C&O 631 ASSIGNMENT 1
Due Friday, May 26, 4:30 p.m., in MC 6024

1. Suppose that $x_1^k + \ldots + x_n^k = k^2$, for $k = 1, \ldots, n$. Evaluate

$$x_1^{n+1} + \ldots + x_n^{n+1}.$$

2. Prove the *hook formula*, that

$$f^\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)},$$

as given in the class notes (you may use the degree formula).

3. Let $a_1, \ldots, a_n$ be arbitrary positive integers. Evaluate

$$\det\left(\binom{a_i + m + j}{a_i}\right)_{i,j=1,\ldots,n}.$$

4. Prove that

$$\det\begin{pmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \ldots & -n+1 \\ p_n & p_{n-1} & p_{n-2} & \ldots & p_1 \end{pmatrix}_{n\times n} = n!\, h_n, \qquad n \geq 1.$$

5. (a) A *tournament* is an orientation of the complete graph. Let $\mathcal{T}_n$ be the set of all tournaments on vertices $\{1, \ldots, n\}$. For $t \in \mathcal{T}_n$, let $o_j(t)$ be the out-degree of vertex j, $j = 1, \ldots, n$, and let $M(t)$ be the number of oriented edges whose source is larger than sink. If

$$T(x_1, \ldots, x_n; u) = \sum_{t\in\mathcal{T}_n} u^{M(t)} x_1^{o_1(t)} \cdots x_n^{o_n(t)},$$

prove that

$$T(x_1, \ldots, x_n; u) = \prod_{1\leq i<j\leq n} (x_i + u x_j).$$

(b) Let $V(x_1, \ldots, x_n) = \det\left(x_i^{n-j}\right)_{i,j=1,\ldots,n}$, for $n \geq 1$, the Vandermonde determinant. Give a combinatorial proof that

$$V(x_1, \ldots, x_n) = \prod_{1\leq i<j\leq n} (x_i - x_j),$$

by finding a sign-reversing involution for tournaments. (**Hint:** Consider transitive and non-transitive tournaments separately.)

(c) Let ρ_m denote the partition $(m, m-1, \ldots, 2, 1)$. Prove that

$$s_{\rho_m}(x_1, \ldots, x_{m+1}) = \prod_{1\leq i<j\leq m+1} (x_i + x_j).$$

(d) **Bonus:** Can you find a bijection between tableaux of shape ρ_m and tournaments to prove part (c) combinatorially ?