

CHARACTERIZING THE NUMBER OF COLOURED m -ARY PARTITIONS MODULO m , WITH AND WITHOUT GAPS

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ABSTRACT. In a pair of recent papers, Andrews, Fraenkel and Sellers provide a complete characterization for the number of m -ary partitions modulo m , with and without gaps. In this paper we extend these results to the case of coloured m -ary partitions, with and without gaps. Our method of proof is different, giving explicit expansions for the generating functions modulo m .

1. INTRODUCTION

An m -ary partition is an integer partition in which each part is a nonnegative integer power of a fixed integer $m \geq 2$. An m -ary partition *without gaps* is an m -ary partition in which m^j must occur as a part whenever m^{j+1} occurs as a part, for every nonnegative integer j .

Recently, Andrews, Fraenkel and Sellers [AFS15] found an explicit expression that characterizes the number of m -ary partitions of a nonnegative integer n modulo m ; remarkably, this expression depended only on the coefficients in the base m representation of n . Subsequently Andrews, Fraenkel and Sellers [AFS16] followed this up with a similar result for the number of m -ary partitions without gaps, of a nonnegative integer n modulo m ; again, they were able to obtain a (more complicated) explicit expression, and again this expression depended only on the coefficients in the base m representation of n . See also Edgar [E16] and Ekhad and Zeilberger [EZ15] for more on these results.

The study of congruences for integer partition numbers has a long history, starting with the work of Ramanujan (see, e.g., [R19]). For the special case of m -ary partitions, a number of authors have studied congruence properties, including Churchhouse [C69] for $m = 2$, Rødseth [R70] for m a prime, and Andrews [A71] for arbitrary positive integers $m \geq 2$. The numbers of m -ary partitions without gaps had been previously considered by Bessenrodt, Olsson and Sellers [BOS13] for $m = 2$.

In this note, we consider m -ary partitions, with and without gaps, in which the parts are *coloured*. To specify the number of colours for parts of each size, we let $\mathbf{k} = (k_0, k_1, \dots)$ for positive integers k_0, k_1, \dots , and say that an m -ary partition is \mathbf{k} -coloured when there are k_j colours for the part m^j , for $j \geq 0$. This means that there are k_j different kinds of parts of the same size m^j . Let $b_m^{(\mathbf{k})}(n)$ denote the number of \mathbf{k} -coloured m -ary partitions of n , and let $c_m^{(\mathbf{k})}(n)$ denote the number of \mathbf{k} -coloured m -ary partitions of n without gaps. For the latter, some part m^j of any colour must occur as a part whenever some part m^{j+1} of any colour (not necessarily the same colour) occurs as a part, for every nonnegative integer j .

We extend the results of Andrews, Fraenkel and Sellers in [AFS15] and [AFS16] to the case of \mathbf{k} -coloured m -ary partitions, where m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$. Our method of proof is different, giving explicit expansions for the generating functions modulo m . These expansions depend on the following simple result.

Proposition 1.1. *For positive integers m, a with m relatively prime to $(a - 1)!$, we have*

$$(1 - q)^{-a} \equiv (1 - q^m)^{-1} \sum_{\ell=0}^{m-1} \binom{a-1+\ell}{a-1} q^\ell \pmod{m}.$$

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Proof. From the binomial theorem we have

$$(1-q)^{-a} = \sum_{\ell=0}^{\infty} \binom{a-1+\ell}{a-1} q^{\ell}.$$

Now using the falling factorial notation $(a-1+\ell)_{a-1} = (a-1+\ell)(a-2+\ell)\cdots(1+\ell)$ we have

$$\binom{a-1+\ell}{a-1} = ((a-1)!)^{-1} (a-1+\ell)_{a-1}.$$

But

$$(a-1+\ell+m)_{a-1} \equiv (a-1+\ell)_{a-1} \pmod{m},$$

for any integer ℓ , and $((a-1)!)^{-1}$ exists in \mathbb{Z}_m since m is relatively prime to $(a-1)!$, which gives

$$(1) \quad \binom{a-1+\ell+m}{a-1} \equiv \binom{a-1+\ell}{a-1} \pmod{m},$$

and the result follows. \square

2. COLOURED m -ARY PARTITIONS

In this section we consider the following generating function for the numbers $b_m^{(\mathbf{k})}(n)$ of \mathbf{k} -coloured m -ary partitions:

$$B_m^{(\mathbf{k})}(q) = \sum_{n=0}^{\infty} b_m^{(\mathbf{k})}(n) q^n = \prod_{j=0}^{\infty} (1 - q^{m^j})^{-k_j}.$$

The following result gives an explicit expansion for $B_m^{(\mathbf{k})}(q)$ modulo m .

Theorem 2.1. *If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have*

$$B_m^{(\mathbf{k})}(q) \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0} \right) \prod_{j=1}^{\infty} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j+\ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m}.$$

Proof. Consider the finite product

$$P_i = \prod_{j=0}^i (1 - q^{m^j})^{-k_j}, \quad i \geq 0.$$

We prove that

$$(2) \quad P_i \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0} \right) (1 - q^{m^{i+1}})^{-1} \prod_{j=1}^i \left(\sum_{\ell_j=0}^{m-1} \binom{k_j+\ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m},$$

by induction on i . As a base case, the result for $i = 0$ follows immediately from Proposition 1.1 with $a = k_0$. Now assume that (2) holds for some choice of $i \geq 0$, and we obtain

$$\begin{aligned} P_{i+1} &= \prod_{j=0}^{i+1} (1 - q^{m^j})^{-k_j} = (1 - q^{m^{i+1}})^{-k_{i+1}} P_i \\ &\equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0} \right) (1 - q^{m^{i+1}})^{-k_{i+1}-1} \prod_{j=1}^i \left(\sum_{\ell_j=0}^{m-1} \binom{k_j+\ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m} \\ &\equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0} \right) (1 - q^{m^{i+2}})^{-1} \prod_{j=1}^{i+1} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j+\ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m}, \end{aligned}$$

where the second last equivalence follows from the induction hypothesis, and the last equivalence follows from Proposition 1.1 with $a = k_{i+1} + 1$, $q = q^{m^{i+1}}$.

This completes the proof of (2) by induction on i , and the result follows immediately since

$$B_m^{(\mathbf{k})}(q) = \lim_{i \rightarrow \infty} P_i.$$

□

Now we give the explicit expression for the coefficients modulo m that follows from the above expansion of the generating function $B_m^{(\mathbf{k})}(q)$.

Corollary 2.2. *For $n \geq 0$, suppose that the base m representation of n is given by*

$$n = d_0 + d_1 m + \dots + d_t m^t, \quad 0 \leq t.$$

If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have

$$b_m^{(\mathbf{k})}(n) \equiv \binom{k_0 - 1 + d_0}{k_0 - 1} \prod_{j=1}^t \binom{k_j + d_j}{k_j} \pmod{m}.$$

Proof. In the expansion of the series $B_m^{(\mathbf{k})}(q)$ given in Theorem 2.1, the monomial q^n arises uniquely with the specializations $\ell_j = d_j$, $j = 0, \dots, t$ and $\ell_j = 0$, $j \geq t$. But with these specializations, we have $\binom{k_j + \ell_j}{k_j} = \binom{k_j}{k_j} = 1$, and the result follows immediately. □

Specializing the expression given in Corollary 2.2 to the case $k_j = 1$ for $j \geq 0$ provides an alternative proof to Andrews, Fraenkel and Sellers' characterization of m -ary partitions modulo m , which was given as Theorem 1 of [AFS15].

3. COLOURED m -ARY PARTITIONS WITHOUT GAPS

In this section we consider the following generating function for the numbers $c_m^{(\mathbf{k})}(n)$ of \mathbf{k} -coloured m -ary partitions without gaps:

$$C_m^{(\mathbf{k})}(q) = 1 + \sum_{n=0}^{\infty} c_m^{(\mathbf{k})}(n) q^n = 1 + \sum_{i=0}^{\infty} \prod_{j=0}^i \left((1 - q^{m^j})^{-k_j} - 1 \right).$$

The following result gives an explicit expansion for $C_m^{(\mathbf{k})}(q)$ modulo m .

Theorem 3.1. *If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have*

$$C_m^{(\mathbf{k})}(q) \equiv 1 + \left(\sum_{\ell_0=1}^m \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \sum_{i=0}^{\infty} (1 - q^{m^{i+1}})^{-1} \prod_{j=1}^i \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j + \ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m}.$$

Proof. Consider the finite product

$$R_i = \prod_{j=0}^i \left((1 - q^{m^j})^{-k_j} - 1 \right), \quad i \geq 0.$$

We prove that

$$(3) \quad R_i \equiv \left(\sum_{\ell_0=1}^m \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) (1 - q^{m^{i+1}})^{-1} \prod_{j=1}^i \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j + \ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m},$$

by induction on i . As a base case, the result for $i = 0$ follows immediately from Proposition 1.1 with $a = k_0$. Now assume that (3) holds for some choice of $i \geq 0$, and we obtain

$$\begin{aligned}
R_{i+1} &= \prod_{j=0}^{i+1} \left((1 - q^{m^j})^{-k_j} - 1 \right) = \left((1 - q^{m^{i+1}})^{-k_{i+1}} - 1 \right) R_i \\
&\equiv \left(\sum_{\ell_0=1}^m \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \left\{ (1 - q^{m^{i+1}})^{-k_{i+1}-1} - (1 - q^{m^{i+1}})^{-1} \right\} \\
&\quad \times \prod_{j=1}^i \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j + \ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m} \\
&\equiv \left(\sum_{\ell_0=1}^m \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) (1 - q^{m^{i+2}})^{-1} \prod_{j=1}^{i+1} \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j + \ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m},
\end{aligned}$$

where the second last equivalence follows from the induction hypothesis, and the last equivalence follows from Proposition 1.1 with $a = k_{i+1} + 1$, $q = q^{m^{i+1}}$ and $a = 1$, $q = q^{m^{i+1}}$.

This completes the proof of (3) by induction on i , and the result follows immediately since

$$C_m^{(\mathbf{k})}(q) = 1 + \sum_{i=0}^{\infty} R_i.$$

□

Corollary 3.2. For $n \geq 1$, suppose that n is divisible by m , with base m representation given by

$$n = d_s m^s + \dots + d_t m^t, \quad 1 \leq s \leq t,$$

where $1 \leq d_s \leq m - 1$, and $0 \leq d_{s+1}, \dots, d_t \leq m - 1$. If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then for $0 \leq d_0 \leq m - 1$ we have

$$c_m^{(\mathbf{k})}(n - d_0) \equiv \binom{k_0 - 1 - d_0}{k_0 - 1} \left(\varepsilon_s + (-1)^{s-1} \left\{ \binom{k_s + d_s - 1}{k_s} - 1 \right\} \sum_{i=s}^t \prod_{j=s+1}^i \left\{ \binom{k_j + d_j}{k_j} - 1 \right\} \right) \pmod{m},$$

where $\varepsilon_s = 0$ if s is even, and $\varepsilon_s = 1$ if s is odd.

Proof. First note that we have

$$n - d_0 = m - d_0 + (m - 1)m^1 + \dots + (m - 1)m^{s-1} + (d_s - 1)m^s + d_{s+1}m^{s+1} + \dots + d_t m^t.$$

Now consider the following specializations: $\ell_0 = m - d_0$, $\ell_j = m - 1$, $j = 1, \dots, s - 1$, $\ell_s = d_s - 1$, $\ell_j = d_j$, $j = s + 1, \dots, t$, and $\ell_j = 0$, $j > t$. Then, in the expansion of the series $C_m^{(\mathbf{k})}(q)$ given in Theorem 3.1, the monomial q^n arises once for each $i \geq 0$, in particular with the above specializations truncated to ℓ_0, \dots, ℓ_i . But with these specializations we have

- for $j = 0$:

$$\binom{k_j - 1 + \ell_j}{k_j - 1} - 1 = \binom{k_0 - 1 + m - d_0}{k_0 - 1} - 1 = \binom{k_0 - 1 - d_0}{k_0 - 1}, \quad \text{from (1),}$$

- for $j = 1, \dots, s - 1$:

$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j - 1}{k_j} - 1 = 0 - 1 = -1,$$

and

$$\sum_{i=0}^{s-1} \prod_{j=1}^i \left\{ \binom{k_j + \ell_j}{k_j} - 1 \right\} = \sum_{i=0}^{s-1} (-1)^i = \varepsilon_s,$$

- for $j = s$:

$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_s + d_s - 1}{k_s} - 1,$$

- for $j = s + 1, \dots, t$:

$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j + d_j}{k_j} - 1,$$

- for $j > t$:

$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j}{k_j} - 1 = 1 - 1 = 0.$$

The result follows straightforwardly from Theorem 3.1. \square

Specializing the expression given in Corollary 3.2 to the case $k_j = 1$ for $j \geq 0$ provides an alternative proof to Andrews, Fraenkel and Sellers' characterization of m -ary partitions modulo m without gaps, which was given as Theorem 2.1 of [AFS16].

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