C&O 631 ASSIGNMENT 2
Suggested Solutions

1. Prove, for any partitions α, β, that

$$\sum_\lambda s_{\lambda/\alpha}(x_1, x_2, \dots)s_{\lambda/\beta}(y_1, y_2, \dots) = \frac{\sum_\mu s_{\beta/\mu}(x_1, x_2, \dots)s_{\alpha/\mu}(y_1, y_2, \dots)}{\prod_{i,j\geq 1}(1 - x_i y_j)}.$$

Solution: Considering countable sets of variables $(w_1, \dots)$, $(x_1, \dots)$, $(y_1, \dots)$, $(z_1, \dots)$, multiply on both sides by $s_\alpha(w_1, \dots)s_\beta(z_1, \dots)$ and sum over all partitions α, β, to obtain on the left hand side

$$\begin{aligned}
&\sum_\lambda \sum_\alpha s_{\lambda/\alpha}(x_1, \dots)s_\alpha(w_1, \dots) \sum_\beta s_{\lambda/\beta}(y_1, \dots)s_\beta(z_1, \dots) \\
&= \sum_\lambda s_\lambda(x_1, \dots w_1, \dots)s_\lambda(y_1, \dots z_1, \dots) \\
&= \prod_{i,j\geq 1}(1 - x_i y_j)^{-1}(1 - x_i z_j)^{-1}(1 - w_i y_j)^{-1}(1 - w_i z_j)^{-1}
\end{aligned}$$

where for the first equality we have used equation (31) on page 25 of the Course Notes. But on the right hand side we obtain

$$\begin{aligned}
&\sum_\mu \sum_\beta s_{\beta/\mu}(x_1, \dots)s_\beta(z_1, \dots) \sum_\alpha s_{\alpha/\mu}(y_1, \dots)s_\alpha(w_1, \dots) \prod_{i,j\geq 1}(1 - x_i y_j)^{-1} \\
&= \sum_\mu s_\mu(z_1, \dots)s_\mu(w_1, \dots) \prod_{i,j\geq 1}(1 - x_i y_j)^{-1}(1 - x_i z_j)^{-1}(1 - w_i y_j)^{-1} \\
&= \prod_{i,j\geq 1}(1 - x_i y_j)^{-1}(1 - x_i z_j)^{-1}(1 - w_i y_j)^{-1}(1 - w_i z_j)^{-1},
\end{aligned}$$

where for the first equality we have used the summation near the top of page 26 of the Course Notes, and for the second equality we have used Theorem 8.2 from the Course Notes. The result follows immediately.

2. Determine the number of permutations of $1, 2, \dots, n^2$ with no increasing subsequence of length $n+1$ and no decreasing subsequence of length $n+1$.

Solution: From the Robinson-Schensted correspondence, the number of such permutations is given by $\left(f^\lambda\right)^2$, where $\lambda = (n, \dots, n)$ is the partition of n^2 with n parts, all equal to n (since this is the only partition of n^2 with largest part smaller than $n+1$ and number of parts smaller than $n+1$). In the hook formula for f^λ, we have denominator

$$\prod_{x\in\lambda} h(x) = \prod_{i=1}^{n} \frac{(n+i-1)!}{(i-1)!},$$

since $\frac{(n+i-1)!}{(i-1)!}$ is the product of the hook-lengths in the ith row of the Ferrers graph. We conclude that the required number of permutations is given by

$$(n^2)!^2 \prod_{i=1}^{n} \frac{(i-1)!^2}{(n+i-1)!^2}.$$

3. Prove the *hook-content* generating function result

$$s_\lambda(x^1,\dots x^n,0,\dots) = x^{\sum_{i\geq 1} i\lambda_i}\,\frac{\prod_{\alpha\in\lambda}(1-x^{n+c(\alpha)})}{\prod_{\alpha\in\lambda}(1-x^{h(\alpha)})}.$$

You may use the rational function expression for $s_\lambda(x^1,\dots x^n,0,\dots)$ given in the Course Notes for the lecture of May 31.

Solution: The formula in the Course Notes (corrected by replacing $i-j$ in the denominator to $j-i$) gives

$$s_\lambda(x^1,\dots x^n,0,\dots) = x^{\sum_{i\geq 1} i\lambda_i} F,$$

where

$$F = \frac{\prod_{1\leq i<j\leq n}(1-x^{\lambda_i-\lambda_j-i+j})}{\prod_{1\leq i<j\leq n}(1-x^{j-i})} = \frac{\prod_{1\leq i<j\leq n}(1-x^{\lambda_i-\lambda_j-i+j})}{\prod_{i=1}^{n}\prod_{\ell=1}^{n-i}(1-x^{\ell})}.$$

But, using the argument given in the solution to Problem 2 on Assignment 1, replacing m by n, we obtain

$$\prod_{\alpha\in\lambda}(1-x^{h(\alpha)}) = \frac{\prod_{i=1}^{n}\prod_{\ell=1}^{\lambda_i+n-i}(1-x^{\ell})}{\prod_{1\leq i<j\leq n}(1-x^{\lambda_i-\lambda_j-i+j})}.$$

Substituting above, we get

$$F = \frac{\prod_{i=1}^{n}\prod_{\ell=n-i+1}^{\lambda_i+n-i}(1-x^{\ell})}{\prod_{\alpha\in\lambda}(1-x^{h(\alpha)})},$$

and the result follows since the values of $n+c(\alpha)$ for the cells α in row i are given by $n+1-i,\dots,n+\lambda_i-i$.

4. On pages 9 – 11 of the Course Notes, we give a nonintersecting path proof of the determinantal identity

$$s_\lambda(x_1,\dots,x_n) = \det\Big(h_{\lambda_j-j+i}(x_1,\dots,x_n)\Big)_{i,j=1,\dots m}, \tag{1}$$

where $\lambda=(\lambda_1,\dots,\lambda_m)$ is a partition with at most m parts.

(a) Describe how to modify the proof of (1) to prove that

$$s_{\lambda/\mu}(x_1,x_2,\dots) = \det\Big(h_{\lambda_j-\mu_i-j+i}(x_1,x_2,\dots)\Big)_{i,j=1,\dots m},$$

where $\lambda=(\lambda_1,\dots,\lambda_m)$ and $\mu=(\mu_1,\dots,\mu_m)$ are partitions with at most m parts.

(b) Describe how to modify the proof of (1) to prove that

$$s_{\lambda/\mu}(x_1,x_2,\dots) = \det\Big(e_{\lambda'_j-\mu'_i-j+i}(x_1,x_2,\dots)\Big)_{i,j=1,\dots,\lambda_1},$$

where λ and μ are partitions, and their conjugates $\lambda'=(\lambda'_1,\dots,\lambda'_m)$ and $\mu'=(\mu'_1,\dots,\mu'_m)$ are partitions with at most λ_1 parts.

(c) Deduce from parts (a) and (b) that $\omega\Big(s_{\lambda/\mu}\Big) = s_{\lambda'/\mu'}$.

Solution: (a) The only modification required is to change the starting points to $P_j = (m - j + \mu_j, 1)$ and the ending points to $Q - j = (m - j + \lambda_j, \infty)$, for $j = 1, \ldots, m$.

(b) Three modifications are sufficient: First, associate one path with each *column*, and note that there are thus λ_1 paths, with the jth such path of length $\lambda'_j - \mu'_j$. Second, use starting points $P_j = (-j + \mu'_j, j - \mu'_j + 1)$ and ending points $Q_j = (-j + \lambda'_j, \infty)$, for $j = 1, \ldots, \lambda_1$). Third, mark each horizontal step from (i, j) to $(i + 1, j)$, by x_{i+j}, for $i + j \geq 1$.

(c) From part (a) we have

$$\omega\Big(s_{\lambda/\mu}\Big) = \omega\Big(\det\Big(h_{\lambda_j-\mu_i-j+i}\Big)_{i,j=1,\ldots m}\Big) = \det\Big(e_{\lambda_j-\mu_i-j+i}\Big)_{i,j=1,\ldots m} = s_{\lambda'/\mu'},$$

where the final equality follows from part (b), giving the required result.

5. Let ρ_m denote the partition $(m, m-1, \ldots, 2, 1)$. Prove that

$$s^2_{\rho_m} = s_{\rho_{m+1}/(1)} s_{\rho_{m-1}} - s_{\rho_{m+1}} s_{\rho_{m-1}/(1)}, \qquad m \geq 2.$$

You may use the result of Problem 4(a) on this Assignment.

Solution: For $m \geq 2$, let A be the $m \times m$ matrix with (i, j)-entry $h_{m+1-2j+i}$, B be the $m \times m$ matrix with (i, j)-entry $h_{m-1-2j+i}$, and C be the $m \times m$ matrix with $(i, 1)$-entry h_{m+1+i} and all other entries equal to 0 (so, e.g., $\det A = s_{\rho_m}$, and $\det B = \det C = 0$ since both B and C have mth columns consisting entirely of 0's). Let D be the $2m \times 2m$ matrix given in block matrix form in terms of A, B, C by

$$D = \left(\begin{array}{c|c} A & B \\ \hline C & A \end{array}\right).$$

We evaluate $\det D$ in two different ways: First, by the Laplace expansion in rows $1, \ldots, m$, we obtain

$$\det D = (\det A)^2 = s^2_{\rho_m},$$

since all other terms are 0 (columns $2, \ldots, m$ must be chosen for the $m \times m$ submatrix on rows $1, \ldots m$, since these columns consist entirely of 0's on rows $m + 1, \ldots, 2m$; if column 1 is also chosen for the $m \times m$ submatrix on rows $1, \ldots m$, then we get the term above; otherwise, for $j = 1, \ldots, m$, if column $m + j$ is also chosen for the $m \times m$ submatrix on rows $1, \ldots m$, then column $m + j$ is equal to column $j + 1$ for $j = 1, \ldots, m - 1$, and column $2m$ consists entirely of 0's, so the $m \times m$ subdeterminant on rows $1, \ldots m$ is equal to 0 in all of these cases).

Second, consider the Laplace expansion in columns $2, \ldots, m$: none of rows $m + 1, \ldots, 2m$ can be chosen for the $m - 1 \times m - 1$ submatrix on columns $2, \ldots, m$, since these rows consist entirely of 0's on columns $2, \ldots, m$; but this means that rows $m + 1, \ldots, 2m$ must be chosen for the $m + 1 \times m + 1$ submatrix on columns $1, m + 1, \ldots, 2m$, together with row j for some $j = 1, \ldots m$. But, for $j = 3, \ldots, m$, row j is identical to row $m + j - 2$ on columns $1, m + 1, \ldots, 2m$, so the subdeterminant on columns $1, m + 1, \ldots, 2m$ is equal to 0 for $j = 3, \ldots, m$. Thus there are two terms in the Laplace expansion, corresponding to $j = 1$ and $j = 2$, giving

$$\det D = s_{\rho_{m+1}/(1)} s_{\rho_{m-1}} - s_{\rho_{m+1}} s_{\rho_{m-1}/(1)},$$

using the result of Problem 4(a) for each of the two terms. The result follows by equating these two expressions for $\det D$.