C&O 631 ASSIGNMENT 3
Due Friday, July 14, 4:30 p.m., in MC 6024

REVISED June 30

1. Let $Y_k = M^{(\lambda)}((k, k+1))$ be the $f^{(\lambda)} \times f^{(\lambda)}$ matrix defined in Theorem 16.1(B) on page 45 of the Course Notes (Young's seminormal representation). For each partition λ of $n \geq 1$, prove that the Y_k, $k = 1, \dots, n-1$ satisfy the Coxeter relations given on page 46 of the Course Notes.

2. For the partition $\alpha = (\alpha_1, \dots, \alpha_m)$ of n, let the ith partial sum of the parts be given by $s_i = \sum_{j=1}^{i} \alpha_j$, $i = 1, \dots m$. Let $\sigma_\alpha = (s_1, \dots, 1)(s_2, \dots, s_1 + 1) \cdots (s_m, \dots, s_{m-1} + 1)$ be the permutation with directed cycles $(s_1, \dots, 1)$, $(s_2, \dots, s_1 + 1)$, $\dots$ $(s_m, \dots, s_{m-1} + 1)$. That is, σ_α is a particular element of conjugacy class $C^{(\alpha)}$.

For another partition λ of n, and a Young tableau T of shape λ, define the product

$$\Omega_\alpha^{(\lambda)}(T) = \prod_{\substack{1 \leq j \leq n \\ j \neq s_1, \dots, s_m}} \frac{1}{d_T(j, j+1)},$$

(a) Prove that the (i,i)-entry of the matrix $M^{(\lambda)}(\sigma_\alpha)$ is given by

$$\Omega_\alpha^{(\lambda)}(T_i).$$

(NOTE the revision to the Course Notes on pages 45 to 47, in which we use the convention of multiplying permutations from left to right.)

(b) Deduce the Murnaghan-Nakayama rule for

$$\operatorname{trace} M^{(\lambda)}(\sigma_\alpha)$$

from the result in part (a), and from the following result:

LEMMA: We say that a tableau T of skew shape λ/μ with n cells is a *Young* tableau when the integers $1, \dots, n$ appear once each in the cells. Given a Young tableau T of skew shape λ/μ, we denote the cell containing integer j by $T^{-1}(k)$, $k = 1, \dots, n$. Introduce indeterminates x_α, $\alpha \in \lambda/\mu$, so there is one of these for each cell in the skew shape. In fact, we'll use the notation $x_{i,j}$ for the cell in row i and column j. Define

$$P^{(\lambda/\mu)}(T) = \prod_{k=1}^{n-1} \frac{1}{x_{T^{-1}(k+1)} - x_{T^{-1}(k)}}.$$

Then we have

$$\sum_T P^{(\lambda/\mu)}(T) = \begin{cases} \frac{A}{B \cdot C}, & \text{if } \lambda/\mu \text{ is edge connected,} \\ 0, & \text{otherwise,} \end{cases}$$

where the summation on the left is over all Young tableaux T of skew shape λ/μ, and on the right we have the three products

$$A = \prod_{(i,j),(i+1,j+1)\in\lambda/\mu} (x_{i+1,j+1} - x_{i,j}) \,,$$

$$B = \prod_{(i,j),(i,j+1)\in\lambda/\mu} (x_{i,j+1} - x_{i,j}) \,,$$

$$C = \prod_{(i,j),(i+1,j)\in\lambda/\mu} (x_{i+1,j} - x_{i,j}) \,.$$

Note, for example, that the range for the product A is all pairs $(i,j),(i+1,j+1)$ such that cells (i,j) and $(i+1,j+1)$ are *both* contained in the skew shape λ/μ.

(c) BONUS: Prove the above LEMMA.