C&O 631 ASSIGNMENT 4
Due Friday, July 28, 4:30 p.m., in MC 6024

1. Prove that the mapping ψ defined on pages 62 and 63 of the Course Notes is an involution.

2. Conside the symmetric functions u_λ defined on page 57 of the Course Notes.

(a) Prove that $u_{(n)} = -p_n$, $n \geq 1$.

(b) Prove that the number of ordered factorizations of $(1\,2\,\ldots\,n)$ into m $(k+1)$-cycles, where $n = km+1$, is given by

$$n^{m-1}.$$

(A $(k+1)$-cycle is a permutation with a cycle of length $k+1$, together with $n-k-1$ fixed points.)

3 (a) For $n \geq 1$, let b_n be the number of equivalence classes of factorizations of $(1\,2\,\ldots\,n)$ into $n-1$ transpositions as considered on pages 63 - 65 of the Course Notes. For $n \geq 2$ and each such factorization f, it is known that, for a unique choice of p, q with $1 \leq p < q \leq n$,

$$f \equiv f_1 \cdot f_2 \cdot (1\,p) \cdot f_3,$$

where f_1 is a minimal transposition factorization of $(1\,(q+1)\,\ldots\,n)$, f_2 is a minimal transposition factorization of $(2\,3\,\ldots\,p)$, and f_3 is a minimal transposition factorization of $(p\,(p+1)\,\ldots\,q)$. Deduce from this that $B(x) = \sum_{n\geq 1} b_n x^{n-1}$ satisfies the functional equation

$$B(x) = 1 + x\,B(x)^3.$$

(b) Deduce from part (a) that

$$b_n = \frac{1}{2n-1}\binom{3n-3}{n-1}, \qquad n \geq 1.$$

(c) BONUS: Prove the canonical representation of equivalence classes given in part (a).