C&O 631 ASSIGNMENT 2
Due Friday, June 16, 4:30 p.m., in MC 6024

1. Prove, for any partitions α, β, that

$$\sum_{\lambda} s_{\lambda/\alpha}(x_1, x_2, \dots) s_{\lambda/\beta}(y_1, y_2, \dots) = \frac{\sum_{\mu} s_{\beta/\mu}(x_1, x_2, \dots) s_{\alpha/\mu}(y_1, y_2, \dots)}{\prod_{i,j \geq 1}(1 - x_i y_j)}.$$

2. Determine the number of permutations of $1, 2, \dots, n^2$ with no increasing subsequence of length $n+1$ and no decreasing subsequence of length $n+1$.

3. Prove the *hook-content* generating function result

$$s_\lambda(x^1, \dots x^n, 0, \dots) = x^{\sum_{i \geq 1} i\lambda_i} \frac{\prod_{\alpha \in \lambda}(1 - x^{n+c(\alpha)})}{\prod_{\alpha \in \lambda}(1 - x^{h(\alpha)})}.$$

You may use the rational function expression for $s_\lambda(x^1, \dots x^n, 0, \dots)$ given in the Course Notes for the lecture of May 31.

4. On pages 9 – 11 of the Course Notes, we give a nonintersecting path proof of the determinantal identity

$$s_\lambda(x_1, \dots, x_n) = \det\Big(h_{\lambda_j - j + i}(x_1, \dots, x_n)\Big)_{i,j=1,\dots m}, \qquad (1)$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$ is a partition with at most m parts.

(a) Describe how to modify the proof of (1) to prove that

$$s_{\lambda/\mu}(x_1, x_2, \dots) = \det\Big(h_{\lambda_j - \mu_i - j + i}(x_1, x_2, \dots)\Big)_{i,j=1,\dots m},$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_m)$ are partitions with at most m parts.

(b) Describe how to modify the proof of (1) to prove that

$$s_{\lambda/\mu}(x_1, x_2, \dots) = \det\Big(e_{\lambda'_j - \mu'_i - j + i}(x_1, x_2, \dots)\Big)_{i,j=1,\dots,\lambda_1},$$

where λ and μ are partitions, and their conjugates $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ and $\mu' = (\mu'_1, \dots, \mu'_m)$ are partitions with at most λ_1 parts.

(c) Deduce from parts (a) and (b) that $\omega\Big(s_{\lambda/\mu}\Big) = s_{\lambda'/\mu'}$.

5. Let ρ_m denote the partition $(m, m-1, \dots, 2, 1)$. Prove that

$$s_{\rho_m}^2 = s_{\rho_{m+1}/(1)} s_{\rho_{m-1}} - s_{\rho_{m+1}} s_{\rho_{m-1}/(1)}, \qquad m \geq 2.$$

You may use the result of Problem 4(a) on this Assignment.