C&O 631 ASSIGNMENT 1
Suggested Solutions

1. Suppose that $x_1^k + \ldots + x_n^k = k^2$, for $k = 1, \ldots, n$. Evaluate

$$x_1^{n+1} + \ldots + x_n^{n+1}.$$

Solution: We have

$$\begin{aligned}
0 = \left[t^{n+1}\right] E(-t) &= \left[t^{n+1}\right] \exp\Big(-\sum_{k\geq 1} \frac{p_k}{k} t^k\Big) \\
&= \left[t^{n+1}\right] \exp\Big(-\Big(\frac{p_{n+1}-(n+1)^2}{n+1}\Big)t^{n+1}\Big) \exp\Big(-\sum_{k\geq 1} \frac{k^2}{k} t^k\Big) \\
&= \left[t^{n+1}\right] \exp\Big(-\Big(\frac{p_{n+1}}{n+1}-(n+1)\Big)t^{n+1}\Big) \exp\Big(-\sum_{k\geq 1} k t^k\Big) \\
&= -\frac{p_{n+1}}{n+1} + (n+1) + \left[t^{n+1}\right] \exp\Big(-\frac{t}{(1-t)^2}\Big).
\end{aligned}$$

But

$$\begin{aligned}
\left[t^{n+1}\right] \exp\Big(-\frac{t}{(1-t)^2}\Big) &= \sum_{k=1}^{n+1} \frac{(-1)^k}{k!} \left[t^{n+1}\right] t^k (1-t)^{-2k} \\
&= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{1}{k!} \binom{-2k}{n+1-k} \\
&= \sum_{k=1}^{n+1} \frac{(-1)^k}{k!} \binom{n+k}{2k-1},
\end{aligned}$$

since $\binom{-a}{b} = (-1)^b \binom{a+b-1}{a-1}$. Thus, rearranging, we obtain

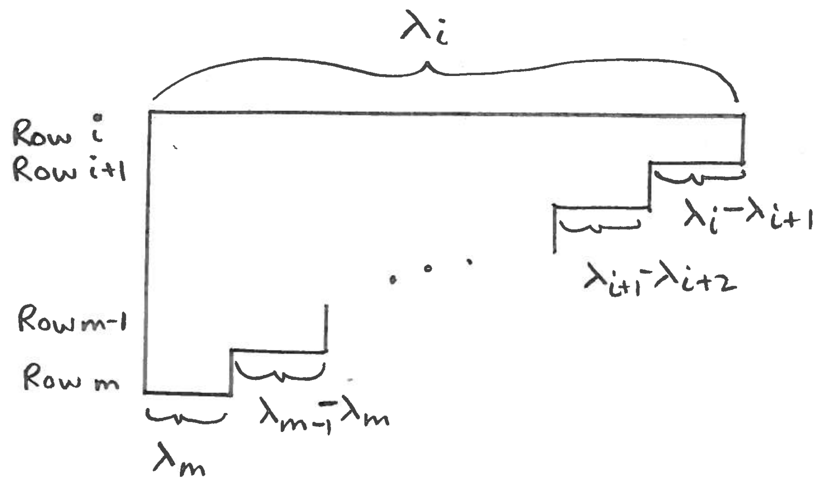
$$x_1^{n+1} + \ldots + x_n^{n+1} = p_{n+1} = (n+1)^2 + (n+1) \sum_{k=1}^{n+1} \frac{(-1)^k}{k!} \binom{n+k}{2k-1}.$$

2. Prove the *hook formula*, that

$$f^\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)},$$

as given in the class notes (you may use the degree formula).

Solution: Let $\lambda = (\lambda_1, \ldots, \lambda_m)$, where $\lambda_1 \geq \ldots \geq \lambda_m \geq 1$. Then from the diagram below, the product of the hook lengths of the cells in row i



is given by

$$(\lambda_i+m-i)_{\lambda_m}(\lambda_i-\lambda_m+m-i-1)_{\lambda_{m-1}-\lambda_m}\cdots(\lambda_i-\lambda_{i+2}+1)_{\lambda_{i+1}-\lambda_{i+2}}(\lambda_i-\lambda_{i+1})!,$$

using falling factorial notation. But this product equals

$$\frac{(\lambda_i+m-i)!}{\prod_{i<j\le m}(\lambda_i-\lambda_j+j-i)},$$

and multiplying this over $i=1,\ldots,m$ (to account for all rows), we get

$$\prod_{x\in\lambda} h(x) = \prod_{i=1}^{m} \frac{(\lambda_i+m-i)!}{\prod_{i<j\le m}(\lambda_i-\lambda_j+j-i)} = \frac{\prod_{i=1}^{m}(\lambda_i+m-i)!}{\prod_{1\le i<j\le m}(\lambda_i-\lambda_j+j-i)},$$

and the result follows from the degree formula.

3. Let $a_1,\ldots,a_n$ be arbitrary positive integers. Evaluate

$$\det\left(\binom{a_i+m+j}{a_i}\right)_{i,j=1,\ldots,n}.$$

Solution: Let D be the given determinant. Now we have

$$\binom{a_i+m+j}{a_i} = \frac{(m+2)^{(a_i)}\,(a_i+m+2)^{(j-1)}}{a_i!\,(m+2)^{(j-1)}}, \qquad i,j=1,\ldots,n,$$

where $x^{(k)}$ is the *rising factorial*, defined by $x^{(k)}=x(x+1)\cdots(x+k-1)$, $k\ge 1$, and $x^{(0)}=1$. Then, factoring $\frac{(m+2)^{(a_i)}}{a_i!}$ from row i, $i=1,\ldots,n$, and $\frac{1}{(m+2)^{(j-1)}}$ from column j, $j = 1, \ldots, n$, we obtain

$$D = \left(\prod_{\ell=1}^{n} \frac{(m+2)^{(a_\ell)}}{a_\ell!(m+2)^{(\ell-1)}} \right) \det \left((y_i)^{(j-1)} \right)_{i,j=1,\ldots,n},$$

where $y_i = a_i + m - 2$. Now, reversing the columns, we obtain

$$\begin{aligned} \det \left((y_i)^{(j-1)} \right)_{i,j=1,\ldots,n} &= (-1)^{\binom{n}{2}} \det \left((y_i)^{(n-j)} \right)_{i,j=1,\ldots,n} \\ &= (-1)^{\binom{n}{2}} \prod_{1 \le i < j \le n} (y_i - y_j) = \prod_{1 \le i < j \le n} (y_j - y_i), \end{aligned}$$

where for the second equality we have used the monic polynomial result from page 17 of the Course Notes (since $x^{(k)}$ is a monic polynomial of degree k). But $y_j - y_i = a_j - a_i$, and we conclude that

$$D = \left(\prod_{\ell=1}^{n} \frac{(m+2)^{(a_\ell)}}{a_\ell!(m+2)^{(\ell-1)}} \right) \prod_{1 \le i < j \le n} (a_j - a_i).$$

(Note that, as a function of m, the denominator of the above quotient expression for D has zeroes at $m = -2, -3, \ldots, -n$. However, $D = 0$ for each of these values (the first $k-1$ columns are identically 0 when $m = -k$. Thus use $D = 0$ when $m = -2, -3, \ldots, -n$, and the above formula for D otherwise.)

4. Prove that

$$\det \begin{pmatrix} p_1 & -1 & 0 & \ldots & 0 \\ p_2 & p_1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \ldots & -n+1 \\ p_n & p_{n-1} & p_{n-2} & \ldots & p_1 \end{pmatrix}_{n \times n} = n!\, h_n, \qquad n \ge 1.$$

Solution: From page 21 of the Course Notes, we have

$$m h_m - \sum_{k=1}^{m} p_k\, h_{m-k} = p_m, \qquad m \ge 1.$$

Then these equations for $m = 1, \ldots, n$ can be written as the matrix equation $A\vec{h} = \vec{p}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -p_1 & 2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_{n-2} & -p_{n-3} & \ldots & n-1 & 0 \\ -p_{n-1} & -p_{n-2} & \ldots & -p_1 & n \end{pmatrix}_{n \times n},$$

$$\vec{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \\ h_n \end{pmatrix}_{n\times 1}, \qquad \vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{pmatrix}_{n\times 1}.$$

Then from Cramer's Rule we obtain

$$h_n = (\det A)^{-1} \cdot \det \begin{pmatrix} 1 & 0 & 0 & \dots & p_1 \\ -p_1 & 2 & 0 & \dots & p_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_{n-2} & -p_{n-3} & \dots & n-1 & p_{n-1} \\ -p_{n-1} & -p_{n-2} & \dots & -p_1 & p_n \end{pmatrix}_{n\times n}.$$

But $\det A = n!$, and the other determinant above can be transformed to the required determinant by first: multiplying each of columns $1, \dots, n-1$ by -1, which changes the determinant by a factor of $(-1)^{n-1}$, and second: cycling the columns so that column j becomes column $j+1$ for $j = 1, \dots, n-1$, and column n becomes column 1, which also changes the determinant by a factor of $(-1)^{n-1}$ (so these factors cancel each other).

5. (a) A *tournament* is an orientation of the complete graph. Let $\mathcal{T}_n$ be the set of all tournaments on vertices $\{1, \dots, n\}$. For $t \in \mathcal{T}_n$, let $o_j(t)$ be the out-degree of vertex j, $j = 1, \dots, n$, and let $M(t)$ be the number of oriented edges whose source is larger than sink. If

$$T(x_1, \dots, x_n; u) = \sum_{t \in \mathcal{T}_n} u^{M(t)} x_1^{o_1(t)} \cdots x_n^{o_n(t)},$$

prove that

$$T(x_1, \dots, x_n; u) = \prod_{1 \le i < j \le n} (x_i + u x_j).$$

(b) Let $V(x_1, \dots, x_n) = \det \left(x_i^{n-j}\right)_{i,j=1,\dots,n}$, for $n \ge 1$, the Vandermonde determinant. Give a combinatorial proof that

$$V(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j),$$

by finding a sign-reversing involution for tournaments. (**Hint:** Consider transitive and non-transitive tournaments separately.)

(c) Let ρ_m denote the partition $(m, m-1, \dots, 2, 1)$. Prove that

$$s_{\rho_m}(x_1, \dots, x_{m+1}) = \prod_{1 \le i < j \le m+1} (x_i + x_j).$$

(d) **Bonus:** Can you find a bijection between tableaux of shape ρ_m and tournaments to prove part (c) combinatorially ?

Solution: (a) For $1 \le i < j \le n$ and $t \in \mathcal{T}_n$, let $S_{ij}(t) = 1$ if edge $\{i,j\}$ is directed from j to i in t, and $S_{ij}(t) = 0$ otherwise. Then we have

$$M(t) = \sum_{1 \le i < j \le n} S_{ij}(t),$$

and the result follows.

(b) We are asked for a combinatorial proof that

$$V(x_1, \dots, x_n) = T(x_1, \dots, x_n; -1). \tag{1}$$

Now a transitive tournament has all edges directed from $\sigma(i)$ to $\sigma(j)$ for $1 \le i < j \le n$, for some permutation $\sigma \in S(n)$. In this tournament t, vertex $\sigma(i)$ has out-degree $n-i$, for $i = 1, \dots, n$, and $M(t) = \mathrm{Inv}(\sigma)$, the number of *inversions* in σ, so we have

$$(-1)^{M(t)} = (-1)^{\mathrm{Inv}(\sigma)} = \mathrm{sgn}(\sigma).$$

But this implies that the contribution to $T(x_1, \dots, x_n; -1)$ from the set of $n!$ transitive tournaments is

$$\sum_{\sigma \in S(n)} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{\sigma(i)}^{n-i} = V(x_1, \dots, x_n),$$

so we will complete the combinatorial proof of (1) if we can show that the contribution to $T(x_1, \dots, x_n; -1)$ from the set of $2^{\binom{n}{2}} - n!$ non-transitive tournaments is 0.

We will do this by proving that the contributions of the non-transitive tournaments cancel in pairs. First note that there must be a directed triangle in a non-transitive tournament (since there must be a directed cycle of some length ≥ 3; the shortest such cycle must be a triangle, since each chord on a longer cycle will create a shorter cycle). Now if we reverse the direction of the three edges on any directed triangle in a non-transitive tournament t, then we obtain another non-transitive tournament t' with the same out-degrees on all vertices as in t, and with opposite sign (since the vertices in such a triangle can be written as $i\,j\,k$ in order around the cycle, where i is the smallest vertex on the triangle, so we have $i < j$ and $i < k$; thus if $j < k$ the contribution to $M(t)$ from this triangle is 1, and the contribution to $M(t')$ from the reversed triangle is 2; on the other hand, if $j > k$ the contribution to $M(t)$ from this triangle is 2, and the contribution to $M(t')$ from the reversed triangle is 1. In both cases $M(t)$ and $M(t')$ have opposite odd-even parity, so $(-1)^{M(t)}$ and $(-1)^{M(t')}$ have opposite sign, and hence the contributions of t and t' cancel.)

We now describe an involution on the set of non-transitive tournaments, that reverses a (canonical) single directed triangle: For a non-transitive tournament t on $n \geq 3$ vertices, let t_i be the sub-tournament induced by vertices $\{1, \ldots, i\}$, for $i = 1, \ldots, n$. Let k be the minimum choice such that t_k is non-transitive. Then $3 \leq k \leq n$ (for the lower bound, since no tournament on 1 or 2 vertices can have a directed triangle; for the upper bound, since $t = t_n$ is non-transitive). Let ρ be the permutation in $S(k-1)$ that corresponds to the transitive tournament t_{k-1}: so vertex $\rho(j)$ has out-degree $k-1-j$ in t_{k-1}, for $j = 1, \ldots, k-1$. Now since t_k is non-transitive, there is at least one directed triangle in t_k, and since t_{k-1} is transitive, each such triangle passes through vertex k, and must have vertices $k, \rho(i), \rho(j)$ in order around the directed cycle, where $i < j$ (since all edges in t_{k-1} are directed from some $\rho(i)$ to $\rho(j)$ with $i < j$). Thus there is an edge directed from k to $\rho(i)$ and an edge directed from $\rho(j)$ to k for some $i < j$. But this means that there exists ℓ chosen from $2, \ldots, k-1$ so that there is an edge directed from k to $\rho(\ell - 1)$, and an edge directed from $\rho(\ell)$ to k (e.g., look for a switch in direction on the edge joined to k between $\rho(i)$ and $\rho(j)$). Let m be the minimum such ℓ; our canonical directed triangle is $k, \rho(m-1), \rho(m)$. When we reverse this triangle in t to obtain t', note that t'_{k-1} is transitive, corresponding to permutation $\rho' = \rho\,(m-1\,m)$ (i.e., multiplying ρ on the right by the transposition $(m-1\,m)$), while t'_k is non-transitive (indeed, it has the directed triangle with vertices $k, \rho(m), \rho(m-1)$, which is also denoted $k, \rho'(m-1), \rho'(m)$ when written in terms in ρ'). The values of k and m are the same when applying this construction to t', and we have an involution, as claimed. This completes the proof that the contribution to $T(x_1, \ldots, x_n; -1)$ from the set of $2^{\binom{n}{2}} - n!$ non-transitive tournaments is 0.

(c) Using the ratio of determinant form for a Schur function in a finite set of variables, we obtain

$$s_{\rho_m}(x_1, \ldots, x_{m+1}) = \frac{\det\left(x_i^{2(n-j)}\right)_{i,j=1,\ldots,m+1}}{\det\left(x_i^{n-j}\right)_{i,j=1,\ldots,m+1}} = \frac{\det\left(y_i^{n-j}\right)_{i,j=1,\ldots,m+1}}{\det\left(x_i^{n-j}\right)_{i,j=1,\ldots,m+1}},$$

where $y_i = x_i^2$, $i = 1, \ldots, m+1$. Thus we have a Vandermonde determinant in both the numerator and the denominator, and writing these in their product form gives

$$s_{\rho_m}(x_1,\ldots,x_{m+1}) = \frac{\prod_{1\le i<j\le m+1}(y_i-y_j)}{\prod_{1\le i<j\le m+1}(x_i-x_j)} = \frac{\prod_{1\le i<j\le m+1}\left(x_i^2-x_j^2\right)}{\prod_{1\le i<j\le m+1}(x_i-x_j)}$$
$$= \prod_{1\le i<j\le m+1}\frac{x_i^2-x_j^2}{x_i-x_j} = \prod_{1\le i<j\le m+1}(x_i+x_j),$$

as required.