C&O 631 ASSIGNMENT 4
Suggested Solutions

1. Prove that the mapping ψ defined on pages 62 and 63 of the Course Notes is an involution.

Solution: Suppose that we are in Case 1, so $\alpha = \emptyset$ or $\alpha_m < s_i$, and we have $\psi((\alpha, s)) = (\alpha', t)$, where $s \equiv s_i t$ and $\alpha' = \alpha \cup \{s_i\}$. Now suppose that t_j exists such that t_j is the largest element in t that commutes with all elements of α', and that can appear leftmost in a string equivalent to t (with $t \equiv t_j t'$). Then $s \equiv s_i t_j t' \equiv t_j s_i t'$ (for the second equivalence, $s_i \in \alpha'$, so t_j commutes with s_i), so we must have $t_j < s_i$. But this implies that $\psi((\alpha', t)) = (\alpha, s)$. On the other hand, if no such t_j exists, then we also have $\psi((\alpha', t)) = (\alpha, s)$.

Otherwise, suppose that we are in Case 2, so α_m exists, and we have $\psi((\alpha, s)) = (\alpha'', s'')$, where $\alpha'' = \alpha \setminus \{\alpha_m\}$ and $s'' = \alpha_m s''$. Then by construction α_m is the largest element in s'' that commutes with everything in α'', and that can appear leftmost in a string equivalent to s''. Also by construction, either $\alpha'' = \emptyset$ or its largest element is α_{m-1} where $\alpha_{m-1} < \alpha_m$. Hence in both cases we have $\psi(\alpha'', s'') = (\alpha, s)$.

2. Consider the symmetric functions u_λ defined on page 57 of the Course Notes.

(a) Prove that $u_{(n)} = -p_n$, $n \geq 1$.

(b) Prove that the number of ordered factorizations of $(1\,2\,\ldots\,n)$ into m $(k+1)$-cycles, where $n = km + 1$, is given by

$$n^{m-1}.$$

(A $(k+1)$-cycle is a permutation with a cycle of length $k+1$, together with $n-k-1$ fixed points.)

Solution: (a) At the bottom of page 56 of the Course Notes, we have the formula

$$h_n^* = \sum_{\lambda \vdash n} (-1)^{l(\lambda)} (n+1)^{l(\lambda)-1} \frac{p_\lambda}{z(\lambda)}, \qquad n \geq 1, \tag{1}$$

where we have used the evaluation $|C^{(\lambda)}| = n!/z(\lambda)$. This formula immediately gives

$$\langle h_n^*, p_n \rangle = (-1)^1 (n+1)^0 = -1,$$

since $\langle p_\lambda, p_\nu \rangle = z(\lambda)\delta_{\lambda,\nu}$. The formula also implies that if μ has more than one part, then $h_\mu^* = h_{\mu_1}^* \cdots$ is a linear combination of p_α's in which all partitions α that appear have more than one part. But this means that

$$\langle h_\mu^*, p_n \rangle = 0$$

for all $\mu \vdash n$ with more than one part. Combining these two expressions gives

$$\langle h_\mu^*, -p_n \rangle = \delta_{\mu,(n)},$$

and we conclude from the definition $\langle h_\mu^*, u_\lambda \rangle = \delta_{\mu,\lambda}$ of the u_λ that $u_{(n)} = -p_n$, $n \geq 1$.

(b) From Corollary 20.2 on page 61 of the Course Notes, the required number is given by

$$\begin{aligned}\left[u_{(n-1)}\right] u_{(k)}^m &= \langle h_{n-1}^*, u_{(k)}^m \rangle = \langle h_{n-1}^*, (-p_k)^m \rangle = (-1)^m \langle h_{n-1}^*, p_k^m \rangle \\ &= (-1)^m (-1)^m (n-1+1)^{m-1} = n^{m-1},\end{aligned}$$

where the second equality follows from part (a), and the fourth equality follows from formula (1).

3 (a) For $n \geq 1$, let b_n be the number of equivalence classes of factorizations of $(1\,2\,\ldots\,n)$ into $n-1$ transpositions as considered on pages 63 - 65 of the Course Notes. For $n \geq 2$ and each such factorization f, it is known that, for a unique choice of p, q with $1 \leq p < q \leq n$,

$$f \equiv f_1 \cdot f_2 \cdot (1\,p) \cdot f_3,$$

where f_1 is a minimal transposition factorization of $(1\,(q+1)\,\ldots\,n)$, f_2 is a minimal transposition factorization of $(2\,3\,\ldots\,p)$, and f_3 is a minimal transposition factorization of $(p\,(p+1)\,\ldots\,q)$. Deduce from this that $B(x) = \sum_{n\geq 1} b_n x^{n-1}$ satisfies the functional equation

$$B(x) = 1 + x\,B(x)^3.$$

(b) Deduce from part (a) that

$$b_n = \frac{1}{2n-1}\binom{3n-3}{n-1}, \qquad n \geq 1.$$

(c) BONUS: Prove the canonical representation of equivalence classes given in part (a).

Solution: [TYPO: The range of values for p, q should be $1 < p \leq q \leq n$.] (a) From the equivalence given, we immediately deduce that

$$b_n = \sum_{1<p\leq q\leq n} b_{n-q+1} b_{p-1} b_{q-p+1} = \sum_{\substack{i,j,k\geq 1 \\ i+j+k=n+1}} b_i b_j b_k, \qquad n \geq 2,$$

where for the second equality, we have changed summation variables to $i = n-q+1$, $j = p-1$, $k = q-p+1$. Multiplying this equation by x^{n-1}, and summing over $n \geq 2$, we obtain

$$\sum_{n\geq 2} b_n x^{n-1} = x \sum_{i\geq 1} b_i x^{i-1} \sum_{j\geq 1} b_j x^{j-1} \sum_{k\geq 1} b_k x^{k-1}.$$

But $b_1 = 1$, since there is a single, empty factorization of (1) into 0 transpositions. Thus adding b_1 to the left hand side of the above equation, and 1 to the right hand side, we get

$$B(x) = 1 + xB(x)^3.$$

(b) Let $A(x) = B(x) - 1$, so $B(x) = A(x) + 1$. Then the cubic equation in part (a) for $B(x)$ becomes the cubic equation

$$A = x(1+A)^3,$$

for $A = A(x)$. Then for $n \geq 2$, Lagrange's Implicit Function Theorem gives

$$b_n = [x^{n-1}]A = \frac{1}{n-1}[z^{n-2}]\left((1+z)^3\right)^{n-1} = \frac{1}{n-1}[z^{n-2}](1+z)^{3n-3} = \frac{1}{n-1}\binom{3n-3}{n-2}.$$

But for $n \geq 2$, we have $\frac{1}{n-1}\binom{3n-3}{n-2} = \frac{(3n-3)!}{(n-1)!(2n-1)!} = \frac{1}{2n-1}\binom{3n-3}{n-1}$, and since $b_1 = 1$, this formula works for all $n \geq 1$, as required.