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0}\right) \prod_{j=1}^{\infty} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j+\ell_j}{k_j} q^{\ell_j m^j}\right) \pmod{m}.$$

Proof. Consider the finite product

$$P_i = \prod_{j=0}^{i} \left(1-q^{m^j}\right)^{-k_j}, \qquad i \geq 0.$$

We prove that

$$P_i \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0}\right) \left(1-q^{m^{i+1}}\right)^{-1} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j+\ell_j}{k_j} q^{\ell_j m^j}\right) \pmod{m}, \tag{2}$$

by induction on i. As a base case, the result for $i=0$ follows immediately from Proposition 1.1 with $a=k_0$. Now assume that (2) holds for some choice of $i \geq 0$, and we obtain

$$\begin{aligned}
P_{i+1} &= \prod_{j=0}^{i+1} \left(1-q^{m^j}\right)^{-k_j} = \left(1-q^{m^{i+1}}\right)^{-k_{i+1}} P_i \\
&\equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0}\right) \left(1-q^{m^{i+1}}\right)^{-k_{i+1}-1} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j+\ell_j}{k_j} q^{\ell_j m^j}\right) \pmod{m} \\
&\equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0}\right) \left(1-q^{m^{i+2}}\right)^{-1} \prod_{j=1}^{i+1} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j+\ell_j}{k_j} q^{\ell_j m^j}\right) \pmod{m},
\end{aligned}$$

where the second last equivalence follows from the induction hypothesis, and the last equivalence follows from Proposition 1.1 with $a = k_{i+1}+1$, $q = q^{m^{i+1}}$.

This completes the proof of (2) by induction on i, and the result follows immediately since

$$B_m^{(\mathbf{k})}(q) = \lim_{i\to\infty} P_i.$$

□

Now we give the explicit expression for the coefficients modulo m that follows from the above expansion of the generating function $B_m^{(\mathbf{k})}(q)$.

Corollary 2.2. *For $n \geq 0$, suppose that the base m representation of n is given by*

$$n = d_0 + d_1 m + \ldots + d_t m^t, \qquad 0 \leq t.$$

If m is relatively prime to $(k_0-1)!$ and to $k_j!$ for $j \geq 1$, then we have

$$b_m^{(\mathbf{k})}(n) \equiv \binom{k_0 - 1 + d_0}{k_0 - 1} \prod_{j=1}^{t} \binom{k_j + d_j}{k_j} \pmod{m}.$$

Proof. In the expansion of the series $B_m^{(\mathbf{k})}(q)$ given in Theorem 2.1, the monomial q^n arises uniquely with the specializations $\ell_j = d_j$, $j = 0, \ldots, t$ and $\ell_j = 0$, $j \geq t$. But with these specializations, we have $\binom{k_j+\ell_j}{k_j} = \binom{k_j}{k_j} = 1$, and the result follows immediately. □

Specializing the expression given in Corollary 2.2 to the case $k_j = 1$ for $j \geq 0$ provides an alternative proof to Andrews, Fraenkel and Sellers' characterization of $m-$ary partitions modulo m, which was given as Theorem 1 of [AFS15].

3. Coloured m-ary partitions without gaps

In this section we consider the following generating function for the numbers $c_m^{(\mathbf{k})}(n)$ of $\mathbf{k}$-coloured m-ary partitions without gaps:

$$C_m^{(\mathbf{k})}(q) = 1 + \sum_{n=0}^{\infty} c_m^{(\mathbf{k})}(n) q^n = 1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \left(\left(1 - q^{m^j}\right)^{-k_j} - 1 \right).$$

The following result gives an explicit expansion for $C_m^{(\mathbf{k})}(q)$ modulo m.

Theorem 3.1. *If m is relatively prime to $(k_0-1)!$ and to $k_j!$ for $j \geq 1$, then we have*

$$C_m^{(\mathbf{k})}(q) \equiv 1 + \left(\sum_{\ell_0=1}^{m} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \sum_{i=0}^{\infty} \left(1 - q^{m^{i+1}}\right)^{-1} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j + \ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m}.$$

Proof. Consider the finite product

$$R_i = \prod_{j=0}^{i} \left(\left(1 - q^{m^j}\right)^{-k_j} - 1 \right), \qquad i \geq 0.$$

We prove that

$$R_i \equiv \left(\sum_{\ell_0=1}^{m} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \left(1 - q^{m^{i+1}}\right)^{-1} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j + \ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m}, \tag{3}$$

by induction on i. As a base case, the result for $i=0$ follows immediately from Proposition 1.1 with $a=k_0$. Now assume that (3) holds for some choice of $i \geq 0$, and we obtain

$$
\begin{aligned}
R_{i+1} &= \prod_{j=0}^{i+1}\left(\left(1-q^{m^j}\right)^{-k_j}-1\right) = \left(\left(1-q^{m^{i+1}}\right)^{-k_{i+1}}-1\right) R_i \\
&\equiv \left(\sum_{\ell_0=1}^{m}\binom{k_0-1+\ell_0}{k_0-1}q^{\ell_0}\right)\left\{\left(1-q^{m^{i+1}}\right)^{-k_{i+1}-1}-\left(1-q^{m^{i+1}}\right)^{-1}\right\} \\
&\qquad\qquad \times \prod_{j=1}^{i}\left(\sum_{\ell_j=0}^{m-1}\left\{\binom{k_j+\ell_j}{k_j}-1\right\}q^{\ell_j m^j}\right) \pmod{m} \\
&\equiv \left(\sum_{\ell_0=1}^{m}\binom{k_0-1+\ell_0}{k_0-1}q^{\ell_0}\right)\left(1-q^{m^{i+2}}\right)^{-1}\prod_{j=1}^{i+1}\left(\sum_{\ell_j=0}^{m-1}\left\{\binom{k_j+\ell_j}{k_j}-1\right\}q^{\ell_j m^j}\right) \pmod{m},
\end{aligned}
$$

where the second last equivalence follows from the induction hypothesis, and the last equivalence follows from Proposition 1.1 with $a=k_{i+1}+1$, $q=q^{m^{i+1}}$ and $a=1$, $q=q^{m^{i+1}}$.

This completes the proof of (3) by induction on i, and the result follows immediately since

$$C_m^{(\mathbf{k})}(q) = 1+\sum_{i=0}^{\infty} R_i.$$

□

Corollary 3.2. *For $n \geq 1$, suppose that n is divisible by m, with base m representation given by*

$$n = d_s m^s + \ldots + d_t m^t, \qquad 1 \leq s \leq t,$$

where $1 \leq d_s \leq m-1$, and $0 \leq d_{s+1},\ldots,d_t \leq m-1$. If m is relatively prime to $(k_0-1)!$ and to $k_j!$ for $j \geq 1$, then for $0 \leq d_0 \leq m-1$ we have

$$c_m^{(\mathbf{k})}(n-d_0) \equiv \binom{k_0-1-d_0}{k_0-1}\left(\varepsilon_s + (-1)^{s-1}\left\{\binom{k_s+d_s-1}{k_s}-1\right\}\sum_{i=s}^{t}\prod_{j=s+1}^{i}\left\{\binom{k_j+d_j}{k_j}-1\right\}\right) \pmod{m},$$

where $\varepsilon_s = 0$ if s is even, and $\varepsilon_s = 1$ if s is odd.

Proof. First note that we have

$$n-d_0 = m-d_0+(m-1)m^1+\ldots+(m-1)m^{s-1}+(d_s-1)m^s+d_{s+1}m^{s+1}+\ldots+d_t m^t.$$

Now consider the following specializations: $\ell_0 = m-d_0$, $\ell_j = m-1$, $j=1,\ldots,s-1$, $\ell_s = d_s-1$, $\ell_j = d_j$, $j=s+1,\ldots,t$, and $\ell_j = 0$, $j>t$. Then, in the expansion of the series $C_m^{(\mathbf{k})}(q)$ given in Theorem 3.1, the monomial q^n arises once for each $i \geq 0$, in particular with the above specializations truncated to $\ell_0,\ldots,\ell_i$. But with these specializations we have

- for $j=0$:
$$\binom{k_j-1+\ell_j}{k_j-1}-1 = \binom{k_0-1+m-d_0}{k_0-1} = \binom{k_0-1-d_0}{k_0-1}, \qquad \text{from (1)},$$
- for $j=1,\ldots,s-1$:
$$\binom{k_j+\ell_j}{k_j}-1 = \binom{k_j-1}{k_j}-1 = 0-1 = -1,$$
and
$$\sum_{i=0}^{s-1}\prod_{j=1}^{i}\left\{\binom{k_j+\ell_j}{k_j}-1\right\} = \sum_{i=0}^{s-1}(-1)^i = \varepsilon_s,$$

- for $j = s$:
$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_s + d_s - 1}{k_s} - 1,$$
- for $j = s+1, \ldots, t$:
$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j + d_j}{k_j} - 1,$$
- for $j > t$:
$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j}{k_j} - 1 = 1 - 1 = 0.$$

The result follows straightforwardly from Theorem 3.1. □

Specializing the expression given in Corollary 3.2 to the case $k_j = 1$ for $j \geq 0$ provides an alternative proof to Andrews, Fraenkel and Sellers' characterization of $m-$ary partitions modulo m without gaps, which was given as Theorem 2.1 of [AFS16].

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