C&O 631 NOTES

1. Formal power series

The series that we shall use in this course are *formal power series*, not the power series of real variables that have been studied in calculus courses. A formal power series is given by $A(x) = \sum_{i\geq 0} a_i x^i$, where $a_i = [x^i]A(x)$, the *coefficient of* x^i, is a complex number, for $i \geq 0$. The basic rule for $A(x)$ is that a_i is determined finitely for each finite i. Let $B(x) = \sum_{i\geq 0} b_i x^i$. Then $A(x) = B(x)$ if and only if $a_i = b_i$ for all $i \geq 0$, and we define sum and product by

$$A(x) + B(x) = \sum_{i\geq 0}(a_i + b_i)x^i, \qquad A(x)B(x) = \sum_{i\geq 0}\left(\sum_{j=0}^{i} a_j b_{i-j}\right)x^i,$$

and a special case of product is the scalar product $c\,A(x) = \sum_{i\geq 0}(c\,a_i)x^i$, for a complex number c. We write $A(0) = a_0$, and unless A is a polynomial, this is the only evaluation we allow. If $b_0 = 0$, then we define the composition

$$A(B(x)) = \sum_{i\geq 0} a_i B(x)^i = \sum_{n\geq 0} \sum_{\substack{i\geq 0, j_1,\ldots,j_i\geq 1 \\ j_1+\ldots+j_i=n}} a_i b_{j_1}\ldots b_{j_i} x^n,$$

and note that the summations above are finite. Note that we only allow substitutions of this type - where we substitute a constant-free $B(x)$ for x.

Now suppose $A(0) = 1$. Then if $B(x)$ is a *multiplicative inverse* of $A(x)$, we have (since multiplication of complex numbers is commutative, so is multiplication of $A(x)$ with $B(x)$, so there is no difference between a left-inverse and a right-inverse) $\sum_{i\geq 0} a_i x^i \sum_{j\geq 0} b_j x^j = 1$, and equating coefficients of x^n on both sides, for $n \geq 0$, we obtain

$$\begin{aligned} b_0 &= 1 \\ a_1 b_0 + b_1 &= 0 \\ a_2 b_0 + a_1 b_1 + b_2 &= 0, \end{aligned}$$

where the nth equation is $a_n b_0 + a_{n-1} b_1 + \ldots + b_0 = 0$, $n \geq 1$. But this gives $b_0 = 1$, and allows us to determine b_n uniquely in terms of $b_0, \ldots, b_{n-1}$, for each $n \geq 1$, so, by induction on n, $B(x)$ is unique. Applying this process to obtain the multiplicative inverse of $A(x) = 1 - x$, we obtain $b_n = 1$, $n \geq 0$, by induction on n, or $(1-x)^{-1} = \sum_{i\geq 0} x^i$. But substitution into this, for an arbitrary $A(x)$ with $A(0) = 1$, gives

$$A(x)^{-1} = (1 - (1 - A(x)))^{-1} = 1 + \sum_{i\geq 1} (1 - A(x))^i,$$

which is therefore the *unique* multiplicative inverse of $A(x)$.

We define *differentiation* and *integration* operators by

$$\frac{d}{dx}A(x) = \sum_{i\geq 1} i a_i x^{i-1}, \qquad I_x\, A(x) = \sum_{i\geq 0} \frac{a_i}{i+1} x^{i+1}.$$

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Now note that we have uniqueness for solution of differential equations: if $\frac{d}{dx}A(x) = \frac{d}{dx}B(x)$ and $A(0) = B(0)$, then $A(x) = B(x)$. Now

$$\frac{d}{dx}(A(x) + B(x)) = \sum_{i\geq 1} i(a_i + b_i)x^{i-1} = \sum_{i\geq 1} ia_i x^{i-1} + \sum_{i\geq 1} ib_i x^{i-1} = \frac{d}{dx}A(x) + \frac{d}{dx}B(x),$$

so the differentiation satisfies the sum rule, and

$$\begin{aligned}\frac{d}{dx}(A(x)B(x)) &= \sum_{i\geq 1}\sum_{j=0}^{i} ia_j b_{i-j} x^{i-1} \\ &= \sum_{i\geq 1}\sum_{j=0}^{i}(j + i - j)a_j b_{i-j} x^{i-1} \\ &= \left(\frac{d}{dx}A(x)\right) B(x) + A(x)\left(\frac{d}{dx}B(x)\right),\end{aligned}$$

and differentiation satisfies the product rule. Induction on n then gives $\frac{d}{dx}B(x)^n = nB(x)^{n-1}\frac{d}{dx}B(x)$ for positive integers n, which allows us to prove the *chain rule*:

$$\frac{d}{dx}A(B(x)) = A'(B(x))\frac{d}{dx}B(x).$$

To differentiate $A(x)^{-1}$, where $A(0) = 1$, we apply the product rule to $A(x) \cdot A(x)^{-1} = 1$, to obtain

$$A(x)^{-1}\frac{d}{dx}A(x) + A(x)\frac{d}{dx}A(x)^{-1} = 0, \tag{1}$$

and thus conclude that

$$\frac{d}{dx}A(x)^{-1} = -A(x)^{-2}\frac{d}{dx}A(x).$$

We now define three special series

$$\varepsilon(x) = \sum_{n\geq 0}\frac{1}{n!}x^n, \quad \lambda(x) = \sum_{n\geq 1}\frac{1}{n}x^n, \quad B_a(x) = \sum_{n\geq 0}\frac{a(a-1)\ldots(a-n+1)}{n!}x^n, \tag{2}$$

where a is a complex number parameter in $B_a(x)$. Our object is to show that $\varepsilon(x), \lambda(x), B_a(x)$ have the properties of the familiar functions $e^x, \ln(1-x)^{-1}, (1+x)^a$, respectively. (Except that we will NOT be able to consider, for example, $\varepsilon(\varepsilon(x))$, since it uses composition with a series with constant term 1.) First, note that $\frac{d}{dx}\varepsilon(x) = \varepsilon(x)$. Then, for example, we can prove that $\varepsilon(x)\varepsilon(-x) = 1$, since $\varepsilon(x)\varepsilon(-x)$ has constant term $\varepsilon(0)\varepsilon(-0) = 1$, and

$$\frac{d}{dx}(\varepsilon(x)\varepsilon(-x)) = \varepsilon(x)\varepsilon(-x) - \varepsilon(x)\varepsilon(-x) = 0,$$

where we have used the product rule and chain rule. The result follows by the uniqueness of solution of differential equations (since 1 also has constant term 1 and derivative 0). Also, we have $\frac{d}{dx}\lambda(x) = \sum_{n\geq 0} x^n = (1-x)^{-1}$, so

$$\frac{d}{dx}\left(\lambda(1-\varepsilon(-x))\right) = \left(\varepsilon(-x)\right)^{-1}\varepsilon(-x) = 1,$$

by the chain rule, and $\lambda(1-\varepsilon(-0)) = 0$, and we conclude that $\lambda(1-\varepsilon(-x)) = x$, by uniqueness of solution of differential equations. (The series $1-\varepsilon(-x)$ has constant term 0, so the composition $\lambda(1-\varepsilon(-x))$ is valid.) Similarly, we prove that $\varepsilon(\lambda(x)) = (1-x)^{-1}$, using $\varepsilon(\lambda(0)) = 1$, and

$$\frac{d}{dx}\left((1-x)\varepsilon(\lambda(x))\right) = -\varepsilon(\lambda(x)) + (1-x)\varepsilon(\lambda(x))(1-x)^{-1} = 0,$$

using the product rule and chain rule. Also, if $A(0) = 1$, the chain rule together with (1) gives

$$\frac{d}{dx}\lambda(1-A(x)^{-1}) = A(x)\frac{d}{dx}\left(1-A(x)^{-1}\right) = A(x)^{-1}\frac{d}{dx}A(x). \tag{3}$$

For the series $B_a(x)$, we have $\frac{d}{dx}B_a(x) = a\,B_{a-1}(x)$, and we omit further details of these computations. In summary, we have demonstrated above that the series $\varepsilon(x), \lambda(x), B_a(x)$ defined in (2) have most of the properties of the familiar functions $e^x, \ln(1-x)^{-1}, (1+x)^a$, respectively, and we shall write

$$e^x = \sum_{n\geq 0}\frac{1}{n!}x^n, \quad \ln(1-x)^{-1} = \sum_{n\geq 1}\frac{1}{n}x^n, \quad (1+x)^a = \sum_{n\geq 0}\frac{a(a-1)\ldots(a-n+1)}{n!}x^n, \tag{4}$$

where, as usual, the only substitutions that we allow for x are constant-free formal power series.

For example, in terms of these familiar functions, (3) becomes the familiar logarithmic differentiation rule

$$\frac{d}{dx}\ln A(x) = A(x)^{-1}\frac{d}{dx}A(x),$$

where $A(0) = 1$. As another example, consider the problem of finding the nth root of a formal power series: begin by noting that it is easy to verify that there are no zero divisors for formal power series, and this fact allows us to establish that nth roots are unique, at least with given constant term, as follows. Suppose $A(0) = B(0) = 1$, and $A(x)^n = B(x)^n$, for some positive integer n. Then we have

$$0 = A(x)^n - B(x)^n = (A(x)-B(x))(A(x)^{n-1} + A(x)^{n-2}B(x) + \ldots + B(x)^{n-1}).$$

Now the constant term in the second factor is $A(0)^{n-1} + A(0)^{n-2}B(0) + \ldots + B(0)^{n-1} = n \neq 0$, so we conclude that $A(x) - B(x) = 0$, since there are no zero divisors, which gives $A(x) = B(x)$, as required. But we can determine the nth root of $A(x)$ with $A(0) = 1$ by substitution in the binomial series $B_{\frac{1}{n}}(x) = (1+x)^{\frac{1}{n}}$, to obtain

$$A(x)^{\frac{1}{n}} = (1 + (A(x)-1))^{\frac{1}{n}} = 1 + \sum_{i\geq 1}\frac{\frac{1}{n}(\frac{1}{n}-1)\ldots(\frac{1}{n}-i+1)}{i!}(A(x)-1)^i,$$

which is therefore the *unique* nth root with constant term 1.

We introduce trigonometric series by defining

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots,$$

and then proving the properties of these series from properties of the series $\varepsilon(x) = e^x$, by

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2},$$

so, for example, we have

$$\begin{aligned}\sin(x)^2+\cos(x)^2 &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2+\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2\\ &= \tfrac{1}{4}\left(-(e^{2ix}-2+e^{-2ix})+(e^{2ix}+2+e^{-2ix})\right)=1.\end{aligned}$$

Then, noting that $\cos(x)$ has constant term 1, so it is invertible, we define

$$\tan(x)=\frac{\sin(x)}{\cos(x)}=x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots=x+\frac{2x^3}{3!}+\frac{16x^5}{5!}+\ldots,$$

$$\sec(x)=\frac{1}{\cos(x)}=1+\frac{x^2}{2}+\frac{5x^4}{24}+\ldots=1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\ldots.$$

Various special functions are of interest combinatorially. For example, the coefficients in these formal power series for *sec* and *tan*, scaled by the factorial as in the above expressions, give the number of permutations of a special kind, called *alternating* permutations.

In a similar way, consider *formal Laurent series*, given by $A(x)=\sum_{i\geq -k}a_ix^i$, for some finite negative integer $-k$. We proceed as for formal power series above, with little modification required. To handle multiplicative inverses, suppose that $A(x)=a_{-k}x^{-k}B(x)$, where a_{-k} is invertible, and $B(x)$ is a formal power series with $B(0)=1$. Then define

$$A(x)^{-1}=a_{-k}^{-1}x^kB(x)^{-1}.$$

We define differentiation in the same way as for formal power series, by $\frac{d}{dx}A(x)=\sum_{i\geq -k}ia_ix^{i-1}$. The *formal residue* of a formal Laurent series is given by the coefficient of x^{-1}. This has a nice invariance property, given in the following result.

Theorem 1.1. *Suppose that $A(x)$ is a formal Laurent series, and that $B(x)=\sum_{i\geq m}b_ix^i$ is a formal power series with m a positive integer, and b_m invertible. Then*

$$[x^{-1}]A(x)=\frac{1}{m}[x^{-1}]A(B(x))\frac{d}{dx}B(x).$$

(Note, that for formal Laurent series $A(x)$, the composition $A(B(x))$ is well-formed only when $B(x)$ is a formal power series with constant term 0.)

Proof. Let $A(x)=\sum_{n\geq -k}a_nx^n$, so on the right hand side of the result we have

$$\frac{1}{m}[x^{-1}]\sum_{n\geq -k}a_nB(x)^n\frac{d}{dx}B(x).$$

For $n\geq 0$, note that $B(x)^nB'(x)$ is a formal power series in x, so we have

$$\frac{1}{m}[x^{-1}]B(x)^nB'(x)=0$$

in this case.

For $-k<n<-1$, we have

$$\frac{1}{m}[x^{-1}]B(x)^nB'(x)=\frac{1}{m}[x^{-1}]\frac{1}{n+1}\left(B(x)^{n+1}\right)'=0,$$

since $[x^{-1}]f'(x)=0$ for any formal Laurent series $f(x)$.

Finally, for $n=-1$, let $B(x)=b_m x^m H(x)$, so $H(x)$ is a formal power series with $H(0)=1$. Then we have

$$\begin{aligned}\frac{1}{m}[x^{-1}]B(x)^n B'(x) &= \frac{1}{m}[x^{-1}]\frac{b_m x_m H'(x)+mb_m x^{m-1}H(x)}{b_m x^m H(x)}\\ &= \frac{1}{m}[x^{-1}]\left(\frac{H'(x)}{H(x)}+mx^{-1}H(x)\right)\\ &= \frac{1}{m}[x^{-1}]\left((\ln H(x))'+mx^{-1}H(x)\right)\\ &= \frac{1}{m}(0+m)=1.\end{aligned}$$

The result follows by combining the results for these cases. □

Before we apply Theorem 1.1 to prove Lagrange's Theorem, we prove that the formal power series $F(x)=\sum_{i\geq 1} f_i x^i$ has a unique compositional inverse whenever f_1 is invertible. Suppose the compositional inverse is given by $G(x)=\sum_{i\geq 1} g_i x^i$. Then equate coefficients of x^n in $F(G(x))=x$, to obtain the equation, $n\geq 1$,

$$\sum_{i=1}^{n} f_i \sum_{j_1+\ldots+j_i=n} g_{j_1}\cdots g_{j_i}=\delta_{n,1},$$

(where, on the right hand side, we obtain 1 if $n=1$ and 0 otherwise.) When $n=1$, this gives $g_1=f_1^{-1}$, and when $n>1$, this gives

$$g_n=-f_1^{-1}\sum_{i=2}^{n} f_i \sum_{j_1+\ldots+j_i=n} g_{j_1}\cdots g_{j_i},$$

which allows us to recursively obtain each coefficient g_n finitely in terms of $f_1,\ldots,f_n$.

Lagrange's Implicit Function Theorem. We have the functional equation $w=t\phi(w)$, for some formal power series ϕ with an invertible constant term. Rewrite this functional equation as $\Phi(w)=t$, where $\Phi(w)=w/\phi(w)$, and let Ψ be the compositional inverse of Φ, so we have $w=\Psi(t)$.

Now consider any formal Laurent series f. Then we have, for $n\neq 0$,

$$[t^n]f(w)=[t^{-1}]t^{-n-1}f(w)=[t^{-1}]t^{-n-1}f(\Psi(t)),$$

and we can apply Theorem 1.1, with $B=\Phi$ (for which we have $m=1$), to obtain

$$[t^n]f(w)=[w^{-1}]\Phi(w)^{-n-1}f(w)\Phi'(w)=-\frac{1}{n}[w^{-1}]f(w)\left(\Phi^{-n}(w)\right)'.$$

But $[w^{-1}]f(w)g'(w)=-[w^{-1}]f'(w)g(w)$ for any formal Laurent series f,g, since

$$[w^{-1}]\left(f(w)g(w)\right)'=0,$$

so we have

$$(5)\qquad [t^n]f(w)=\frac{1}{n}[w^{-1}]f'(w)\Phi^{-n}(w)=\frac{1}{n}[w^{-1}]f'(w)\phi^n(w)w^{-n}=\frac{1}{n}[w^{n-1}]f'(w)\phi^n(w),$$

for $n\geq 1$, where $w=t\phi(w)$, which is the usual statement of Lagrange's Implicit Function Theorem. For an analytic treatment of Lagrange's Theorem, see Section 5.1 of Wilf's book *Generatingfunctionology*.

2. Lecture of May 1

We now turn to *symmetric functions*. For fixed positive integer n, we consider formal power series in $x_1, \ldots, x_n$. A formal power series $A(x_1, \ldots, x_n)$ is *symmetric* if $A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = A(x_1, \ldots, x_n)$ for all permutations σ of $\{1, \ldots, n\}$ (so the word "symmetric" here refers to the symmetric group $S(n)$, which consists of all permutations of $\{1, \ldots, n\}$).

We begin with the simplest example. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, so λ is a *partition* with at most n parts (the number of parts in λ is given by the number of positive λ_i's). Then the *monomial* symmetric function $m_\lambda = m_\lambda(x_1, \ldots, x_n)$ is defined to be the

$$m_\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} + \ldots, \tag{6}$$

in which we add the minimal set of monomials that, together with the monomial $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$, creates a symmetric function. In general, m_λ is a sum of $\frac{n!}{|\mathrm{aut}\lambda|}$ distinct monomials, where $|\mathrm{aut}\lambda|$ is the number of permutations that fix the n-tuple $(\lambda_1, \ldots, \lambda_n)$. For example, we have $m_{(2,2,2)}(x_1, x_2, x_3) = x_1^2 x_2^2 x_3^2$, and

$$m_{(3,2,0)}(x_1, x_2, x_3) = x_1^3 x_2^2 + x_1^3 x_3^2 + x_1^2 x_2^3 + x_2^3 x_3^2 + x_1^2 x_3^3 + x_2^2 x_3^3.$$

Second, the *elementary* symmetric functions $e_k = e_k(x_1, \ldots, x_n)$, $k \geq 0$, are defined by $e_k = m_{(1,\ldots,1,0,\ldots,0)}$, where $(1, \ldots, 1, 0, \ldots, 0)$ has k 1's and $n-k$ 0's. Equivalently, using a generating function, we have

$$\sum_{k \geq 0} e_k t^k = \prod_{j=1}^{n} (1 + x_j t). \tag{7}$$

Thus e_k is the sum of $\binom{n}{k}$ monomials with exponents 0 or 1, each of which corresponds to a k-subset of $\{1 \ldots, n\}$, the k-subset specifying which of the indeterminates have exponent 1.

Third, the *homogeneous* (or *complete*) symmetric functions $h_k = h_k(x_1, \ldots, x_n)$, $k \geq 0$, are defined by

$$h_k = \sum m_\lambda,$$

where the summation is over all $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 + \cdots + \lambda_n = k$. But this means that h_k is the sum of all monomials in $x_1, \ldots, x_n$ of (total) degree k, so, equivalently, using a generating function, we have

$$\sum_{k \geq 0} h_k t^k = \prod_{j=1}^{n} (1 - x_j t)^{-1} = \prod_{j=1}^{n} (1 + x_j t + x_j^2 t^2 + \ldots). \tag{8}$$

Thus h_k is the sum of all of the $\binom{n+k-1}{k}$ monomials with non-negative exponents totalling k.

Fourth, the *power sum* symmetric functions $p_k = p_k(x_1, \ldots, x_n)$, $k \geq 0$, are defined by $p_k = m_{(k,0,\ldots,0)}$, so we have $p_0 = 1$ and

$$p_k = \sum_{j=1}^{n} x_j^k, \quad k \geq 1. \tag{9}$$

In terms of generating functions, we obtain

$$\sum_{k \geq 1} p_k \frac{t^k}{k} = \sum_{k \geq 1} \sum_{j=1}^{n} x_j^k \frac{t^k}{k} = \sum_{j=1}^{n} \sum_{k \geq 1} x_j^k \frac{t^k}{k} = \sum_{j=1}^{n} \ln(1 - x_j t)^{-1} = \ln \sum_{i \geq 0} h_i t^i.$$

As a fifth example, we now consider a less familiar example of symmetric functions, the Schur functions. We shall make extensive use of *determinants* in this presentation, for which it will be useful to recall the definition as a summation over the symmetric group:

$$\det\left(a_{i,j}\right)_{i,j=1,\ldots,n} = \sum_{\rho \in S(n)} \operatorname{sgn}(\rho) \prod_{i=1}^{n} a_{i,\rho(i)}.$$

Then for $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, the *Schur* symmetric function $s_\lambda = s_\lambda(x_1, \ldots, x_n)$ is defined by

$$s_\lambda = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1,\ldots,n}}{\det\left(x_i^{n-j}\right)_{i,j=1,\ldots,n}}. \tag{10}$$

This is symmetric because

$$s_\lambda(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \frac{\operatorname{sgn}(\sigma)}{\operatorname{sgn}(\sigma)} s_\lambda(x_1, \ldots, x_n),$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ (since in both the numerator and denominator determinants of s_λ, to evaluate at $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$, we simply permute the rows by σ).

The denominator determinant in (10) is well-known, as the *Vandermonde* determinant. Call it $V(x_1, \ldots, x_n)$. Note that V is a polynomial in $x_1, \ldots, x_n$, and has value 0 if $x_k = x_\ell$ for any $k \neq \ell$ (since this makes rows k and ℓ identical). Therefore V is divisible by

$$\prod_{1 \leq k < \ell \leq n} (x_k - x_\ell). \tag{11}$$

But V is homogeneous of total degree $(n-1)+\ldots+1+0 = \binom{n}{2}$, and (11) is also homogeneous of total degree $\binom{n}{2}$, so we conclude that

$$V(x_1, \ldots, x_n) = c \cdot \prod_{1 \leq k < \ell \leq n} (x_k - x_\ell),$$

for some constant c. Now, the monomial $x_1^{n-1} \cdots x_{n-1}^1 x_n^0$ appears in V with coefficient 1, since it is produced uniquely in the expansion of the determinant as the product of the entries on the main diagonal (for which ρ is the identity permutation, with sgn $+1$); this monomial also appears in the expansion of (11) with coefficient 1, since it is produced uniquely in the expansion of the product by selecting the x_k from each $x_k - x_\ell$. This implies that $c = 1$, giving

$$V(x_1, \ldots, x_n) = \prod_{1 \leq k < \ell \leq n} (x_k - x_\ell). \tag{12}$$

Now the same argument as we used for the denominator proves that the numerator determinant in (10) is a polynomial divisible by (11), which implies that s_λ is a symmetric *polynomial* in $x_1, \ldots, x_n$ (not just a symmetric rational function), since the denominator in (10) perfectly divides the numerator. Moreover, since the numerator of (10) is a homogeneous polynomial of total degree $\lambda_1 + \ldots + \lambda_n + \binom{n}{2}$, we conclude that the polynomial s_λ itself is homogeneous, of total degree $\lambda_1 + \ldots + \lambda_n$.

We now give a second, combinatorial definition of the polynomial s_λ (and will then prove that these definitions are consistent). The *Ferrers graph* of the partition λ is an array of boxes (called *cells*), with λ_i cells in row i (indexed from the top), for each positive λ_i, aligned at the left. For example, the Ferrers graph of $(5, 3, 2)$ is given on the left hand side of Figure 1.

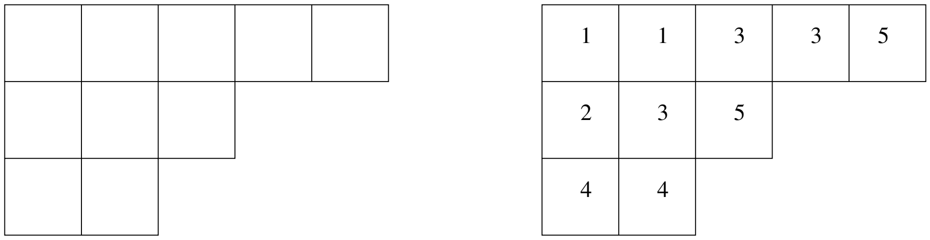

FIGURE 1. A Ferrers graph and tableau.

A *tableau* of *shape* λ is obtained by placing positive integers (here the positive integers are between 1 and n) into the cells of the Ferrers graph of λ (one in each cell), subject to the condition that the integers weakly increase from left to right across each row, and strictly increase from top to bottom down each column. An example of a tableau of shape $(5, 3, 2)$ is given on the right hand side of Figure 1.

Our combinatorial definition of $s_\lambda(x_1, \ldots, x_n)$ is now given by

$$s_\lambda = \sum_T \mathrm{wt}\,(T), \qquad \mathrm{wt}\,(T) = \prod_{j=1}^{n} x_j^{n_j(T)}, \tag{13}$$

where the summation is over all tableaux T of shape λ, and $n_j(T)$ is the number of times j appears in a cell of T. For example, the tableau given in Figure 1 contributes the monomial $x_1^2 x_2 x_3^3 x_4^2 x_5^2$ to $s_{(5,3,2)}(x_1, \ldots, x_n)$ (for any $n \geq 5$).

3. LECTURE OF MAY 3

Note that the combinatorial object defined in (13) is a homogeneous polynomial of total degree $\lambda_1 + \ldots \lambda_n$ (where $\lambda = (\lambda_1, \ldots . \lambda_n)$), since each monomial has exponents totalling the number of cells in the Ferrers graph of λ.

It is also straightforward to prove directly that (13) is a symmetric polynomial, as follows. For fixed i, $1 \leq i \leq n-1$, and any tableau T of shape λ, consider the cells that contain i or $i+1$. These cells must appear in T as disjoint collections, each of the form that appears in Figure 2 – horizontal segments containing a weakly increasing sequence of i's and $i+1$'s,

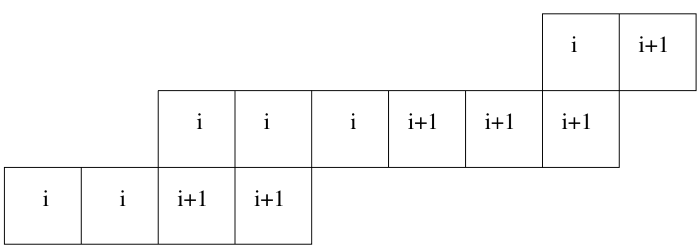

FIGURE 2. Cells containing i or $i+1$ in a tableau.

with vertical pairs containing one i and one $i+1$ in between. Define a mapping ϕ_i on the set of such tableaux, whose action on T is to replace each horizontal segment with j i's followed by k $i+1$'s, by a horizontal segment with k i's followed by j $i+1$'s. (For example, there are three horizontal segments in Figure 2 – the first from the left would be replaced with two $i+1$'s, the middle by two i's and one $i+1$, and the third by a single i.) Clearly ϕ_i is an involution (which means that $\phi_i(\phi_i(T)) = T$), whose fixed points have equal numbers of i's and $i+1$'s in all horizontal segments. Thus ϕ_i is a bijection on the set of tableaux that interchanges the number of i's and $i+1$'s, and hence (13) is invariant under the adjacent transposition $(i, i+1)$ (note that the fixed points of ϕ_i are themselves symmetric under the action of $(i, i+1)$). But every permutation can be written as a product of adjacent transpositions, and we conclude that (13) is a symmetric function.

Now we turn to proving that these two definitions of the Schur function really are identical. The first step is to consider tableaux from a geometric point of view, as a collection of *lattice paths*, whose unit steps that are either horizontal (from left to right) or vertical (from bottom to top). In particular, for $m \le n$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$, with $\lambda_1 \ge \ldots \ge \lambda_m \ge 1$, for $j = 1, \ldots, m$, we construct path π_j, from starting point $P_j = (m-j, 1)$ to ending point $Q_j = (m-j+\lambda_j, n)$, to correspond to tableau T of shape λ, in the following way. There is a horizontal step in π_j at height k for each cell containing a k in row j of T. The vertical steps are filled in uniquely. For example, the three paths corresponding to the tableau given in Figure 1 are illustrated in Figure 3. Now, for each fixed λ, let

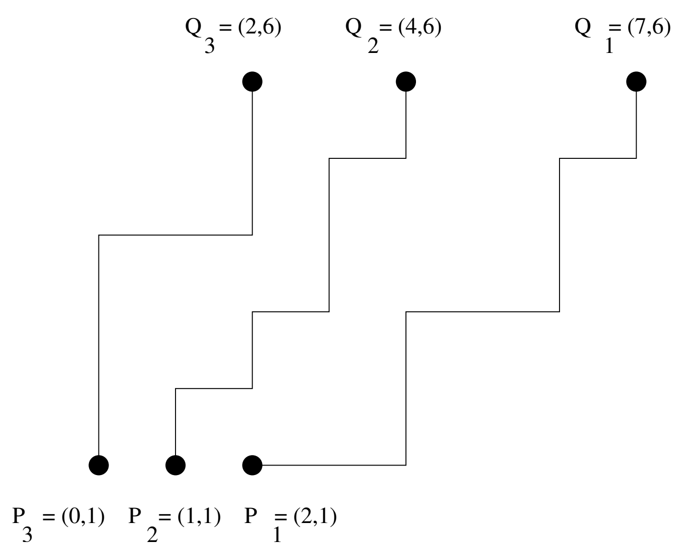

FIGURE 3. Three paths corresponding to a tableau.

$$\mathcal{S} = \{(\sigma, \pi_1 \ldots, \pi_m) : \sigma \in S(m), \pi_i \text{ a lattice path from } P_i \text{ to } Q_{\sigma(i)}, i = 1, \ldots, m\},$$

$$\Phi_{\mathcal{S}} = \sum_{(\sigma, \pi_1 \ldots, \pi_m) \in \mathcal{S}} \mathrm{W}(\sigma, \pi_1 \ldots, \pi_m),$$

where $S(m)$ is the symmetric group on $\{1, \ldots, m\}$, and we further define

$$\mathrm{W}(\sigma, \pi_1 \ldots, \pi_m) = \mathrm{sgn}(\sigma) \prod_{i=1}^{m} \prod_{j=1}^{n} x_j^{N_{i,j}},$$

and $N_{i,j}$ is the number of horizontal steps at height j in π_i. But

$$\sum_{\pi_i}\prod_{j=1}^{n} x_j^{N_{i,j}} = \sum_{a_1+\ldots+a_n=\lambda_{\sigma(i)}-\sigma_i+i} x_1^{a_1}\cdots x_n^{a_n} = h_{\lambda_{\sigma(i)}-\sigma(i)+i}(x_1,\ldots,x_n),$$

where the sum is over all paths π_i from P_i to $Q_{\sigma(i)}$, since each such path has $m-\sigma(i)+\lambda_{\sigma(i)}-(m-i)=\lambda_{\sigma(i)}-\sigma(i)+i$ horizontal steps, and exactly one of these paths has a_j at height j for each $a_1,\ldots,a_n\geq 0$ with $a_1+\ldots+a_n=\lambda_{\sigma(i)}-\sigma(i)+i$. Thus we have

$$\Phi_{\mathcal{S}} = \sum_{\sigma\in S_m}\operatorname{sgn}(\sigma)\prod_{i=1}^{m} h_{\lambda_{\sigma(i)}-\sigma(i)+i}(x_1,\ldots,x_n) = \det\left(h_{\lambda_j-j+i}(x_1,\ldots,x_n)\right)_{i,j=1,\ldots,m}. \tag{14}$$

Now partition the set $\mathcal{S}$ into two subsets – $\mathcal{S}_{\text{int}}$, in which some pair of the paths $\pi_1,\ldots,\pi_m$ intersects, and $\mathcal{S}_{\text{nonint}}$, in which no pair of the paths $\pi_1,\ldots,\pi_m$ intersects. We immediately have

$$\Phi_{\mathcal{S}} = \Phi_{\mathcal{S}_{\text{nonint}}} + \Phi_{\mathcal{S}_{\text{int}}}, \tag{15}$$

since this partitions the set $\mathcal{S}$.

Now, if $(\sigma,\pi_1,\ldots,\pi_m)\in\mathcal{S}_{\text{nonint}}$, then σ must be the identity permutation, since otherwise, we must have $\sigma(i)>\sigma(j)$ for some $i<j$, but in this case π_i must intersect π_j (note that P_i is to the right of P_j, but $Q_{\sigma(i)}$ is to the left of $Q_{\sigma(j)}$). Thus we have $\operatorname{sgn}(\sigma)=1$ for all elements of $\mathcal{S}_{\text{nonint}}$, and we'll leave it as an exercise to prove that

$$\Phi_{\mathcal{S}_{\text{nonint}}} = s_\lambda, \tag{16}$$

where s_λ is as defined in (13).

We now prove that $\Phi_{\mathcal{S}_{\text{int}}}=0$, by defining a map $\phi:\mathcal{S}_{\text{int}}\to\mathcal{S}_{\text{int}}$, with $(\sigma,\pi_1,\ldots,\pi_m)\mapsto(\sigma',\pi_1',\ldots,\pi_m')$, given as follows. For $(\sigma,\pi_1,\ldots,\pi_m)\in\mathcal{S}_{\text{int}}$,

(1) Find the minimum $i=1,\ldots,m$ such that π_i intersects π_ℓ for some $\ell\neq i$,
(2) Find the first point R on π_i (*i.e.*, starting at P_i) that is also on π_ℓ for some $\ell\neq i$,
(3) Find the minimum $j=1,\ldots,m$ such that π_i intersects π_j at R.

Then define $\pi_\ell'=\pi_i$ for $\ell\neq i,j$; π_i' consists of the portion of π_i from P_i to R, concatenated with the portion of π_j from R to $Q_{\sigma(j)}$; π_j' consists of the portion of π_j from P_j to R, concatenated with the portion of π_i from R to $Q_{\sigma(i)}$. We'll leave it as an exercise to prove that this mapping is well-defined, and that it is an involution (which means that ϕ^2 is the identity map) and a bijection. Now note that

$$\sigma'=\sigma(i\,j)=(\sigma(i)\,\sigma(j))\sigma,$$

so $\operatorname{sgn}(\sigma')=-\operatorname{sgn}(\sigma)$. But since the paths $\pi_1,\ldots,\pi_m$ and $\pi_1',\ldots,\pi_m'$ have exactly the same (multiset of) steps, and therefore the same set of horizontal steps, we immediately have $W(\sigma,\pi_1,\ldots,\pi_m)=-W(\sigma',\pi_1',\ldots,\pi_m')$, and this gives

$$\Phi_{\mathcal{S}_{\text{int}}} = \sum_{(\sigma,\pi_1,\ldots,\pi_m)\in\mathcal{S}_{\text{int}}} W(\sigma,\pi_1,\ldots,\pi_m) = -\sum_{(\sigma',\pi_1',\ldots,\pi_m')\in\mathcal{S}_{\text{int}}} W(\sigma',\pi_1',\ldots,\pi_m') = -\Phi_{\mathcal{S}_{\text{int}}},$$

where we have changed the range of summation by applying the bijection ϕ. But this gives $2\Phi_{\mathcal{S}_{\text{int}}}=0$ and we conclude that

$$\Phi_{\mathcal{S}_{\text{int}}} = 0. \tag{17}$$

Combining (14), (15), (16) and (17), we obtain

$$s_\lambda = \det\left(h_{\lambda_j-j+i}(x_1,\ldots,x_n)\right)_{i,j=1,\ldots,m}, \tag{18}$$

where $\lambda = (\lambda_1,\ldots,\lambda_m)$. We interpret $h_k = 0$ when $k < 0$ (combinatorially, in the above path argument, we would have an empty set of paths). The convention that $h_0 = 1$ agrees with our path combinatorics, since it corresponds to a single, vertical path with no horizontal steps. Note that we do not need the restriction that λ_m is positive. The path argument would work as above if $\lambda_m = 0$, and the determinant of the right hand side of (18) would have a last column with 1 on the main diagonal, and 0's above – then a cofactor expansion on the last column gives the same determinant on $m-1$ rows and columns, for the partition $(\lambda_1,\ldots,\lambda_{m-1})$. The determinant in (18) is well-known, and called the *Jacobi-Trudi determinant*.

The cancellation argument given above to establish result (18) can be used more generally to prove that, under reasonably broad conditions, the generating function for m-tuples of lattice paths with no pairwise intersection, where the ith such path starts at point P_i and ends at point Q_i, $i = 1,\ldots,m$, is given by

$$\det\left(G(P_i,Q_j)\right)_{i,j=1,\ldots,m},$$

where $G(P_i,Q_j)$ is the generating function for lattice paths from P_i to Q_j. This has been studied in CO 630 in the Winter term of 2017, and is often called *Gessel-Viennot* methodology (although it was previously known in the probability literature, where it was attributed to *Karlin-McGregor*).

We now prove that the definitions of the Schur function s_λ given in (10) and (13) are consistent, by proving that the Jacobi-Trudi determinant given in (18), when m is replaced by n (we can set $\lambda_i = 0$ for $i = m+1,\ldots,n$ without changing the value of the Jacobi-Trudi determinant, for reasons given following (18)), is equal to the ratio of determinants given in (10). First, define polynomials $e_a^{(i)}$ of degree a in $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n$ by

$$\sum_{a=0}^{n-1} e_a^{(i)} t^a = \prod_{\substack{j=1,\\ j\neq i}}^{n} (1+x_j t), \qquad i = 1,\ldots,n.$$

Then, applying (8), we have

$$\sum_{a=0}^{n-1} e_a^{(i)}(-1)^a t^a \sum_{\ell\geq 0} h_\ell t^\ell = \prod_{\substack{j=1,\\ j\neq i}}^{n}(1-x_j t)\prod_{m=1}^{n}(1-x_m t)^{-1} = (1-x_i t)^{-1} = \sum_{s\geq 0} x_i^s t^s.$$

Now, for any sequence $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and $j = 1,\ldots n$, equate coefficients of t^{λ_j-j+n} on both sides of the above equation, to obtain

$$\sum_{k=1}^{n} e_{n-k}^{(i)}(-1)^{n-k}\, h_{\lambda_j-j+k} = x_i^{\lambda_j-j+n}, \qquad i,j = 1,\ldots,n,$$

(where we have reindexed using summation variable $k = n-a$). Now these n^2 equations, together, can be written as the single matrix equation

$$\left(e_{n-j}^{(i)}(-1)^{n-j}\right)_{i,j=1,\ldots,n}\left(h_{\lambda_j-j+i}\right)_{i,j=1,\ldots,n} = \left(x_i^{\lambda_j-j+n}\right)_{i,j=1,\ldots,n}. \tag{19}$$

If we take determinants of the matrix equation (19) in the case $\lambda_1 = \ldots = \lambda_n = 0$, we obtain

$$\det\left(e^{(i)}_{n-j}(-1)^{n-j}\right)_{i,j=1,\ldots,n} \det\left(h_{j-i}\right)_{i,j=1,\ldots,n} = \det\left(x_i^{n-j}\right)_{i,j=1,\ldots,n}.$$

But $\left(h_{j-i}\right)_{i,j=1,\ldots,n}$ is a matrix with 1's on the main diagonal (since $h_0 = 1$), and 0's below the main diagonal (since $h_\ell = 0$ for any $\ell < 0$), so we have $\det\left(h_{j-i}\right)_{i,j=1,\ldots,n} = 1$, and we conclude that

$$\det\left(e^{(i)}_{n-j}(-1)^{n-j}\right)_{i,j=1,\ldots,n} = \det\left(x_i^{n-j}\right)_{i,j=1,\ldots,n}. \tag{20}$$

Now if we take determinants of the matrix equation (19) for arbitrary λ and apply (20), we obtain

$$\det\left(h_{\lambda_j - j + i}\right)_{i,j=1,\ldots,n} = \frac{\det\left(x_i^{\lambda_j - j + n}\right)_{i,j=1,\ldots,n}}{\det\left(x_i^{n-j}\right)_{i,j=1,\ldots,n}},$$

completing the proof that (10) and (13) are consistent, via (18).

4. Lecture of May 8

A tableau with n cells, in which each symbol $1, \ldots, n$ appears in a cell exactly once, is called a *Young tableau*. There is a famous bijection between the set of permutations of $\{1, \ldots, n\}$ and ordered pairs of Young tableaux of the same shape λ, where λ varies over all paritions of n. This bijection involves the following iterative procedure: consider a tableau T with distinct elements, and a number i not contained in T; To *row-insert* i in T, we *row-merge* i in row 1 of T, by finding the smallest j in row 1 with $j > i$, replacing j by i, and row-merging j in row 2; if there is no such j, add i (in a new cell) at the end of row 1 and STOP. Repeat until STOP (this will STOP at a previously empty row, if not sooner).

Robinson-Schensted Algorithm Consider a permutation σ of $\{1, \ldots, n\}$ and a pair (P_0, Q_0) of empty tableaux (with no cells, corresponding to the empty partition). For i from 1 to n, row-insert $\sigma(i)$ in P_{i-1}, and place i in a new cell added to Q_{i-1}, in the same cell in which P_i differs from P_{i-1}. CLAIM: (P_n, Q_n) is an ordered pair of Young tableaux on elements $\{1, \ldots, n\}$, of the same shape, and this is a bijection.

As an example of the Robinson-Schensted Algorithm with $n = 7$, consider σ specified by $\sigma(1)\ldots\sigma(7) = 4\,2\,3\,6\,5\,1\,7$. Then, by following the algorithm, we obtain:

$$P_1 = \begin{matrix} 4 \end{matrix} \qquad Q_1 = \begin{matrix} 1 \end{matrix}, \qquad P_2 = \begin{matrix} 2 \\ 4 \end{matrix} \qquad Q_2 = \begin{matrix} 1 \\ 2 \end{matrix}$$

$$P_3 = \begin{matrix} 2 & 3 \\ 4 & \end{matrix} \qquad Q_3 = \begin{matrix} 1 & 3 \\ 2 & \end{matrix}$$

$$P_4 = \begin{matrix} 2 & 3 & 6 \\ 4 & & \end{matrix} \qquad Q_4 = \begin{matrix} 1 & 3 & 4 \\ 2 & & \end{matrix}$$

$$P_5 = \begin{matrix} 2 & 3 & 5 \\ 4 & 6 & \end{matrix} \qquad Q_5 = \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \end{matrix}$$

$$P_6 = \begin{matrix} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{matrix} \qquad Q_6 = \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & & \end{matrix}$$

$$P_7 = \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{matrix} \qquad Q_7 = \begin{matrix} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{matrix}$$

In this case, we have mapped σ to the pair (P_7, Q_7) of Young tableaux, both of the same shape $(4, 2, 1)$. As another example, if we apply the Robinson-Schensted Algorithm to the permutation $\pi = 6\,2\,3\,1\,5\,4\,7$, then we obtain the pair (Q_7, P_7) of Young tableaux (i.e., the roles of P and Q Are interchanged). Note that this permutation π is the inverse of the permutation σ above; in general we claim that if a permutation is mapped to the pair (P, Q) of Young tableaux of the same shape, then σ^{-1} is mapped to the pair (Q, P). As two other examples of the Robinson-Schensted Algorithm, consider applying it to the identity permutation $1\,2\,3\,4\,5\,6\,7$, in which case we obtain both P and Q equal to the Young tableau with a single row of 7 cells, with the elements $1, \ldots, 7$ in increasing order along that row, or consider applying it to the reverse permutation $7\,6\,5\,4\,3\,2\,1$, in which case we obtain both P and Q equal to the Young tableau with a single column of 7 cells, with the elements $1, \ldots, 7$ in increasing order down that column.

Now we prove that the Robinson-Schensted Algorithm works, although we will not prove the statement about the inverse and interchanging P and Q. We use the term "bumping" to describe what happens to element j in some row when $i < j$ is inserted in its place.

Lemma 4.1. *Suppose that we row-insert i in T, and the bumped element in row ℓ is b_ℓ, originally in column c_ℓ, $\ell = 1, \ldots, r$ (i.e., there are r "bumps"), and b_r is placed in column c_{r+1} of row $r + 1$. Then $c_1 \geq \ldots \geq c_{r+1}$ (and $i < b_1 < \ldots < b_r$).*

Proof. We have $i < b_1 < \ldots < b_r$ by construction, since a bumped element is larger than the inserted element at each stage. The element in row $\ell + 1$ and column c_ℓ (if it exists) is larger than b_ℓ (since b_ℓ originally occurred in row ℓ and column c_ℓ). Thus b_ℓ cannot be placed to the right of column c_ℓ in row $\ell + 1$, so we have $c_{\ell+1} \leq c_\ell$. □

Corollary 4.2. *Row-inserting i in T creates a tableau.*

Proof. By construction, the rows are always strictly increasing, so we check only the column strictness. Let $b_0 = i$, and let T' be the array created by inserting i in T. Let $t_{\ell,m}$, $t'_{\ell,m}$ be the elements in position ℓ, m (i.e., in row ℓ, column m) of T and T', respectively. Then $T'_{\ell,c_\ell} = b_{\ell-1}$ for $\ell = 1, \ldots r + 1$, and these are the only positions in which T' differs from T. But we have, from Lemma 4.1,

$$t'_{\ell-1,c_\ell} \leq t'_{\ell-1,c_{\ell-1}} = b_{\ell-2} < b_{\ell-1} = t'_{\ell,c_\ell},$$

which gives $t'_{\ell-1,c_\ell} < t'_{\ell,c_\ell}$, for $\ell = 2, \ldots, r + 1$. Also, we have, from Lemma 4.1,

$$t'_{\ell+1,c_\ell} \geq t'_{\ell+1,c_{\ell+1}} = b_\ell > b_{\ell-1} = t'_{\ell,c_\ell},$$

which gives $t'_{\ell+1,c_\ell} > t'_{\ell,c_\ell}$, when there is an element in position $\ell + 1, c_\ell$ of T', for $\ell = 1, \ldots, r$. But, by construction, there is no element in position $r + 2, c_{r+1}$ of T', so we have checked completely that T' satisfies column-strictness. Also, since $c_{r+1} \leq c_r$, then the new cell in T', in position $r + 1, c_{r+1}$, occurs below a existing cell in T, since $c_{r+1} \leq c_r$ (and the cell in position r, c_r is occupied in T, and therefore in T'). Thus T' is a tableau. □

Theorem 4.3. *The Robinson-Schensted Algorithm is a bijection between permutations on* $\{1,\ldots,n\}$ *and ordered pairs of Young tableaux of the same shape, where the shape varies over all partitions of* n.

Proof. In Corollary 4.2, we have established at every stage that P_i is a tableau, for $i = 1,\ldots,n$. As part of that proof, we established that the cell in which P_i differs from P_{i-1} occurs at the end of a row, and directly below another cell, so Q_i is also a tableau at every stage. It is easy to invert this Algorithm, since P_{i-1} can be recovered from P_i by noting that the cell containing i in Q_i is the cell in which P_i differs from P_{i-1}, and then the row-insertion process is easy to invert. This is a bijection because reversing as described above always yields a tableau P_{i-1} (check inequalities), at every stage, and so we obtain a unique permutation σ corresponding to each pair of tableaux (P_n, Q_n). □

We have established that the Robinson-Schensted Algorithm works, but haven't proved more refined properties, which we will prove now (at least sketch how). First, if $\sigma \mapsto (P,Q)$ then $\sigma^{-1} \mapsto (Q,P)$. Second, the length of the longest increasing subsequence in σ is equal to the length of the first row in P (and Q); the length of the longest decreasing subsequence in σ is equal to the length of the first column in P (and Q).

To prove these facts, it is convenient to give a more geometrical (but equivalent) version of the Algorithm. Given a permutation σ of $\{1,\ldots,n\}$, place vertices at positions $(i,\sigma(i))$ in $\mathbb{R}^2$. Let the quarter plane given by the intersection of $x \geq a$ and $y \geq b$ be the *shadow* of vertex (a,b). Find the union of the shadows of the n vertices $(1,\sigma(1)),\ldots,(n,\sigma(n))$, and let the piecewise linear boundary of this union be the first "shadow-line". Remove the vertices on this shadow-line, and repeat until no vertices remain, to obtain a set of shadow-lines, with each vertex on a unique shadow-line, called the "shadow diagram". For example, the shadow-lines corresponding to $\sigma(1)\ldots\sigma(7) = 4\,2\,3\,6\,5\,1\,7$, are pictured in Figure 4 (this is the permutation for which we illustrated the Robinson-Schensted Algorithm on pages 12, 13). Note that the shadow diagram of σ^{-1} is obtained from the shadow diagram of σ by

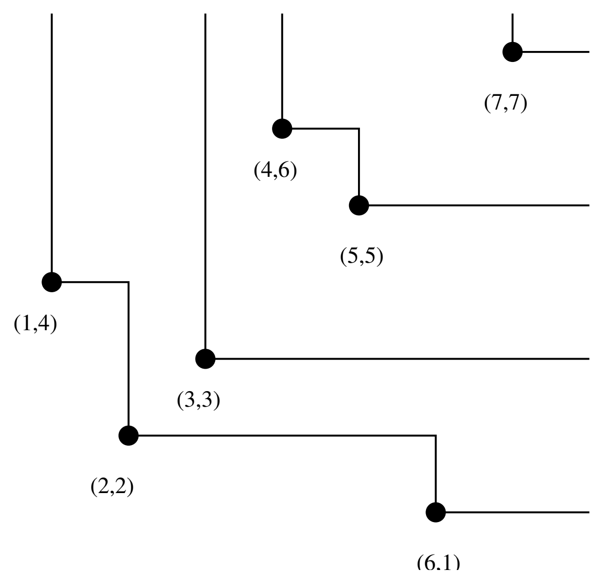

FIGURE 4. A shadow diagram.

reflecting about the line $y = x$. Note also that the entries in the first row of P are precisely the y-coordinates of the rightmost vertices in the shadow-lines (in the example, 1 3 5 7), and the entries in the first row of Q are precisely the x-coordinates of the leftmost vertices in the shadow-lines (in the example, 1 3 4 7). These latter properties are very easy to prove by induction on the stage in the Robinson-Schensted Algorithm: consider the shadow diagram for points $(1, \sigma(1), \ldots, (k, \sigma(k))$ (obtained visually in the above example by removing the half plane $x \geq k + 0.5$ from the figure). Then it is immediate that the Algorithm's rule for how to bump from the first row of P_k is matched precisely by the y-coordinates in the rightmost vertices in the corresponding shadow diagrams (and, trivially, by the x-coordinates of the leftmost vertices in the corresponding shadow diagrams). To understand the remaining rows in P and Q, repeat with the northeast corners of the shadow-lines as a new set of vertices (in Figure 4, these vertices are $(2, 4), (6, 2), (5, 6)$). These give the second rows of P and Q in the same way, and every row can be obtained by iterating this "northeast corner" construction (in Figure 4, another, final iteration yields the point $(6, 4)$, to correspond to the third rows). An intricate, but straightforward induction gives a proof of these facts. The reflection about $y = x$ clearly replaces the permutation σ by the permutation σ^{-1}, but also interchanges the roles of P and Q.

5. Lecture of May 10

If permutation σ maps to a pair of Young tableaux of shape λ, then we have described above how the length of the longest increasing subsequence is equal to λ_1, the length of the first row of the tableaux. How about the length of the other rows ? If s is a subsequence, the let its length be denoted by $|s|$. Then we have the following result of Greene: for each $j \geq 1$,

$$\lambda_1 + \ldots + \lambda_j = \max\{|s_1| + \ldots + |s_j|\},$$

where $s_1, \ldots, s_j$ are disjoint increasing substrings in σ.

For example, if $\sigma = 2\,4\,7\,9\,5\,1\,3\,6\,8$, then applying the Robinson-Schensted Algorithm, we obtain

$$P = \begin{array}{ccccc} 1 & 3 & 5 & 6 & 8 \\ 2 & 4 & 9 & & \\ 7 & & & & \end{array},$$

so here we have $\lambda = (5, 3, 1)$. Now the length of the longest increasing subsequence in σ is indeed $\lambda_1 = 5$ (the unique such subsequence is 2 4 5 6 8), but when this subsequence is omitted, the resulting sequence, 7 9 1 3, has no increasing subsequence of length 3 (the longest has length 2). But the two disjoint increasing subsequences $s_1 = 2\,4\,7\,9$ and $s_2 = 1\,3\,6\,8$, both of length 4, do achieve the value $\lambda_1 + \lambda_2 = 5 + 3 = 8 = 4 + 4$ in Greene's result above.

The enumeration of permutations with given longest increasing subsequence length is carried out by the above bijection with pairs of Young tableaux. For example, the famous result that the average length of longest increasing subsequence in permutations on $\{1, \ldots, n\}$ is asymptotically equal to $2\sqrt{n}$ was obtained by evaluating this average as

$$\frac{1}{n!} \sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2,$$

where f^λ denotes the number of Young tableaux of shape λ, and "$\lambda \vdash n$" means "λ is a partition of n". In the summation, the term $\left(f^\lambda\right)^2$ accounts for the number of pairs (P, Q) of Young tableaux of shape λ, which correspond to a permutation in which the longest increasing subsequence has length λ_1 (the length of the first row of λ). For more information about this problem see the paper: R. P. Stanley, *Increasing and decreasing subsequences and their variants*, Proc. ICM, 2006.

Indeed, even more simply, if our claim that Robinson-Schensted Algorithm is a bijection is true, then we must have

$$\sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2 = n! \tag{21}$$

Also, if the effect of the Robinson-Schensted Algorithm on the inverse of the permutation is to interchange the tableaux, then a permutation which is its own inverse (this is an involution) must be mapped to two copies of the same Young tableau (try the algorithm on such a permutation,, for example, 3 5 1 4 2), so we must have

$$\sum_{\lambda \vdash n} f^{\lambda} = a_n, \tag{22}$$

where a_n is the number of involutions on $\{1, \dots, n\}$. But an involution must have only fixed points and two-cycles in its disjoint cycle representation, so from the exponential generating function methods of previous enumeration courses, we have

$$\begin{aligned}
a_n &= \left[\frac{x^n}{n!}\right] \exp\left(x + \frac{x^2}{2}\right) = \left[\frac{x^n}{n!}\right] \sum_{i \geq 0} \frac{\left(x + \frac{x^2}{2}\right)^i}{i!} \\
&= \left[\frac{x^n}{n!}\right] \sum_{i \geq 0} \frac{x^i}{i!} \sum_{j \geq 0} \binom{i}{j} \left(\frac{x}{2}\right)^j \\
&= \left[\frac{x^n}{n!}\right] \sum_{i \geq 0} \sum_{j \geq 0} \frac{1}{j!(i-j)!2^j} x^{i+j} \\
&= n! \sum_{\substack{i,j \geq 0 \\ i+j=n}} \frac{1}{j!(i-j)!2^j} \\
&= \sum_{j \geq 0} \frac{n!}{j!(n-2j)!2^j},
\end{aligned}$$

where we have set $i = n - j$ to give the final summation formula, which gives, for example, $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 10$.

If we consider the partitions of 4, then it is easy to check that the numbers of Young tableaux are given by $f^{(4)} = f^{(1,1,1,1)} = 1$, $f^{(3,1)} = f^{(2,1,1)} = 3$, and $f^{(2,2)} = 2$. Then we get

$$\sum_{\lambda \vdash 4} \left(f^{\lambda}\right)^2 = 1^2 + 1^2 + 3^2 + 3^2 + 2^2 = 24 = 4!,$$

which confirms (21) for $n = 4$. We also have

$$\sum_{\lambda \vdash n} f^{\lambda} = 1 + 1 + 3 + 3 + 2 = 10 = a_4,$$

which confirms (22) for $n = 4$.

Now we will find a formula for f^{λ}, the number of Young tableaux of shape λ for an arbitrary partition λ. We begin by noting that, from (13) and (18), we have

$$f^{\lambda} = [x_1 \cdots x_n] \det\left(h_{\lambda_j - j + i}\right)_{i,j=1,\dots,m}, \tag{23}$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition of n with $m \le n$ parts. We now introduce some notation that will be useful in determining f^λ from (23). For polynomials F_1, F_2 in $x_1, \ldots, x_n$, we say that $F_1 \equiv F_2$ if every monomial in $F_1 - F_2$ has an exponent on at least one of $x_1, \ldots, x_n$ that is equal to 2 or more. Then we immediately have

$$h_k \equiv e_k \equiv \frac{p_1^k}{k!}, \qquad k \ge 0,$$

where we have used the multinomial theorem for the last $\equiv$. But if $F_1 \equiv F_2$ and $G_1 \equiv G_2$, we have $F_1 \cdot G_1 \equiv F_2 \cdot G_2$, and $[x_1 \cdots x_n]F_1 = [x_1 \cdots x_n]F_2$, so from (23), we obtain

$$\begin{aligned}
f^\lambda &= [x_1 \cdots x_n] \det \left(\frac{p_1^{\lambda_j - j + i}}{(\lambda_j - j + i)!} \right)_{i,j=1,\ldots,m} \\
&= [x_1 \cdots x_n] p_1^n \det \left(\frac{1}{(\lambda_j - j + i)!} \right)_{i,j=1,\ldots,m} \\
&= n! \det \left(\frac{1}{(\lambda_j - j + i)!} \right)_{i,j=1,\ldots,m} \\
&= \frac{n!}{\prod_{\ell=1}^m (\lambda_\ell - \ell + m)!} \det \left((\lambda_j - j + m)_{m-i} \right)_{i,j=1,\ldots,m}
\end{aligned}$$

where, for the second equality, we have factored $p_1^{\lambda_j - j}$ out of column j, $j = 1, \ldots, m$, and factored p_1^i out of row i, $i = 1, \ldots, m$, and then used the fact that $\lambda_1 + \ldots + \lambda_m = n$. For the third equality we have used the multinomial theorem, and for the fourth equality we have factored $((\lambda_j - j + m)!)^{-1}$ out of column j, $j = 1, \ldots, m$. In the last determinant, we are using the falling factorial notation $(x)_k = x(x-1)\cdots(x-k+1)$ for $k \ge 1$, and $(x)_0 = 1$. To evaluate this last determinant, we use the following lemma. (Note that $(x)_k$ is a *monic* polynomial of degree k in x – which means that the coefficient of the monomial of highest degree is equal to 1.)

Lemma 5.1. *If $P_k(x)$ is a monic polynomial of degree k, $k \ge 0$ (and $P_k(x) = 0$ for $k < 0$), then*

$$\det \left(P_{m-j}(x_i) \right)_{i,j=1,\ldots,m} = \prod_{1 \le i < j \le m} (x_i - x_j).$$

Proof. Let $P_k(x) = \sum_{j=0}^k P_{k,j} x^j$, $k \ge 0$, so $P_{k,k} = 1$, and let $P_{k,j} = 0$ for $j > k$ or $k < 0$. Then we have

$$\begin{aligned}
\left(P_{m-j}(x_i) \right)_{i,j=1,\ldots,m} &= \left(\sum_{\ell=1}^m P_{m-j,m-\ell} x_i^{m-\ell} \right)_{i,j=1,\ldots,m} \\
&= \left(x_i^{m-j} \right)_{i,j=1,\ldots,m} \left(P_{m-j,m-i} \right)_{i,j=1,\ldots,m}.
\end{aligned}$$

But $\left(P_{m-j,m-i} \right)_{i,j=1,\ldots,m}$ has 0's above the diagonal, and 1's on the diagonal, so it has determinant equal to 1. The result follows from the Vandermonde determinant evaluation (12). ☐

Applying Lemma 5.1 with $x_j = \lambda_j - j + m$, we have

$$\begin{aligned} f^\lambda &= \frac{n!}{\prod_{\ell=1}^m (\lambda_\ell - \ell + m)!} \det\left((x_j)_{m-i}\right)_{i,j=1,\ldots,m} \\ &= \frac{n!}{\prod_{\ell=1}^m (\lambda_\ell - \ell + m)!} \det\left((x_i)_{m-j}\right)_{i,j=1,\ldots,m} \\ &= \frac{n!}{\prod_{\ell=1}^m (\lambda_\ell - \ell + m)!} \prod_{1 \le i < j \le m} (\lambda_i - i + m - \lambda_j + j - m), \end{aligned}$$

and hence we obtain

$$f^\lambda = \frac{n!}{\prod_{\ell=1}^m (\lambda_\ell - \ell + m)!} \prod_{1 \le i < j \le m} (\lambda_i - i - \lambda_j + j), \tag{24}$$

which is often called the *degree* formula for f^λ. For example, if $\lambda = (3, 2)$, then (24) gives

$$f^{(3,2)} = \frac{5!}{4!2!} \cdot 2 = 5,$$

and it is easy to check that the number of Young tableaux of shape $(3, 2)$ is indeed 5.

Another famous formula for f^λ is called the *hook* formula, which states that

$$f^\lambda = \frac{n!}{\prod_{x \in \lambda} h(x)}, \tag{25}$$

where $x \in \lambda$ means that x is a cell in the Ferrers graph of λ. The value of $h(x)$, often called the "hook-length" for cell x, is the number of cells strictly to the right of and in the same row as x, plus the number of cells strictly below and in the same column as x, plus 1 (for the cell x itself). The term "hook-length" is used because $h(x)$ is the number of cells in the subarray (shaped like a "hook") of the Ferrers graph consisting of cell x together with those to the right of x and those below x. For example, when $\lambda = (3, 2)$, the hook formula (25) gives

$$f^{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5,$$

in agreement with the degree formula in this case.

6. Lecture of May 15

We now consider symmetric functions in a countable set of indeterminates $x_1, x_2, \ldots$. We begin with the monomial symmetric functions m_λ, defined as for the finite variable case by by

$$m_\lambda = \sum x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(\ell)}^{\lambda_\ell},$$

summed over all *distinct* monomials, where σ is a permutation of the positive integers, and λ is a partition with parts $\lambda_1 \ge \ldots \ge \lambda_\ell$. Note that ℓ, the number of parts of the partition λ, is finite, and λ_1, the largest part of λ, is finite, though both ℓ and λ_1 can be arbitrarily large. If λ is a partition of k (i.e., $\lambda_1 + \ldots + \lambda_\ell = k$), then m_λ is a symmetric function of *degree* k. We define Λ^k, symmetric functions of degree k, to be the vector space over the integers with basis $\{m_\lambda : \lambda \vdash k\}$. Of course, independently of this basis, we can recognize that a symmetric function $f(x_1, x_2, \ldots)$ has degree k exactly when $f(tx_1, tx_2, \ldots) = t^k f(x_1, x_2, \ldots)$.

We now consider Schur functions $s_\lambda(x_1, x_2, \ldots)$ in the countable set of indeterminates $x_1, x_2, \ldots$, by extending the tableau definition (13). Note that this is symmetric in $x_1, x_2, \ldots$, by the argument on pages 8 and 9.

For our treatment of Schur functions, it will be convenient to introduce some further notation for partitions. Let $\widetilde{\lambda}$ denote the *conjugate* of λ, which is the partition whose Ferrers graph is the reflection about the main diagonal of the Ferrers graph of λ. Algebraically, this is equivalent to

$$\widetilde{\lambda}_j = |\{i : \lambda_i \geq j, i \geq 1| , \qquad j \geq 1. \tag{26}$$

For example, if $\lambda = (4, 3, 1)$, then $\widetilde{\lambda} = (3, 2, 2, 1)$. Also, define the following total order $\prec$ for partitions, called *Reverse Lexicographic Order*: write the parts of the partitions in weakly decreasing order, and order them lexicographically, as in dictionary order; then $\lambda \prec \mu$ when $\lambda_j = \mu_j$ for $j < i$, and $\lambda_i < \mu_i$, for some $i \geq 1$ (where we have $\lambda_1 \geq \lambda_2 \geq \ldots$, and $\mu_1 \geq \mu_2 \geq \ldots$). For example, this means that $44222 \prec 443111 \prec 4442 \prec 5333$. Also, for the partitions of 5, we have

$$11111 \prec 2111 \prec 221 \prec 311 \prec 32 \prec 41 \prec 5.$$

Now we use the tableau definition to write the Schur function s_λ as a linear combination of monomial symmetric functions. Note that, in a tableau of shape λ, the integer i can never be placed in a cell in row j for any $1 \leq i < j$, because of column strictness. But this implies immediately that if a tableau contains μ_i i's, for $i \geq 1$, with $\mu_1 \geq \mu_2 \geq \ldots$, then either $\mu = \lambda$ (and the unique such tableau has all i's in row i, for every $i \geq 1$), or we have $\mu_i = \lambda_i$ for $i = 1, \ldots, m-1$ and $\mu_m < \lambda_m$, for some $m \geq 1$. Thus we have

$$s_\lambda = m_\lambda + \sum_{\mu \prec \lambda} a_{\lambda,\mu} m_\mu, \tag{27}$$

where "$\prec$" is Reverse Lexicographic Order. Now if $s^{(k)}$ is a column vector whose entries are the s_λ for $\lambda \vdash k$, arranged in increasing reverse lexicographic order, and $m^{(k)}$ is a column vector whose entries are the m_μ for $\mu \vdash k$, arranged in increasing reverse lexicographic order, then from (27) we obtain the matrix equation $s^{(k)} = A^{(k)} m^{(k)}$ for each k, where $A^{(k)}$ is a square matrix with rows and columns indexed by partitions of k; $A^{(k)}$ has 1's on the diagonal, 0's above the diagonal, and nonnegative integers below the diagonal. This means that $\det A^{(k)} = 1$, so we can write each m_μ as a unique linear combination of the s_λ, and Cramer's Rule implies that the coefficients in these linear combinations are integers – so $\{s_\lambda : \lambda \vdash k\}$ is a basis for Λ^k over the integers. (*Cramer's Rule* says in this case that

$$m_\mu = \frac{\det[A^{(k)} : s^{(k)}]_\mu}{\det A^{(k)}} = \det[A^{(k)} : s^{(k)}]_\mu,$$

where $[A^{(k)} : s^{(k)}]_\mu$ is the matrix obtained from $A^{(k)}$ by replacing column μ by the column vector $s^{(k)}$.)

Now define the elementary symmetric functions indexed by a partition in the following multiplicative way: We let $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$, where λ is a partition with parts $\lambda_1, \lambda_2, \ldots$. As for the finite variable case, here we have $e_k = m_{(1,\ldots,1)}$, where there are k 1's in $(1, \ldots, 1)$, $k \geq 1$, and $e_0 = 1$.

Now, we prove that

$$e_{\tilde{\lambda}} = m_\lambda + \sum_{\mu \prec \lambda} a_{\lambda,\mu} m_\mu, \tag{28}$$

where the $a_{\lambda,\mu}$'s are non-negative integers. To prove (28), consider a monomial $x_1^{\mu_1} x_2^{\mu_2} \cdots$, $\mu_1 \geq \mu_2 \geq \ldots$ that appears (i.e., with non-zero coefficient) in the expansion of $e_{\tilde{\lambda}} = e_{\tilde{\lambda}_1} e_{\tilde{\lambda}_2} \cdots$. We will prove that if $\mu_i = \lambda_i$ for $1 \leq i \leq n-1$, then $\mu_n \leq \lambda_n$. But this is immediate from the algebraic statement of the definition of conjugate given in (26) – the condition that $\mu_i = \lambda_i$ for $1 \leq i \leq n-1$ implies that the monomial $x_1 \cdots x_{\tilde{\lambda}_\ell}$ must be chosen from $e_{\tilde{\lambda}_\ell}$ for $\ell > \lambda_n$, and that the monomial that is chosen from $e_{\tilde{\lambda}_\ell}$ must have $x_1 \cdots x_{n-1}$ as a factor for $\ell \leq \lambda_n$. But this means that there are at most λ_n factors that can contain x_n, and so we conclude that $\mu_n \leq \lambda_n$. Iterate this for $n \geq 1$ to obtain the result.

But, if we consider equation (28) over all partitions λ of k, then we obtain a linear system of equations that is unitriangular (i.e., 1's on the diagonal), and therefore invertible, with a unitriangular inverse that also has integer entries (once again use Cramer's rule). Thus we can express each m_μ as a unique integer linear combination of e_λ's, and we have proved that $\{e_\lambda : \lambda \vdash k\}$ is a basis for Λ^k over the integers.

Now define $\Lambda = \bigoplus_{k\geq 0} \Lambda^k$ (a vector space direct sum). Note that Λ is naturally also an algebra, since we can multiply symmetric functions in the obvious way. This is a graded algebra (graded by degree) since if $f \in \Lambda^k$ and $g \in \Lambda^\ell$, then $f \cdot g \in \Lambda^{k+\ell}$.

Note that, from this point of view, the fact that $\{e_\lambda : \lambda \vdash k\}$ is a basis for Λ^k over the integers implies that the elements of Λ can be written uniquely as a polynomial in $e_1, e_2, \ldots$ (written $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$), which also implies that the e_i's are algebraicly independent over the integers.

We define the complete symmetric functions indexed by a partition in the same multiplicative way: Let $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$, where λ is a partition with parts $\lambda_1, \lambda_2, \ldots$. As for the finite variable case, here we have $h_k = \sum_{\lambda \vdash k} m_\lambda$, $k \geq 1$, and $h_0 = 1$. In order to consider symmetric functions indexed by a partition, it will he helpful to conserve the mapping ω defined by

$$\omega : \Lambda \to \Lambda : e_i \mapsto h_i, \qquad i \geq 1,$$

which is well-defined since the e_i's are algebraicly independent. Now let

$$H(t) = \sum_{k\geq 0} h_k t^k = \prod_{j\geq 1} (1 - x_j t)^{-1},$$

$$E(t) = \sum_{k\geq 0} e_k t^k = \prod_{j\geq 1} (1 + x_j t),$$

so we immediately have

$$E(t)H(-t) = 1. \tag{29}$$

Now apply ω to this equation (i.e., on both sides, we have formal power series in t, with coefficients from Λ, so we are applying ω to the coefficients), to get

$$H(t)\omega\left(H(-t)\right) = 1,$$

and replace t by $-t$, which gives

$$H(-t)\omega\left(H(t)\right) = 1.$$

Comparing this with (29), we obtain

$$\omega\left(H(t)\right) = \frac{1}{H(-t)} = E(t),$$

and hence we conclude that

$$\omega(h_i) = e_i, \qquad i \geq 1.$$

7. Lecture of May 17

We conclude that ω is an involution on Λ, so it is an *automorphism*. This implies immediately that the h_i's are algebraicly independent, that $\{h_\lambda : \lambda \vdash k\}$ is a basis for Λ^k, and that $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$.

Next we define the power sum symmetric functions in terms of monomials by $p_k = m_k$, $k \geq 1$, and $p_0 = 1$. Again, we let let $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$, where λ is a partition with parts $\lambda_1, \lambda_2, \ldots$. Then we have

$$\log H(t) = \sum_{j\geq 1} \sum_{k \geq 1} x_j^k \frac{t^k}{k} = \sum_{k\geq 1} p_k \frac{t^k}{k}.$$

Differentiating this with respect to t, and multiplying on both sides by $H(t)$, we obtain

$$\sum_{n\geq 1} n h_n t^{n-1} = \sum_{\ell \geq 0} h_\ell t^\ell \sum_{k\geq 1} p_k t^{k-1},$$

and equating coefficients of t^{n-1} on both sides, this gives

$$n h_n = \sum_{k=1}^{n} h_{n-k} p_k, \qquad n \geq 1$$

which means, by induction on n, that we can write h_n, $n \geq 1$, as a polynomial in the p_i's with rational coefficients. But rearranging this equation, we also have

$$p_n = n h_n - \sum_{k=1}^{n-1} h_{n-k} p_k, \qquad n \geq 1,$$

which means, by induction on n, that we can write p_n, $n \geq 1$, as a polynomial in the h_i's with integer coefficients. But since the h_i's are algebraicly independent over the integers, they must also be algebraicly independent over the rationals, so the p_i's are algebraicly independent over the rationals. Thus we have $\{p_\lambda : \lambda \vdash k\}$ is a basis for $\Lambda_{\mathbb{Q}}^k$ (where we write $\Lambda_{\mathbb{Q}}$ for the algebra over the set of rationals $\mathbb{Q}$), and that $\Lambda = \mathbb{Q}[p_1, p_2, \ldots]$. Now, we have

$$\log E(t) = \sum_{k\geq 1} (-1)^{k-1} p_k \frac{t^k}{k},$$

obtained similarly to the expression for $\log H(t)$ above, and applying ω, we obtain

$$\log H(t) = \sum_{k\geq 1} (-1)^{k-1} \omega(p_k) \frac{t^k}{k}.$$

Comparing this to the expression for $\log H(t)$ above, we have $\omega(p_k) = (-1)^{k-1} p_k$, $k \geq 1$, so

$$\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda,$$

for all partitions λ, where $|\lambda| = \lambda_1 + \lambda_2 + \ldots$, and $\ell(\lambda)$ is the number of (positive) parts in λ. This implies that $\{p_\lambda : \lambda \vdash k\}$ is a basis of eigenvectors for $\Lambda^k_{\mathbb{Q}}$.

Now we introduce a bilinear form $\langle\ ,\ \rangle$ on Λ, defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu},$$

where $\delta_{\lambda,\mu} = 1$, if $\lambda = \mu$, and $\delta_{\lambda,\mu} = 0$ otherwise. This means, if $f, g \in \Lambda$ with $f = \sum_\lambda c_\lambda h_\lambda$ and $g = \sum_\mu d_\mu m_\mu$, then

$$\langle f, g \rangle = \sum_\lambda \sum_\mu c_\lambda d_\mu \langle h_\lambda, m_\mu \rangle = \sum_\lambda c_\lambda d_\lambda.$$

Theorem 7.1. *Suppose that $\{u_\lambda\}$ and $\{v_\lambda\}$ are bases for $\Lambda_{\mathbb{Q}}$. Then the following are equivalent:*

$$\text{(i)}\quad \langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}, \quad \text{for all } \lambda, \mu.$$
$$\text{(ii)} \sum_\lambda u_\lambda(x_1, \ldots) v_\lambda(y_1, \ldots) = \prod_{i,j\ge 1} (1 - x_i y_j)^{-1}.$$

Proof. Let $u_\lambda = \sum_\rho a_{\lambda\,\rho} h_\rho$, $v_\mu = \sum_\sigma b_{\mu\,\sigma} m_\sigma$. Then

$$\langle u_\lambda, v_\mu \rangle = \sum_\rho \sum_\sigma a_{\lambda\,\rho} b_{\mu\,\sigma} \langle h_\rho, m_\sigma \rangle = \sum_\rho a_{\lambda\,\rho} b_{\mu\,\rho} = \left(AB^t\right)_{\lambda\,\mu},$$

where A, B are matrices with rows and columns indexed by partitions.

We also have

$$\prod_{i,j\ge 1} (1 - x_i y_j)^{-1} = \prod_{j\ge 1} \sum_{k_j \ge 0} h_{k_j}(x_1, \ldots) y_j^{k_j} = \sum_\lambda h_\lambda(x_1, \ldots) m_\lambda(y_1, \ldots). \tag{30}$$

Continuing, we have

$$\begin{aligned}
\sum_\lambda u_\lambda(x_1, \ldots) v_\lambda(y_1, \ldots) &= \sum_\lambda \left(\sum_\rho a_{\lambda\,\rho} h_\rho(x_1, \ldots) \right) \left(\sum_\sigma b_{\lambda\,\sigma} m_\sigma(y_1, \ldots) \right) \\
&= \sum_{\rho,\sigma} h_\rho(x_1, \ldots) m_\sigma(y_1, \ldots) \sum_\lambda a_{\lambda\,\rho} b_{\lambda\,\sigma} \\
&= \sum_{\rho,\sigma} h_\rho(x_1, \ldots) m_\sigma(y_1, \ldots) \left(A^t B\right)_{\rho\,\sigma}.
\end{aligned}$$

Thus condition (i) is equivalent to $AB^t = I$, and (ii) is equivalent to $A^t B = I$. But this is equivalent (by taking transposes) to $B^t A = I$, and the result follows, since conditions (i) and (ii) are both equivalent to $A^{-1} = B^t$. □

Now suppose that we modify (30) to obtain

$$\prod_{i,j\ge 1} (1 - x_i y_j)^{-1} = \prod_{i\ge 1} \sum_{k_i \ge 0} h_{k_i}(y_1, \ldots) x_i^{k_i} = \sum_\lambda m_\lambda(x_1, \ldots) h_\lambda(y_1, \ldots).$$

Then Theorem 7.1 tells us that $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu} = \delta_{\mu,\lambda} = \langle h_\mu, m_\lambda \rangle$, where the last equality follows from the definition of $\langle\ ,\ \rangle$. But this implies that the bilinear form $\langle\ ,\ \rangle$ is *symmetric* – i.e., $\langle f, g \rangle = \langle g, f \rangle$ for any $f, g \in \Lambda_{\mathbb{Q}}$.

8. Lecture of May 23

For the next result, it is convenient to use the notation

$$z(\lambda) = 1^{i_1} 2^{i_2} \cdots i_1! i_2! \cdots ,$$

where the partition λ has i_j parts equal to j, $j \geq 1$.

Corollary 8.1.

$$\text{(i)} \quad \{p_\lambda\} \text{ is an orthogonal basis, with } \langle p_\lambda, p_\mu \rangle = z(\lambda)\, \delta_{\lambda,\mu},$$

$$\text{(ii)} \quad \langle\ ,\ \rangle \quad \text{is positive definite,}$$

$$\text{(iii)} \quad \omega \text{ is an isometry for } \langle\ ,\ \rangle \text{ (i.e., } \langle \omega(f), \omega(g) \rangle = \langle f, g \rangle, \text{ for all } f, g).$$

Proof. For part(i), we have

$$\begin{aligned}
\prod_{i,j\geq 1} (1 - x_i y_j)^{-1} &= \exp \sum_{i,j\geq 1} \sum_{k\geq 1} \frac{x_i^k y_j^k}{k} \\
&= \exp \sum_{k\geq 1} \frac{p_k(x_1, \ldots) p_k(y_1, \ldots)}{k} = \prod_{k\geq 1} \exp \frac{p_k(x_1, \ldots) p_k(y_1, \ldots)}{k} \\
&= \prod_{k\geq 1} \sum_{i_k \geq 0} \frac{p_k(x_1, \ldots)^{i_k} p_k(y_1, \ldots)^{i_k}}{k^{i_k} i_k!} \\
&= \sum_{i_1, i_2, \ldots \geq 0} \prod_{k\geq 1} \frac{p_k(x_1, \ldots)^{i_k} p_k(y_1, \ldots)^{i_k}}{k^{i_k} i_k!} = \sum_\lambda \frac{p_\lambda(x_1, \ldots) p_\lambda(y_1, \ldots)}{z(\lambda)},
\end{aligned}$$

so Theorem 7.1 gives $\langle p_\lambda / z(\lambda), p_\mu \rangle = \delta_{\lambda,\mu}$, and the result follows, using bilinearity.

For part (ii), consider $f \in \Lambda_{\mathbb{Q}}$, and suppose that $f = \sum_\lambda c_\lambda p_\lambda$. Then

$$\langle f, f \rangle = \langle \sum_\lambda c_\lambda p_\lambda, \sum_\mu c_\mu p_\mu \rangle = \sum_{\lambda,\mu} c_\lambda c_\mu \langle p_\lambda, p_\mu \rangle = \sum_\lambda c_\lambda^2 z(\lambda) \geq 0,$$

and this is equal to 0 if and only if $c_\lambda = 0$ for all partitions λ, in which case $f = 0$, giving the result.

For part (iii), it is sufficient to prove for any basis. Using power sums, we have

$$\langle \omega(p_\lambda), \omega(p_\mu) \rangle = \langle (-1)^{|\lambda| - \ell(\lambda)} p_\lambda, (-1)^{|\mu| - \ell(\mu)} p_\mu \rangle = \langle p_\lambda, p_\mu \rangle,$$

giving the result. □

The fact that $\{s_\lambda :\}$ is an orthonormal basis for $\Lambda_{\mathbb{Q}}$ (which means that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\,\mu}$ for all partitions λ, μ), follows immediately from the following result, by applying Theorem 7.1.

Theorem 8.2.

$$\sum_\lambda s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \prod_{i,j\geq 1} (1 - x_i y_j)^{-1}.$$

Proof. Consider an arbitrary multiset of ordered pairs of positive integers, $\{(i_1, j_1), \ldots, (i_m, j_m)\}$, arranged so that $i_k \leq i_\ell$ for $k < \ell$, and so that $j_k \leq j_\ell$ whenever $i_k = i_\ell$ for $k < \ell$. We map each such multiset to a pair of tableaux of the same shape as follows: Start with a pair (P_0, Q_0) of empty tableaux. For ℓ from 1 to m, row-insert j_ℓ in $P_{\ell-1}$ and place

i_ℓ in a new cell added to $Q_{\ell-1}$, in the same cell in which P_ℓ differs from $P_{\ell-1}$. The multiset is mapped to the pair (P_m, Q_m). For example, suppose

$$\{(i_1, j_1), \ldots, (i_m, j_m)\} = \{(1,1), (1,3), (1,3), (2,2), (2,2), (3,1), (3,3)\}.$$

Then, applying the above algorithm, we obtain:

$$P_1 = \begin{matrix} 1 \end{matrix} \qquad Q_1 = \begin{matrix} 1 \end{matrix}, \qquad P_2 = \begin{matrix} 1 & 3 \end{matrix} \qquad Q_2 = \begin{matrix} 1 & 1 \end{matrix}$$

$$P_3 = \begin{matrix} 1 & 3 & 3 \end{matrix} \qquad Q_3 = \begin{matrix} 1 & 1 & 1 \end{matrix}$$

$$P_4 = \begin{matrix} 1 & 2 & 3 \\ 3 & & \end{matrix} \qquad Q_4 = \begin{matrix} 1 & 1 & 1 \\ 2 & & \end{matrix}$$

$$P_5 = \begin{matrix} 1 & 2 & 2 \\ 3 & 3 & \end{matrix} \qquad Q_5 = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \end{matrix}$$

$$P_6 = \begin{matrix} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{matrix} \qquad Q_6 = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{matrix}$$

$$P_7 = \begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 3 & & \\ 3 & & & \end{matrix} \qquad Q_7 = \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & & \\ 3 & & & \end{matrix}$$

In this case we have mapped the multiset to the pair (P_7, Q_7), which are tableaux of the same shape.

In general, note that, by construction, P_m contains $j_1, \ldots, j_m$ in its cells, and Q_m contains $i_1, \ldots, i_m$ in its cells. The proof that P_ℓ and Q_ℓ are tableaux for every ℓ is identical to the proof given on pages 12 – 14 for the Robinson-Schensted Algorithm. This mapping is reversible by the following fact, straightforward to prove and omitted: if $i_\ell = i_{\ell+1}$, P_ℓ differs from $P_{\ell-1}$ by a cell in column a, and $P_{\ell+1}$ differs from P_ℓ by a cell in column b, then $a < b$. This fact allows us to determine, at each stage k, which cell was most recently added to Q_k, and then it is straighforward to reverse the row-insertion for P_k.

The consequence is that this mapping is a bijection, and we thus obtain (with a_{ij} the frequency of the ordered pair (i,j) in the multiset)

$$\sum_\lambda s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \sum_{\substack{a_{rs} \geq 0, \\ r,s \geq 1}} \prod_{i,j \geq 1} (x_i y_j)^{a_{ij}} = \prod_{i,j \geq 1} \sum_{a_{ij} \geq 0} (x_i y_j)^{a_{ij}},$$

giving the result. □

Now we define a new combinatorial object. The *skew shape* λ/μ is obtained by removing the cells of the Ferrers graph of μ from the Ferrers graph of λ. A tableau of skew shape λ/μ is obtained by placing a positive integer in each cell of the skew shape λ/μ so that the rows are weakly increasing, and the columns are strictly increasing. For example, a skew tableau of skew shape $(5,5,3,2,1)/(3,2)$ is given in Figure 5. We define the *skew Schur function* $s_{\lambda/\mu}$ by

$$s_{\lambda/\mu}(x_1, \ldots) = \sum_T \prod_{j \geq 1} x_j^{n_j(T)},$$

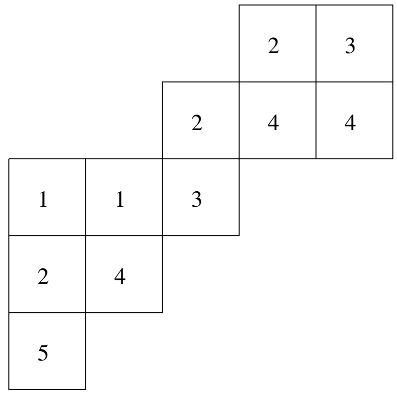

FIGURE 5. A skew tableau, of skew shape $(5, 5, 3, 2, 1)/(3, 2)$.

where the summation is over all tableaux T of skew shape λ/μ, and $n_j(T)$ is the number of times j appears in a cell in T. Note that the skew Schur function is 1 when $\lambda = \mu$, and is 0 when the ith part of μ is bigger than the ith part of λ for any $i \geq 1$ (where we add enough parts equal to 0 to make λ and μ have the same number of parts).

Also, the skew Schur function in a countable set of indeterminates is symmetric, by the argument given on pages 8, 9 for the Schur function in a finite set of variables. Now suppose that we fill the cells of shape λ by elements of the countable set $1', 2', \ldots, 1, 2, \ldots$, with z_i marking the occurrence of i' in a cell, and y_j marking the occurrence of j in a cell. Then the generating function is the Schur function $s_\lambda(z_1, z_2, \ldots, y_1, y_2, \ldots)$. If we order this countable set so that $i' < j$ for all $i, j \geq 1$, then the elements of $1', 2', \ldots$ all occur in cells of some shape μ in the top left hand corner of the tableau, and the the elements $1, 2, \ldots$ all occur in the remaining cells, which have skew shape λ/μ. Thus we immediately obtain

$$s_\lambda(z_1, z_2, \ldots, y_1, y_2, \ldots) = \sum_\mu s_\mu(z_1, z_2, \ldots) s_{\lambda/\mu}(y_1, y_2, \ldots), \tag{31}$$

which can be used to help us prove a number of results about skew Schur functions.

9. LECTURE OF MAY 25

Theorem 9.1.

$$\langle s_\lambda, s_\mu s_\nu \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle$$

Proof. Applying Theorem 8.2, we obtain

$$\begin{aligned} \sum_\lambda s_\lambda(x_1, \ldots) s_\lambda(z_1, \ldots, y_1, \ldots) &= \prod_{i,j \geq 1} (1 - x_i z_j)^{-1} (1 - x_i y_j)^{-1} \\ &= \sum_{\mu,\nu} s_\mu(x_1, \ldots) s_\mu(z_1, \ldots) s_\nu(x_1, \ldots) s_\nu(y_1, \ldots). \end{aligned} \tag{32}$$

Now, if we equate the coefficients of $s_\mu(z_1, \ldots) s_\nu(y_1, \ldots) s_\lambda(x_1, \ldots)$ on both sides of (32), we obtain

$$[s_\nu(y_1, \ldots)] s_{\lambda/\mu}(y_1, \ldots) = [s_\lambda(x_1, \ldots)] s_\mu(x_1, \ldots) s_\nu(x_1, \ldots),$$

where we have used (31) for the left hand side. But, if $f \in \Lambda_{\mathbb{Q}}$, then we have $f = \sum_\mu c_\mu s_\mu$, so

$$\langle s_\lambda, f\rangle = \langle s_\lambda, \sum_\mu c_\mu s_\mu \rangle = \sum_\mu c_\mu \langle s_\lambda, s_\mu\rangle = c_\lambda,$$

and we conclude that

$$[s_\lambda] f = \langle s_\lambda, f\rangle, \tag{33}$$

giving the result. □

Now suppose that we equate coefficients of $s_\mu(z_1, \ldots)$ on both sides of (32). Then, from (31), we obtain

$$\sum_\lambda s_\lambda(x_1, \ldots) s_{\lambda/\mu}(y_1, \ldots) = \frac{s_\mu(x_1, \ldots)}{\prod_{i,j\geq 1}(1 - x_i y_j)},$$

a generalization of Theorem 8.2 that should be helpful for Problem 1 on the upcoming Assignment 2.

In general, Schur functions do not have nice explicit expressions in terms of monomials, but for certain specializations of the variables $x_1, \ldots$ there are nice forms. For example, consider the following result, involving hook-lengths again (see page 18, and Problem 2 on Assignment 1).

Theorem 9.2.

$$s_\lambda(x^1, x^2, x^3, \ldots) = \frac{x^{\sum_{i\geq 1} i\lambda_i}}{\prod_{\alpha \in \lambda}(1 - x^{h(\alpha)})}.$$

Proof. Note that $s_\lambda(x^1, x^2, x^3, \ldots)$ is the generating function for tableaux of shape λ, with the exponent on x giving the sum of the entries in the tableau. Take a tableau of shape λ, and subtract i from every entry in row i, $i \geq 1$ (thus subtracting $\sum_{i\geq 1} i\lambda_i$ from the sum of the entries). The result is a row- and column-weak array of nonnegative integers of shape λ. Let this array be R_0, and let H_0 be an array of shape λ consisting entirely of 0's. We now describe an iterative procedure, where at every stage we have a pair of arrays of shape λ, R_0, H_0, R_1, H_1, ... We call the H_i's *hook arrays*.

Then, for $\ell \geq 1$, find the leftmost column of $R_{\ell-1}$ with a positive entry, and find the bottom cell with a positive entry in that column, as an initial cell. Form a path starting at the initial cell using the following rule: move up to a cell with an equal entry if possible, else move right to the next entry, exiting (and thus terminating the path) if there is no cell to the right. Subtract 1 from every entry in a cell of the path, to obtain R_ℓ. If the path started in column j and exited in row i, add 1 to the cell in row i and column j of $H_{\ell-1}$, to obtain H_ℓ. Stop when R_ℓ consists entirely of 0's. We map the tableau to the terminating hook array.

For example, consider $\lambda = (3, 3, 1)$, and the tableau

$$\begin{matrix} 1 & 2 & 4 \\ 4 & 6 & 6 \\ 6 & & \end{matrix}\ .$$

Then, iterating, we obtain

$$R_0 = \begin{matrix} 0 & 1 & 3 \\ 2 & 4 & 4 \\ 3 & & \end{matrix} \qquad H_0 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & & \end{matrix}$$

$$R_1 = \begin{matrix} 0 & 1 & 3 \\ 2 & 4 & 4 \\ 2 & & \end{matrix} \qquad H_1 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & & \end{matrix}$$

$$R_2 = \begin{matrix} 0 & 1 & 3 \\ 1 & 3 & 3 \\ 1 & & \end{matrix} \qquad H_2 = \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & & \end{matrix}$$

$$R_3 = \begin{matrix} 0 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & & \end{matrix} \qquad H_3 = \begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & & \end{matrix}$$

$$R_4 = \begin{matrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & & \end{matrix} \qquad H_4 = \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & & \end{matrix}$$

$$R_5 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & & \end{matrix} \qquad H_5 = \begin{matrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & & \end{matrix}$$

$$R_6 = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & & \end{matrix} \qquad H_6 = \begin{matrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & & \end{matrix}$$

In this case, we have mapped the original tableau to the terminating hook array H_6.

This procedure, called the Hillman-Grassl correspondence, is a bijection: to reverse it, note that the hook array entries grow from left to right, and then bottom to top. This is straightforward to prove, since it follows by induction that at every stage we have row- and column-weakness, and the path starts as far left as possible. For paths that start in the same column, it is straightforward to prove that they may meet, but never cross (they move weakly up). Note also that the number of entries in a path starting in column j and exiting in row i is always equal to the the hook-length of the cell in row i and column j, so the sum of the entries in the original tableau is equal to

$$\sum_{i\geq 1} i\lambda_i + \sum_{\alpha\in\lambda} k_\alpha h(\alpha),$$

where k_α is the entry in cell α of the associated hook array. Thus we have

$$s_\lambda(x^1, x^2, x^3, \ldots) = x^{\sum_{i\geq 1} i\lambda_i} \sum_{\substack{k_\alpha\geq 0,\\ \alpha\in\lambda}} \prod_{\alpha\in\lambda} x^{k_\alpha h(\alpha)} = x^{\sum_{i\geq 1} i\lambda_i} \prod_{\alpha\in\lambda} \sum_{k_\alpha\geq 0} x^{k_\alpha h(\alpha)},$$

and the result follows. □

One context in which these symmetric function results are helpful is in the enumeration of plane partitions. A *plane partition* of n is a Ferrers graph of some shape, with positive integers summing to n in the cells so that they are weakly decreasing along the rows from left to right, and are weakly decreasing down the columns from top to bottom. For example, a plane partition of 36 is given by

$$\begin{array}{ccc} 5 & 5 & 3 \\ 4 & 3 & 3 \\ 4 & 3 & 1 \\ 3 & & \\ 1 & & \\ 1 & & \end{array} \tag{34}$$

10. Lecture of May 29

To enumerate plane partitions, we decompose them into the Ferrers graphs of partitions, as follows: if r is the plane partition, and m is the maximum integer appearing in the cells of r, then let $r^{(i)}$ be the Ferrers graph formed by the cells of r containing an integer greater than or equal to i, for $i = 1, \ldots, m$. We'll place a 1 in each cell of $r^{(i)}$. For example, the plane partition r of 36 given in (34) above, for which $m = 5$, yields

$$r^{(1)} = \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & & \\ 1 & & \\ 1 & & \end{array}, \quad r^{(2)} = \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{array}, \quad r^{(3)} = \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{array}, \quad r^{(4)} = \begin{array}{cc} 1 & 1 \\ 1 & \\ 1 & \end{array}, \quad r^{(5)} = \begin{array}{cc} 1 & 1 \end{array}$$

Plane partitions are also referred to as *solid* partitions, because they can be viewed as three dimensional Ferrers graphs. For example, represent the integer k in a cell as a column of k unit cells extending out from the page. From this point of view, the partition $r^{(i)}$ is obtained by taking the two dimensional slice of this solid at distance i above the page.

Now we give another notation for a partition. Suppose that the Ferrers graph of partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ has k cells on the main diagonal (i.e., in column j of row j for some positive j) for some $k \leq m$. Then there are α_j cells strictly to the right of the cell on the main diagonal in row j, for $j = 1, \ldots, k$, and there are β_j cells strictly below the cell on the main diagonal in column j, for $j = 1, \ldots, k$, where we have $\alpha_1 \ldots > \alpha_k \geq 0$, and $\beta_1 \ldots > \beta_k \geq 0$. We write $\lambda = ((\alpha_1, \ldots, \alpha_k) \,|(\beta_1, \ldots, \beta_k))$, in *Frobenius* notation.

Now suppose that r is a plane partition that decomposes into the partitions $r^{(1)}, \ldots, r^{(m)}$. For $i = 1, \ldots, m$, let $r^{(i)} = (s^i | t^i)$, in Frobenius notation, and let a, b have columns $s^1 + 1, \ldots, s^m + 1$, $t^1 + 1, \ldots, t^m + 1$, respectively (where $+1$ means to add 1 to all parts). For example, if r is the plane partition given above, then we have

$$a = \begin{array}{ccccc} 3 & 3 & 3 & 2 & 2 \\ 2 & 2 & 2 & & \\ 1 & & & & \end{array}, \qquad b = \begin{array}{ccccc} 6 & 4 & 4 & 3 & 1 \\ 2 & 2 & 2 & & \\ 1 & & & & \end{array}.$$

Note that, by construction, a and b are plane partitions of the same shape, with columns that are *strictly* decreasing. Also, if a, b are plane partitions of A, B, respectively, then we have $A + B - D = n$, where D is the number of cells in a (and in b), and this is bijective. But a and b are precisely tableaux of the same shape, with the integers in reverse order, so we conclude that if P_n is the number of plane partitions of n, $n \geq 0$, then from Theorem 8.2

we have

$$\begin{aligned}\sum_{n\geq 0} P_n x^n &= \sum_{\lambda} s_\lambda(x^1, x^2, x^3, \ldots) s_\lambda(x^{1-1}, x^{2-1}, x^{3-1}, \ldots)\\ &= \prod_{i,j\geq 1} (1-x^{i+j-1})^{-1}\\ &= \prod_{k\geq 1} (1-x^k)^{-k},\end{aligned}$$

a classical result of MacMahon (for the last equality, we need only to determine that the number of solutions to $i+j-1=k$ for $i,j\geq 1$ and each fixed $k\geq 1$, is equal to k).

There are a number of results known for the number of plane partitions under various restrictions. For example, suppose we want to know the number of plane partitions that are symmetric (under reflection about the main diagonal). For example,

$$\begin{matrix} 5 & 4 & 3 & 2\\ 4 & 4 & 2 & 1\\ 3 & 2 & 2 & 1\\ 2 & 1 & 1 & 1\end{matrix} \tag{35}$$

is a symmetric plane partition q of 38. To enumerate these, first decompose into partitions as above. For example, when q is given in (35), we have maximum integer $m=5$, and so we get

$$q^{(1)} = \begin{matrix} 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{matrix}, \quad q^{(2)} = \begin{matrix} 1 & 1 & 1 & 1\\ 1 & 1 & 1 & \\ 1 & 1 & 1 & \\ 1 & & & \end{matrix}, \quad q^{(3)} = \begin{matrix} 1 & 1 & 1\\ 1 & 1 & \\ 1 & & \end{matrix}, \quad q^{(4)} = \begin{matrix} 1 & 1\\ 1 & 1\end{matrix}, \quad q^{(5)} = \begin{matrix} 1 \end{matrix}$$

Note that the symmetry of q implies that the partitions $q^{(i)}$ are also symmetric, or *self-conjugate*; this means that $q^{(i)} = (u^i|u^i)$ in Frobenius notation. Now suppose that q is a symmetric plane partition that decomposes into the partitions $q^{(1)}, \ldots, q^{(m)}$. For $i=1,\ldots,m$, let $q^{(i)} = (u^i|u^i)$, in Frobenius notation, and let c have columns $2u^1+1, \ldots, 2u^m+1$, (where $2u^i+1$ means to multiply each part of u^i by 2, and add 1). For example, if q is the plane partition given above, then we have

$$c = \begin{matrix} 7 & 7 & 5 & 3 & 1\\ 5 & 3 & 1 & 1 & \\ 3 & 1 & & & \\ 1 & & & & \end{matrix}.$$

Note that, by construction, c is a tableau with integers in reverse order, and all integers are odd, so if Q_n is the number of symmetric plane partitions of n, then we have

$$\sum_{n\geq 0} Q_n x^n = \sum_{\lambda} s_\lambda(x^1, 0, x^3, 0, x^5, \ldots),$$

and to complete this evaluation, we could use the result

$$\sum_{\lambda} s_\lambda(x_1, \ldots) = \prod_{i\geq 1} (1-x_i)^{-1} \prod_{1\leq j<k} (1-x_j x_k)^{-1}, \tag{36}$$

which follows as in the proof of Theorem 8.2, by noting that $P = Q$ corresponds exactly to $a_{ij} = a_{ji}$ for all $1 \leq i \leq j$ (this latter requires a detailed proof, extending the result that if the permutation σ corresponds to tableaux (P, Q) under Robinson-Schensted (the case in which the matrix A has ith row and column sum equal to 1 for $i = 1, \ldots, n$, and 0 for $i > n$) then σ^{-1} corresponds to (Q, P)). The extension of the Robinson-Schensted correspondence that we gave in order to prove Theorem 8.2 is called the *Robinson-Schensted-Knuth* correspondence.

Applying (36), we obtain

$$\sum_{n\geq 0} Q_n x^n = \prod_{k\geq 1}(1-x^{2k-1})^{-1} \prod_{m\geq 1}(1-x^{2m})^{-\lfloor \frac{m}{2} \rfloor},$$

where we have used the *floor* function notation $\lfloor x \rfloor$ to denote the largest integer $\leq x$.

We now finish this segment on principal specializations of Schur functions by returning to Theorem 9.2, which says that

$$s_\lambda(x^1, x^2, x^3, \ldots) = \frac{x^{\sum_{i\geq 1} i\lambda_i}}{\prod_{\alpha\in\lambda}(1-x^{h(\alpha)})}.$$

We consider the question of what happens on the right hand side above when on the left hand side we truncate the substitutions $x_i = x^i$ at $i = n$, and substitute $x_i = 0$ thereafter, thus obtaining

$$s_\lambda(x^1, x^2, , \ldots, x^n, 0, 0, \ldots).$$

We'll proceed algebraicly (though there is a combinatorial proof due to Krattenthaler in the literature).

To begin, we note that we can work with the Schur function in a finite number of variables, and thus use the determinant formula (10), which says that for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with at most n parts $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, we have

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1,\ldots,n}}{\det\left(x_i^{n-j}\right)_{i,j=1,\ldots,n}} = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1,\ldots,n}}{\prod_{1\leq i<j\leq n}(x_i - x_j)}.$$

(Note that if λ were to have more than n positive parts, then $s_\lambda(x_1, \ldots, x_n) = 0$, since there are no such tableaux – any corresponding tableau would have at least one strictly increasing column of length more than n, yet there are only the n different choices $1, \ldots, n$ to occupy the cells.) Substituting $x_i = x^i$ for $i = 1, \ldots, n$ in this formula, we obtain

$$s_\lambda(x^1, \ldots, x^n) = \frac{\det\left(x^{i(\lambda_j+n-j)}\right)_{i,j=1,\ldots,n}}{\prod_{1\leq i<j\leq n}(x^{\lambda_i+n-i} - x^{\lambda_j+n-j})}. \tag{37}$$

Now let $y_j = x^{\lambda_j+n-j}$ for $j = 1, \ldots, n$, so the numerator determinant can be written as

$$\det\left(y_j^i\right)_{i,j=1,\ldots,n} = (-1)^{\binom{n}{2}} \det\left(y_j^{n+1-i}\right)_{i,j=1,\ldots,n} = (-1)^{\binom{n}{2}} y_1 \cdots y_n \det\left(y_j^{n-i}\right)_{i,j=1,\ldots,n},$$

where for the first equality we have reversed the rows, and for the second equality we have factored y_j from column j, $j = 1, \ldots, n$. Then, using the Vandermonde determinant, we obtain

$$\det\left(y_j^i\right)_{i,j=1,\dots,n} = (-1)^{\binom{n}{2}} y_1\cdots y_n \prod_{1\le i<j\le n}(y_i - y_j)$$
$$= (-1)^{\binom{n}{2}} x^{\sum_{k=1}^n(\lambda_k+n-k)} \prod_{1\le i<j\le n}(x^{\lambda_i+n-i} - x^{\lambda_j+n-j}).$$

11. Lecture of May 31

Substituting into (37), we have

$$s_\lambda(x^1,\dots,x^n) = (-1)^{\binom{n}{2}} x^{\sum_{k=1}^n(\lambda_k+n-k)} \frac{\prod_{1\le i<j\le n}(x^{\lambda_i+n-i} - x^{\lambda_j+n-j})}{\prod_{1\le i<j\le n}(x^i - x^j)}$$
$$= x^{\sum_{k=1}^n(\lambda_k+n-k)} \frac{\prod_{1\le i<j\le n}(x^{\lambda_j+n-j} - x^{\lambda_i+n-i})}{\prod_{1\le i<j\le n}(x^i - x^j)}$$
$$= \frac{x^{\sum_{k=1}^n k(\lambda_k+n-k)}}{x^{\sum_{k=1}^n k(n-k)}} \frac{\prod_{1\le i<j\le n}(1 - x^{\lambda_i-\lambda_j-i+j})}{\prod_{1\le i<j\le n}(1 - x^{j-i})}$$
$$= x^{\sum_{k=1}^n k\lambda_k} \frac{\prod_{1\le i<j\le n}(1 - x^{\lambda_i-\lambda_j-i+j})}{\prod_{1\le i<j\le n}(1 - x^{j-i})}.$$

Next we'll show that the above expression has a simple combinatorial form by considering a particular example.Let $\lambda = (3,2)$, and $n = 4$. Then the above formula gives

$$s_\lambda(x^1,\dots,x^n) = x^7 \frac{(1-x^2)(1-x^5)(1-x^6)(1-x^3)(1-x^4)(1-x^1)}{(1-x^1)(1-x^2)(1-x^3)(1-x^1)(1-x^2)(1-x^1)}$$
$$= x^7 \frac{(1-x^4)(1-x^5)(1-x^6)}{(1-x^1)(1-x^1)(1-x^2)}$$
$$= x^7 \frac{(1-x^3)(1-x^4)(1-x^4)(1-x^5)(1-x^6)}{(1-x^1)(1-x^1)(1-x^2)(1-x^3)(1-x^4)},$$

where, for the second equality, we have cancelled common factors on the top and bottom, and for the last equality, we have multiplied on the top and bottom by $(1-x^3)(1-x^4)$, to create the product $\prod_{\alpha\in\lambda}(1-x^{h(\alpha)})$ in the denominator. The question is: What is the product in the numerator? The answer involves a combinatorial quantity associated with each cell in a diagram of λ called content. The *content* of cell α in row i and column j of the Ferrers diagram of λ is given by $c(\alpha) = j - i$. The product in the numerator in this case can then be written as $\prod_{\alpha\in\lambda}(1-x^{n+c(\alpha)})$, and in general we have the result

$$s_\lambda(x^1,\dots,x^n) = x^{\sum_{i=1}^n i\lambda_i} \frac{\prod_{\alpha\in\lambda}(1-x^{n+c(\alpha)})}{\prod_{\alpha\in\lambda}(1-x^{h(\alpha)})}.$$

This is a generalization of Theorem 9.2, since as $n\to\infty$, the numerator product goes to 1. You are asked to prove this for arbitrary partition λ, and arbitrary positive integer n in Problem 3 of Assignment 2.

Now we return to plane partitions, where we can discuss a different enumerative combinatorics methodology that is often used in this area, before we return to symmetric functions themselves.. For a plane partition with a rows, at most b columns, and the entries at most

c (where a, b, c are positive integers), it is convenient to represent it as a three dimensional analogue of a Ferrers graph. The third axis gives the value of the entry, and the cells become unit boxes. For example, the plane partition

$$\begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix}$$

of 5, with $a = 3$, $b = c = 2$, is contained in a $3 \times 2 \times 2$ box, and can be drawn in two dimensions as in Figure 6 (taking the perspective that the top left rhombus - the one marked with a T2, is the "top" of the second box in position $(1, 1)$ (row 1, column 1); the three rhombuses marked T1 are the "tops" of the single boxes in positions $(1, 2)$, $(2, 1)$, $(2, 2)$; the two rhombuses marked T0 denote that there are no boxes in positions $(3, 1)$, $(3, 2)$).

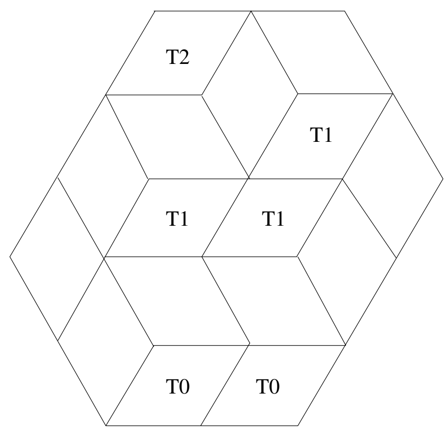

FIGURE 6. Rhombuses representing a plane partition.

Now, cut each of the rhombuses in this drawing (there are three types of rhombuses) into two isomorphic equilateral triangles (i.e., cut across the shorter diagonal), to get the drawing of the triangular grid in Figure 7, of dimensions $3 \times 2 \times 2$, as shown in the drawing. It is not too hard to prove that the set of all plane partitions in a $3 \times 2 \times 2$ box is exactly equivalent to the ways of pairing the triangles in this triangular grid, so that the triangles in each pair have a common side (and thus, together, form a rhombus).

Now, from the triangular grid, form the planar graph whose vertices are drawn in the centres of the triangles, and edges join two triangles that have a common side. This graph is drawn (for $a = 3$, $b = c = 2$) in Figure 8. Note that all interior faces of this planar embedding are hexagons, in general, as well as in the particular case $a = 3$, $b = c = 2$. It is clear that a pairing of the triangles in the triangular grid corresponds exactly to a (perfect) matching of this hexagonal graph, and thus, we can count plane partitions in a box of fixed dimensions by determining the number of perfect matchings in the hexagonal graph.

The problem of counting matchings is well known, especially in physics, and the algebraic key is determinants, especially via the **pfaffian**. For indeterminates $\{a_{ij} : 1 \leq i < j \leq 2n\}$, define the pfaffian by

$$\text{pf}\,(A) = \sum \text{sgn}\,(M) a_{m_{11}m_{12}} a_{m_{21}m_{22}} \cdots a_{m_{n1}m_{n2}}, \tag{38}$$

where the sum is over all *matchings* $M = \{\{m_{11}, m_{12}\}, \{m_{21}, m_{22}\}, \ldots \{m_{n1}, m_{n2}\}\}$, arranged so that $m_{i1} < m_{i2}$, $i = 1, \ldots, n$, and $m_{11} < m_{21} < \ldots < m_{n1}$. We define $\text{sgn}\,(M) = \text{sgn}\,(\sigma)$,

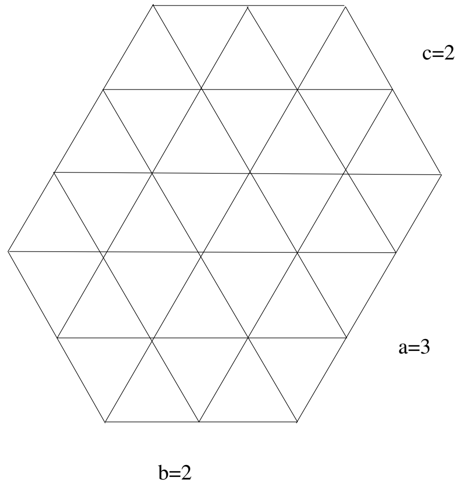

Figure 7. A triangular grid of dimensions $3 \times 2 \times 2$.

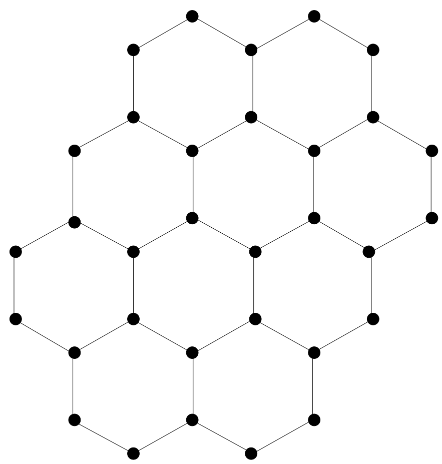

Figure 8. The hexagonal graph of dimensions $3 \times 2 \times 2$.

where σ is the permutation on $\{1, \ldots, 2n\}$ with $\sigma(2i-1) = m_{i1}$, $\sigma(2i) = m_{i2}$, for $i = 1, \ldots, n$. Suppose we define A to be the $2n \times 2n$ skew-symmetric matrix with (i, j)-entry a_{ij} for $i < j$ (so the (i, i)-entry is 0 for all i and the (j, i)-entry is $-a_{ij}$ for $i < j$). Then it is well-known that

$$\det(A) = \mathrm{pf}\,(A)^2. \tag{39}$$

For example, when $n = 2$ this gives

$$\det \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 . \tag{40}$$

The method of using this to count matchings of a simple graph with $2n$ vertices is as follows:

- name the vertices $1, \ldots, 2n$ in some convenient way;

- for a given **orientation** of the edges of the graph (i.e., independently assign a direction to each edge), consider $\det(A)$, where A is the skew-symmetric matrix with $a_{ij} = 1$ (and thus $a_{ji} = -1$) for each edge directed from vertex i to vertex j, and $a_{ij} = a_{ji} = 0$ when vertex i is not adjacent to vertex j;
- choose an orientation so that all non-zero terms in $\mathrm{pf}\,(A)$ (each term has absolute value 0 or 1) have the same (positive or negative) sign;
- for this particular orientation, evaluate $\det(A)$, and take the positive square root to obtain the number of matchings of the graph.

Perhaps surprisingly, such an orientation can be found to make this work for many graphs, in fact for any graph without an edge-subdivision of $K_{3,3}$ as a subgraph.

As a preliminary example, by examining (40), we see that the matchings of K_4 are counted in this way by using the following orientation of K_4: 1 2, 1 4, 2 3, 2 4, 3 1, 3 4 (where "$i\,j$" means that edge $\{i, j\}$ is directed from i to j). Thus we evaluate

$$\det \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix} = 9,$$

and conclude that there are 3 matchings of K_4.

To count matchings in the hexagonal grid, we can orient the edges by directing all edges upwards (this is unambiguous, since no edges are horizontal), When the appropriate determinant is evaluated (due to Kastelyn), this gives the classical result that the number of plane partitions in an $a \times b \times c$ box is given by

$$\frac{H(a+b+c)\,H(a)\,H(b)\,H(c)}{H(a+B)\,H(a+c)\,H(b+c)},$$

where $H(n) = (n-1)!\,(n-2)!\cdots 2!\,1!$

For a planar embedding, it is always possible to orient the edges to count the number of matchings. We need only orient the edges so that there is an odd number of edges oriented clockwise in very face of even length (except the "outside" face). See, e.g., Chapter 7 of Godsil's book "Algebraic Combinatorics". A quick check on the hexagonal and rectangular grids treated previously will show that our orientations satisfied this condition (in these cases, every such face was of even length).

12. Lecture of June 5

Before we move on to other topics, we'll record one fundamental result for pfaffians, that complements the Gessel-Viennot methodology described on page 11 of the Course Notes, for the enumeration of tuples of lattice paths with no pairwise intersection. For ℓ even, lattice points $P_1, \ldots, P_\ell$, and a set of lattice points Q with $|Q| \geq \ell$, let the generating function for ℓ-tuples of lattice paths $\pi_1, \ldots, \pi_\ell$, where π_i starts at P_i and ends at some (unspecified) point in Q, $i = 1, \ldots, \ell$, be given by $\Phi_Q(P_1, \ldots, P_\ell)$. Then Stembridge proved that

$$\Phi_Q(P_1, \ldots, P_\ell) = \mathrm{pf}\ (\Phi_Q(P_i, P_j))_{\ell\times\ell}\,,$$

where the pfaffian $\mathrm{pf}\,(a_{i,j})_{\ell\times\ell}$ is the matching sum defined in (38) (which is the square root of the determinant of the corresponding skew-symmetric matrix), as described on pages 32 and 33 of the Course Notes).

Now we return to more technical material on symmetric functions, and consider the expansion of $h_k s_\mu$ as a linear combination of Schur functions, say of the form

$$h_k\, s_\mu = \sum_\lambda c_\lambda s_\lambda, \tag{41}$$

where k is a positive integer and μ is a partition. We need to find the range of partitions λ for which the coefficient c_λ is non-zero, and for those, the values of each such c_λ. Here k is a scalar, but we can also write h_k as $h_{(k)}$, where (k) is the partition with a single part equal to k. (From the Jacobi-Trudi determinant, we can also write h_k as $s_{(k)}$, but that won't be useful here.) Taking inner products of both sides of (41) with s_λ, we obtain

$$\langle\, h_{(k)} s_\mu, s_\lambda \,\rangle = c_\lambda,$$

so from Theorem 9.1, we then get

$$\langle\, h_{(k)}, s_{\lambda/\mu} \,\rangle = c_\lambda.$$

But the bases $\{h_\nu\}$ and $\{m_\nu\}$ are orthogonal with respect to the inner product, which implies for an any symmetric function f that $\langle\, h_\nu, f \,\rangle = [m_\nu]\, f$, and thus the above equation gives

$$[m_{(k)}]\, s_{\lambda/\mu} = c_\lambda.$$

Now of course $m_{(k)} = p_k = x_1^k + x_2^k + \ldots$, so from the definition of the skew Schur function $s_{\lambda/\mu}$ on page 24 of the Course Notes, for non-zero c_λ the above equation implies that λ/μ must have exactly k cells, and each cell must contain the same positive integer. Thus no column can contain more than one cell (since cells in the same column must contain different integers, and so such shapes never create the monomial $m_{(k)}$). Indeed, one can always insert i in each of the k cells of such a skew shape, in a unique way, for every $i \geq 1$. Thus we conclude that $c_\lambda = 1$ for every λ such that λ/μ contains k cells, and no column of λ/μ contains more than one cell, and $c_\lambda = 0$ otherwise. The following terminology is often used to express this compactly: a skew shape λ/μ with k cells is said to have *size* k for any non-negative integer k; a skew shape λ/μ for which no column of λ/μ contains more than one cell is called a *horizontal strip*. Note that a horizontal strip can consist of many connected components, each of which is a single row. Using this notation, the expansion that we were seeking in (41) becomes (generally referred to as the *Pieri Rule*)

$$h_k\, s_\mu = \sum_\lambda s_\lambda, \tag{42}$$

where the summation is over all partitions λ such that λ/μ is a horizontal strip of size k.

Now suppose that we apply the involution ω to the Pieri Rule. Then we obtain $e_k\, \omega(s_\mu) = \sum \omega(s_\lambda)$, and to proceed further, it will be useful to know how ω acts on Schur functions. The following result, which is closely related to Theorem 8.2, will be helpful for this. The notation λ', for the *conjugate* of the partition λ, appears in the statement of the result.

Theorem 12.1.

$$\sum_\lambda s_\lambda(x_1, \ldots) s_{\lambda'}(y_1, \ldots) = \prod_{i,j\geq 1} (1 + x_i y_j).$$

Proof. Consider an arbitrary set of ordered pairs of positive integers, $\{(i_1, j_1), \ldots, (i_m, j_m)\}$, arranged so that $i_k \leq i_\ell$ for $k < \ell$, and so that $j_k < j_\ell$ whenever $i_k = i_\ell$ for $k < \ell$. We map each such set to a pair of tableaux of conjugate shape as follows: Start with a pair (P_0, Q_0) of empty tableaux. For ℓ from 1 to m, place j_ℓ in $P_{\ell-1}$ using the following modification of row-insertion: to insert i, place i immediately to the right of the right-most element that is strictly smaller than i (with a conventional 0 to the left of the row). Also, place i_ℓ in a new cell added to $Q_{\ell-1}$, in the same cell in which P_ℓ differs from $P_{\ell-1}$. The set is mapped to the pair (P_m, Q_m). For example, suppose

$$\{(i_1, j_1), \ldots, (i_m, j_m)\} = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,2),(3,3)\}.$$

Then, applying the above algorithm, we obtain:

$$P_1 = \begin{matrix} 1 \end{matrix} \qquad Q_1 = \begin{matrix} 1 \end{matrix}, \qquad P_2 = \begin{matrix} 1 & 2 \end{matrix} \qquad Q_2 = \begin{matrix} 1 & 1 \end{matrix}$$

$$P_3 = \begin{matrix} 1 & 2 & 4 \end{matrix} \qquad Q_3 = \begin{matrix} 1 & 1 & 1 \end{matrix}$$

$$P_4 = \begin{matrix} 1 & 2 & 4 \\ 1 & & \end{matrix} \qquad Q_4 = \begin{matrix} 1 & 1 & 1 \\ 2 & & \end{matrix}$$

$$P_5 = \begin{matrix} 1 & 2 & 4 \\ 1 & 2 & \end{matrix} \qquad Q_5 = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \end{matrix}$$

$$P_6 = \begin{matrix} 1 & 2 & 4 \\ 1 & 2 & \\ 2 & & \end{matrix} \qquad Q_6 = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{matrix}$$

$$P_7 = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & & \end{matrix} \qquad Q_7 = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & & \end{matrix}$$

In this case we have mapped the set to the pair (P_7, Q_7), which are arrays of the same shape. However, note that Q_7 is a tableau, but P_7 is row-strict and column-weak, so if we reflect P_7 about the main diagonal, then we obtain a tableau of conjugate shape to Q_7. Note that, by construction, P_m contains $j_1, \ldots, j_m$ in its cells, and Q_m contains $i_1, \ldots, i_m$ in its cells. The proof that Q_ℓ and the reflection of P_ℓ are tableaux of conjugate shape for every ℓ is similar to the proof given on pages 23 and 24 for the corresponding multiset result.

The consequence is that this mapping is a bijection, and we thus obtain (with a_{ij} the frequency of the ordered pair (i,j) in the set)

$$\sum_\lambda s_\lambda(x_1, x_2, \ldots) s_{\lambda'}(y_1, y_2, \ldots) = \sum_{\substack{0 \le a_{rs} \le 1, \\ r,s \ge 1}} \prod_{i,j \ge 1} (x_i y_j)^{a_{ij}} = \prod_{i,j \ge 1} \sum_{0 \le a_{ij} \le 1} (x_i y_j)^{a_{ij}},$$

giving the result. □

The combinatorial bijection given in the above proof is usually referred to as the *dual Robinson-Schensted-Knuth* correspondence.

It is now straightforward to determine the action of ω on Schur functions, and we do so in the following result.

Theorem 12.2. *For all partitions λ, we have*

$$\omega(s_\lambda) = s_{\lambda'}.$$

Proof. Let ω_y denote the action of the involution ω on symmetric functions in indeterminates $y_1, \ldots$. Then applying ω_y to Theorem 8.2, we obtain

$$\begin{aligned}\sum_\lambda s_\lambda(x_1,\ldots)\omega_y\left(s_\lambda(y_1,\ldots)\right) &= \omega_y \prod_{i,j\geq 1}(1-x_iy_j)^{-1} = \omega_y \prod_{i\geq 1}\sum_{k_i\geq 0} h_{k_i}(y_1,\ldots)x_i^{k_i} \\ &= \prod_{i\geq 1}\sum_{k_i\geq 0} e_{k_i}(y_1,\ldots)x_i^{k_i} = \prod_{i,j\geq 1}(1+x_iy_j),\end{aligned}$$

and the result follows immediately from Theorem 12.1 □

Now, to establish how ω acts on skew Schur functions, suppose we have

$$s_{\lambda/\mu} = \sum_\nu a_{\lambda,\mu,\nu} s_\nu. \tag{43}$$

Then, taking inner products of both sides with s_ν, we obtain

$$\langle\, s_{\lambda/\mu}, s_\nu \,\rangle = a_{\lambda,\mu,\nu},$$

so from Theorem 9.1, we get

$$\langle\, s_\lambda, s_\mu\, s_\nu \,\rangle = a_{\lambda,\mu,\nu}.$$

If we apply ω on both sides inside the inner product and use the fact that ω is an isometry for the inner product, we then have

$$\langle\, s_{\lambda'}, s_{\mu'}\, s_{\nu'} \,\rangle = a_{\lambda,\mu,\nu},$$

and applying Theorem 9.1 again, we obtain

$$\langle\, s_{\lambda'/\mu'}, s_{\nu'} \,\rangle = a_{\lambda,\mu,\nu}.$$

13. Lecture of June 7

Now, from this equation we immediately have

$$s_{\lambda'/\mu'} = \sum_\lambda a_{\lambda,\mu,\nu} s_{\nu'}.$$

But, applying ω directly to (43), we obtain

$$\omega\left(s_{\lambda/\mu}\right) = \sum_\nu a_{\lambda,\mu,\nu} s_{\nu'},$$

and comparing these two equations gives

$$\omega\left(s_{\lambda/\mu}\right) = s_{\lambda'/\mu'}, \tag{44}$$

(which of course specializes to Theorem 12.2 when μ is the empty partition).

For a completely different pathway to establishing (44), without using Theorems 12.1 and 12.2, see Assignment 2. In Problem 4, you are asked to prove that the Gessel-Viennot nonintersecting path methodology described on pages 9 – 11 of the Course Notes can be used to prove that the Jacobi-Trudi determinant for the Schur function extends to skew shapes, and has a version involving elementary symmetric functions instead of complete symmetric functions. Then you are asked to deduce (44) from these determinantal results.

Now, returning to the question of what happens when we apply ω to the Pieri Rule (42). The result is

$$e_k\, s_{\mu'} = \sum_{\lambda} s_{\lambda'},$$

where the summation is over all λ such that λ/μ is a horizontal strip of size k. If we replace μ by μ' and λ by λ', we obtain

$$e_k\, s_{\mu} = \sum_{\lambda'} s_{\lambda},$$

where the summation is over all λ' such that λ'/μ' is a horizontal strip of size k. But, translating the summation condition for conjugates (which interchanges rows and columns), we get

$$e_k\, s_{\mu} = \sum_{\lambda} s_{\lambda}, \tag{45}$$

where the summation is over all λ such that λ/μ is a vertical strip of size k (where a *vertical* strip is a skew shape in which no *row* contains more that n one cell). Equation (45) is often referred to as the *dual* Pieri rule, or simply the Pieri rule for a single column (since e_k is the Schur function for a partition whose diagram is a single column with k cells).

As examples of the Pieri formula and its dual, when $\mu = (3,1)$ and $k = 2$, we obtain:

$$h_2\, s_{(3.1)} = s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + s_{(3,2,1)},$$
$$e_2\, s_{3,1)} = s_{(4,2)} + s_{(4,1,1)} + s_{(3,2,1)} + s_{(3,1,1,1)}.$$

We next consider the expansion of $p_k\, s_\mu$ as a linear combination of Schur functions, say of the form

$$p_k\, s_{\mu} = \sum_{\lambda} c_\lambda s_{\lambda}, \tag{46}$$

where k is a positive integer and μ is a partition. We will consider the symmetric functions in a finite collection of variables $x_1, \dots, x_n$, where the choice of n for any particular values of k and μ will be explained. In fact, we will return to our starting point for Schur functions, the ratio of determinants formula given in (10). It will be convenient to introduce some special notation: For an arbitrary n-tuple of integers $v = (v_1, \dots, v_n)$, we define

$$a_v = \det\left(x_i^{v_j}\right)_{i,j=1,\dots,n}. \tag{47}$$

This determinant is often referred to as an *alternant* – it is not a symmetric function since its value is equal to the sign of σ when $x_1, \dots, x_n$ are replaced by $x_{\sigma(1)}, \dots, x_{\sigma(n)}$. Let $\delta_n = (n-1, n-2, \dots, 1, 0)$ be the particular n-tuple of integers with jth entry equal to $n-j$, $j = 1, \dots, n$. Then for any $s = 1, \dots, n-1$ and any n-tuple of integers $v = (v_1, \dots, v_n)$, we obtain the relationship

$$a_{v+\delta_n} = -a_{(v_1,\dots,v_{s-1},v_{s+1}-1,v_s+1,v_{s+2},\dots,v_n)+\delta_n}, \tag{48}$$

by interchanging columns s and $s+1$ in the determinant.

But, using the above alternant notation, (10) can be written

$$s_\lambda = \frac{a_{\lambda+\delta_n}}{a_{\delta_n}},$$

where λ is a partition with at most n parts, and on the right hand side above we interpret λ as the n-tuple $(\lambda_1, \dots, \lambda_n)$, possibly terminating with 0's (when the number of parts is less than n). Of course, in this case, λ an n-tuple of nonnegative integers whose entries appear in weakly decreasing order from left to right. Replacing the Schur functions in (46) by this ratio of determinants, and multiplying on both sides by the Vandermonde determinant a_{δ_n}, we obtain

$$p_k\, a_{\mu+\delta_n} = \sum_{\lambda} c_\lambda a_{\lambda+\delta_n}, \tag{49}$$

and we proceed by considering this expansion in terms of alternants. Thus, we have

$$\begin{aligned} p_k\, a_{\mu+\delta_n} &= \sum_{\ell=1}^{n} x_\ell^k \sum_{\sigma\in S(n)} \mathrm{sgn}\,(\sigma) \prod_{j=1}^{n} x_{\sigma(j)}^{\mu_j+n-j} \\ &= \sum_{\sigma\in S(n)} \mathrm{sgn}\,(\sigma) \sum_{\ell=1}^{n} x_{\sigma(\ell)}^k \prod_{j=1}^{n} x_{\sigma(j)}^{\mu_j+n-j} \\ &= \sum_{\ell=1}^{n} a_{\mu+k\,\varepsilon_\ell+\delta_n}, \end{aligned}$$

where $\varepsilon_\ell = (0,\ldots,0,1,0,\ldots,0)$ is the n-tuple with the ℓth entry equal to 1, and all other entries equal to 0, for $\ell = 1,\ldots,n$. Consider the alternant $a_{\mu+k\,\varepsilon_\ell+\delta_n}$ that arises in the sum on the right hand side. For $\ell = 1,\ldots,n$, the entries in the n-tuple

$$\mu + k\,\varepsilon_\ell = (\mu_1,\ldots,\mu_{\ell-1},\mu_\ell + k,\mu_{\ell+1},\ldots,\mu_n)$$

will be weakly decreasing (and hence $\mu+k\,\varepsilon_\ell$ will be a partition), if and only if $\mu_{\ell-1} \ge \mu_\ell + k$. Otherwise, if $\mu_{\ell-1} < \mu_\ell + k$, we will apply (49) with $s = \ell - 1$, and continue to apply (49) moving to the left in the n-tuple, attempting to make it weakly decreasing left to right. As a result, for $\ell = 1,\ldots,n$, we are able to write $a_{\mu+k\,\varepsilon_\ell+\delta_n}$ as $(-1)^{i-1} a_{\lambda+\delta_n}$ where

$$\lambda = (\mu_1,\ldots,\mu_{\ell-i},\mu_\ell + k - i + 1,\mu_{\ell-i+1} + 1,\ldots,\mu_{\ell-1} + 1,\mu_{\ell+1},\ldots,\mu_n)$$

for some $1 \le i \le \ell$, where $\mu_\ell + k - i + 1 \ge \mu_{\ell-i+1} + 1$, in which one of the following occurs:

Case 1: $i = \ell$,

Case 2: $\mu_{\ell-i} \ge \mu_\ell + k - i + 1$

Case 3: $\mu_{\ell-i} + 1 = \mu_\ell + k - i + 1$

In Case 3, λ is not weakly decreasing, but it is impossible to make it weakly decreasing since if we apply (48) with $s = \ell - i$, then we obtain $a_{\lambda+\delta_n} = -a_{\lambda+\delta_n}$, so we conclude that $a_{\lambda+\delta_n} = 0$ (indeed the corresponding determinant has column $\ell - i$ equal to column $\ell - i + 1$, which gives a direct reason that it has value equal to 0).

In Cases 1 and 2, the skew shape λ/μ has k cells, and is edge-connected but with no 2×2 subset of cells; these are called *border strips* of size k. In the notation above, the border strip has i rows, and in this situation we say that teh border strip has *height* (usually denoted "ht") equal to $i - 1$.

As long as we select $n = m + k$, where m is the number of parts in μ, then we obtain all possible choices of λ; if $n > m + k$, then μ terminates with more than k 0's, and we would thus be in Case 3 for each $\ell > m+k$, with $\lambda = (\mu_1,\ldots,\mu_m,0,\ldots,0,1,\ldots 1)$, which terminates with exactly k 1's.

We conclude that

$$p_k\, s_\mu = \sum_\lambda (-1)^{\mathrm{ht}\,(\lambda/\mu)} s_\lambda, \tag{50}$$

where the summation is over all partitions λ such that λ/μ is a border strip of size k. This is referred to as the *Murnaghan-Nakayama Rule.* For example, with $k = 3$ and $\mu = (4, 3, 1)$, we obtain

$$p_3\, s_{(4,3,1)} = s_{(7,3,1)} - s_{(5,5,1)} - s_{(4,3,2,2)} + s_{(4,3,1,1,1,1)}.$$

14. Lecture of June 14

Applying (50) iteratively, for an arbitrary monomial $p_\alpha = p_{\alpha_1} \cdots p_{\alpha_m}$ in the power sums, and an arbitrary partition μ, we obtain the formula

$$p_\alpha\, s_{\lambda^{(0)}} = \sum (-1)^{\mathrm{ht}\,(\lambda^{(1)}/\lambda^{(0)}) + \ldots + \mathrm{ht}\,(\lambda^{(m)}/\lambda^{(m-1)})} s_{\lambda^{(m)}}, \tag{51}$$

where the summation is over all partitions $\lambda^{(1)}, \ldots, \lambda^{(m)}$ such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a border strip of size α_i, for $i = 1, \ldots, m$. In order to write (51) more compactly, we now introduce a single object that captures all the above information about the sequence of partitions $\lambda^{(0)}, \ldots, \lambda^{(m)}$. Suppose we place integer i in each cell of the border strip $\lambda^{(i)}/\lambda^{(i-1)}$, to obtain B_i, and draw all of $B_1, \ldots, B_m$ in a single figure T. Note that the cells containing i and j for different i and j are disjoint, and that T is simply the skew shape $\lambda^{(m)}/\lambda^{(0)}$ with one of the positive integers $1, \ldots, m$ placed in each cell. In particular, there are α_i occurrences of i for $i = 1, \ldots, m$. We then say that T is a *border strip tableau* of (skew) *shape* $\lambda^{(m)}/\lambda^{(0)}$ and *type* α (note that α need not be a partition, because the parts can appear in any order). For example, one border strip tableau of skew shape $(8, 8, 5, 4, 4, 1)/(2, 1, 1)$ and type $(3, 6, 3, 4, 2, 5, 3)$ is given by

$$\begin{array}{cccccccc} & & 1 & 2 & 2 & 6 & 6 & 7 \\ & 1 & 1 & 2 & 6 & 6 & 7 & 7 \\ & 2 & 2 & 2 & 6 & & & \\ 3 & 3 & 3 & 5 & & & & \\ 4 & 4 & 4 & 5 & & & & \\ 4 & & & & & & & \end{array}.$$

We define the height $\mathrm{ht}\,(T)$ of the border strip tableau T to be the sum over $i = 1, \ldots, m$ of the heights of the border strips B_i, so for example the height of the border strip tableau given above is $1 + 2 + 0 + 1 + 1 + 2 + 1 = 8$. Using this notation, we can write (51) (with $\lambda^{(0)}$ and $\lambda^{(m)}$ replaced by the more generic μ and λ, respectively) as

$$p_\alpha\, s_\mu = \sum_\lambda s_\lambda \sum_T (-1)^{\mathrm{ht}\,(T)},$$

where the inner summation is over all border strip tableaux T of skew shape λ/μ, and type α. The special case in which μ is the empty partition is of particular interest, giving

$$p_\alpha = \sum_\lambda s_\lambda \sum_T (-1)^{\mathrm{ht}\,(T)}, \tag{52}$$

where the inner summation is over all border strip tableaux T of shape λ, and type α. As an example of (52), consider the case $m = 2$ and $\alpha_1 = 3$, $\alpha_2 = 2$. Then there are eight border strip tableaux of type $(3, 2)$, given by

$$\begin{matrix} 1 & 1 & 1 \\ 2 & & \\ 2 & & \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 & 2 & 2 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & \\ 2 & \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & & & \end{matrix} \qquad \begin{matrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 1 & 2 \\ 1 & \end{matrix} \qquad \begin{matrix} 1 & 2 & 2 \\ 1 & & \\ 1 & & \end{matrix} .$$

Thus formula (52) in this case gives

$$p_3\, p_2 = s_{(3,2)} + s_{(5)} + s_{(2,1,1,1)} - s_{(4,1)} - s_{(1,1,1,1,1)} - s_{(2,2,1)},$$

in which the Schur function $s_{(3,1,1)}$ doesn't appear because it has coefficient $1 - 1 = 0$. For comparison, consider the case in which the α_i appear in a different order, say $\alpha_1 = 2$ and $\alpha_2 = 3$. then there are *six* border strip tableaux of type $(2, 3)$, given by

$$\begin{matrix} 1 & 1 \\ 2 & \\ 2 & \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 2 & 2 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 1 & 2 & 2 & 2 \end{matrix} \qquad \begin{matrix} 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{matrix} \qquad \begin{matrix} 1 & 2 & 2 \\ 1 & 2 & \end{matrix} \qquad \begin{matrix} 1 & 2 & 2 & 2 \\ 1 & & & \end{matrix} ,$$

and formula (52) in this case gives

$$p_2\, p_3 = s_{(2,1,1,1)} - s_{(2,2,1)} + s_{(5)} - s_{(1,1,1,1,1)} + s_{(3,2)} - s_{(4,1)}.$$

Of course, these have exactly the same linear combinations of Schur functions on the right hand sides, because $p_3\, p_2 = p_2\, p_3$, though the linear combinations come about in different ways as a sum over border strip tableaux – in the second case the Schur function $s_{(3,1,1)}$ has coefficient 0 because it simply doesn't appear at all in that summation over border strip tableaux. In general we restrict attention in (52) to the case in which α is a partition.

Note that an equivalent way of writing (52) is to use the inner product for symmetric functions, so, using the orthonormality of the s_λ, we obtain

$$\langle\, p_\alpha, s_\lambda \,\rangle = \sum_T (-1)^{\mathrm{ht}\,(T)}, \tag{53}$$

where the inner summation is over all border strip tableaux T of shape λ, and type α. But then, using the orthogonality of the p_α, we also obtain

$$s_\lambda = \sum_\alpha \frac{\langle\, p_\alpha, s_\lambda \,\rangle}{\langle\, p_\alpha, p_\alpha \,\rangle} p_\alpha = \sum_\alpha \langle\, p_\alpha, s_\lambda \,\rangle \frac{p_\alpha}{z(\alpha)}, \tag{54}$$

using the notation $z(\alpha)$ defined on page 23 of the Course Notes. Formula (54) in conjunction with (53) requires us to consider the border strip tableaux of fixed shape λ, with type α that varies. For example, in the case that $\lambda = (2, 1)$, there are five border strip tableaux of shape $(2, 1)$, given by

$$\begin{matrix} 1 & 1 \\ 1 & \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 1 & \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 3 & \end{matrix} \qquad \begin{matrix} 1 & 3 \\ 2 & \end{matrix} .$$

Hence we obtain the expression

$$s_{(2,1)} = -\frac{p_3}{3} + (1-1)\frac{p_2\, p_1}{2} + (1+1)\frac{p_1^3}{3!} = \frac{p_1^3}{3} - \frac{p_3}{3}$$

for the Schur function $s_{(2,1)}$ expanded in power sum monomials.

The reason that so much attention has been focussed on these expansions relating the Schur function and power sum bases for symmetric functions is that the coefficient $\langle p_\alpha, s_\lambda \rangle$ features centrally in the representation theory of the symmetric groups $S(n)$, $n \geq 0$ – in particular, for α and λ partitions of n, the coefficient $\langle p_\alpha, s_\lambda \rangle$ is the *character* of the irreducible representation of $S(n)$ indexed by λ, evaluated at any element of the conjugacy class of $S(n)$ indexed by α. In this context, expression (53) is referred to as the Murnaghan-Nakayama Rule for the characters.

We now consider the basics of representation theory for finite groups, especially the symmetric group. For a finite group G, a linear representation is a family of square invertible matrices M_g, $g \in G$, such that $M_g M_h = M_{g \cdot h}$ (here G is a multiplicative group). That is, the *matrix* multiplication of these matrices is homomorphic with the *group* multiplication of their corresponding group elements. If the matrices are $m \times m$, then we say that m is the *dimension* or *degree* of the representation. Note that this implies $M_e = I_m$ for the identity element e in G, and $M_{g^{-1}} = (M_g)^{-1}$ for any $g \in G$. For any invertible $m \times m$ matrix P, clearly PM_gP^{-1}, $g \in G$ is also a representation. Also, if N_g is also a representation (with dimension not necessarily equal to m) then their block diagonal direct sum $N_g \oplus M_g$, $g \in G$ is another representation.

A key result of representation theory for finite groups is that all representations can be formed uniquely from a special finite set of *irreducible* representations using the above two constructions (similarity via some P, and direct sum of not necessarily different representations). There is one irreducible representation for each conjugacy class in the group. Then the *character* of the representation evaluated at $g \in G$ is equal to trace(M_g). Note that all of the representations obtained from M_g, $g \in G$ by similarity have the same character, since $\operatorname{trace} PM_gP^{-1} = \operatorname{trace} M_g$ for any invertible P (for example, this is an easy consequence of the result for $m \times k$ and $k \times m$ matrices $A_{m\times k}$ and $B_{k\times m}$ that $\operatorname{trace} A\,B = \operatorname{trace} B\,A$).

15. Lecture of June 19

In a finite group G, two group elements a and b are *conjugate* when there exists $x \in G$ such that $b = x \cdot a \cdot x^{-1}$. Note that a and a are conjugate since $a = id \cdot a \cdot id^{-1}$. Note also that b and a are conjugate since, multiplying the equation $b = x \cdot a \cdot x^{-1}$ on the left by x^{-1} and on the right by x, we obtain $a = x^{-1} \cdot b \cdot x = x^{-1} \cdot b \cdot (x^{-1})^{-1}$. Finally, suppose that we have $c = y \cdot b \cdot y^{-1}$, so b and c are conjugate. Then, substituting for b, we obtain $c = y \cdot x \cdot a \cdot x^{-1} \cdot y^{-1} = (y \cdot x) \cdot a \cdot (y \cdot x)^{-1}$, and we conclude that a and c are conjugate. Hence, considered as a binary relation, conjugacy is reflexive, symmetric and transitive, which implies that it is an *equivalence relation*. The elements of the group are thus partitioned into equivalence classes consisting of the elements that are conjugate with each other, called *conjugacy classes*.

Now consider an *abelian* group, like the cyclic group, in which the group multiplication is commutative, so $a \cdot b = b \cdot a$ for all group elements a and b. Then if a and b are conjugate, we have $b = x \cdot a \cdot x^{-1} = a \cdot x \cdot x^{-1} = a \cdot id = a$, so a and b must be equal, which means that all conjugacy classes have size 1.

Similarly, in any finite group G, the conjugacy class containing the identity element id has size 1, since if id and b are conjugate, then for some group element x we have $b = x \cdot id \cdot x^{-1} = x \cdot x^{-1} = id$.

The main example of a finite group we shall consider is the symmetric group $S(n)$ consisting of all permutations of $\{1,\ldots,n\}$, for some fixed $n \geq 0$. Consider the two row representation of a permutation $\sigma \in S(n)$, with $1,2,\ldots,n$ in the first row, and the permuted symbols $\sigma(1),\ldots,\sigma(n)$ in then second row. For example, when $n = 7$, one such two row representation is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 1 & 7 & 5 & 3 \end{pmatrix}.$$

Now regard this permutation σ as a bijective function on the set $\{1,\ldots,n\}$, and construct the *functional digraph* of σ: the vertex-set is $\{1,\ldots,n\}$, and there is an edge directed from i to $\sigma(i)$ for each $i = 1,\ldots,n$. Each connected component of the resulting directed graph is a directed cycle, in which the cycles have length 1 or more, and the directed graph is referred to as the *disjoint cycle representation* of the permutation. For example, the disjoint cycle representation of the permutation in $S(7)$ displayed above has a 3-cycle that sends 1 to 2 to 4, and then back to 1, and a 4-cycle that sends 3 to 6 to 5 to 7, and then back to 3.

Now suppose that a and b are conjugate in $S(n)$, with $b = \sigma a \sigma^{-1}$. If we let $a(i) = j$, then there is an edge directed from vertex i to vertex j in the functional digraph of a. Then we have

$$b(\sigma(i)) = \sigma a \sigma^{-1}(\sigma(i)) = \sigma(a(i)) = \sigma(j),$$

and so we have an edge directed from vertex $\sigma(i)$ to vertex $\sigma(j)$ in the functional digraph of b. But this means that the functional digraph of b is obtained from the functional digraph of a by relabelling vertex i by $\sigma(i)$ for all $i = 1,\ldots,n$. This relabelling cannot change the underlying structure of the digraph, and in particular preserves the unordered collection of lengths of the disjoint cycles. Thus the conjugacy classes of $S(n)$ are indexed by the partitions of n (in which the parts specify the lengths of the disjoint cycles). To determine the size of the conjugacy class with cycle lengths given by partition λ, suppose that λ has i_j parts of size equal to j for each $j \geq 1$. Then from the theory of exponential generating series, the number of permutations in this conjugacy class is given by

$$\left[p_1^{i_1} p_2^{i_2} \cdots \frac{t^n}{n!}\right] \exp\left(\sum_{k\geq 1}(k-1)!\,p_k \frac{t^k}{k!}\right) = \left[p_1^{i_1} p_2^{i_2} \cdots \frac{t^n}{n!}\right] \exp\left(\sum_{k\geq 1} p_k \frac{t^k}{k}\right)$$
$$= \left[p_1^{i_1} p_2^{i_2} \cdots \frac{t^n}{n!}\right] \prod_{k\geq 1} \exp\left(p_k \frac{t^k}{k}\right) = \left[\frac{t^n}{n!}\right] t^n \prod_{k\geq 1} \frac{1}{k^{i_k} i_k!} = \frac{n!}{z(\lambda)},$$

again using the notation $z(\lambda)$ that appears at the top of page 23 of the Course Notes.

Now back to irreducible representations and characters. If g and h are conjugate in G, we have $g = x \cdot h \cdot x^{-1}$ for some $x \in G$, so $M_g = M_x M_h \left(M_x\right)^{-1}$, which gives trace $M_g =$ trace M_h, so characters are constant on conjugacy classes. We suppose that the group G has conjugacy classes $C^{(i)}$, $i = 1,\ldots,k$ (so, for example $\sum_{i=1}^{k} |C^{(i)}| = |G|$). The character of the irreducible representation indexed by the ith conjugacy class is denoted by $\chi^{(i)}$, $i = 1,\ldots,k$, and the value of this character evaluated at any element of conjugacy class $C^{(j)}$ is given by $\chi^{(i)}(j)$, $j = 1,\ldots,k$ (where convenient, we shall also use the notation $\chi^{(i)}(g)$ for the value of $\chi^{(i)}$ evaluated at group element g). The degree of $\chi^{(i)}$ is denoted by $f^{(i)}$, $i = 1,\ldots,k$. Note that $f^{(i)} = \chi^{(i)}(id)$, since the trace of an $m \times m$ identity matrix is equal to m. A basic fact about representations of finite groups is that

$$\sum_{i=1}^{k} \left(f^{(i)}\right)^2 = |G|.$$

This implies immediately for the cyclic group that $f^{(i)} = 1$ for all $i = 1, \ldots, k$, since in this case $k = |G|$. For the symmetric group, it turns out that $f^{(i)}$ is equal to the number of Young tableaux whose shape is give by the ith indexed partition of n.

Suppose that we have an arbitrary representation M_g, $g \in G$ of the group G, with character $\Lambda(g)$. Since all representations are formed by direct sums and similarity from irreducible representations, then we have $\Lambda(g) = \sum_{i=1}^{k} n_i \chi^{(i)}(g)$ for some nonnegative integers n_i, $i = 1, \ldots, k$ (i.e., we have taken the direct sum of n_i copies of the irreducible representation with character $\chi^{(i)}$, $i = 1, \ldots, k$ – using the fact that the trace of the direct sum of square matrices A and B is equal to trace A + trace B).

How do we create representations? We'll now consider a classical construction for the symmetric group due to Alfred Young. In particular we will construct an irreducible representation of $S(n)$ corresponding to the partition λ of n, for each such λ. the degree of this representation $f^{(\lambda)}$ is the number of Young tableaux of shape λ. Suppose S and T are two different Young tableaux of the same shape, and suppose that i appears in different cells of S and T. Then we say i is a *disagreeing* number between S and T. The *last-letter order* for the Young tableaux of the same shape is defined in terms of disagreeing numbers as follows: If ℓ is the largest disagreeing number between S and T, and ℓ appears in a lower indexed row of S than T, then we say that $S < T$. Let the vector space V_λ have ordered basis $(T_1, \ldots, T_{f^{(\lambda)}})$, arranged from left to right in increasing last-letter order. For example, if $\lambda = (3, 2)$, then there are 5 Young tableaux of shape λ, and their last-letter order is displayed below:

$$\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix} \quad < \quad \begin{matrix} 1 & 2 & 5 \\ 3 & 4 & \end{matrix} \quad < \quad \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \end{matrix} \quad < \quad \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix} \quad < \quad \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix} \; .$$

$$T_1 \qquad\qquad T_2 \qquad\qquad T_3 \qquad\qquad T_4 \qquad\qquad T_5$$

Also, for a Young tableaux with n cells and $k = 1, \ldots, n-1$, we define an action of the adjacent transposition $(k, k+1)$ on T by saying that $(k, k+1)T$ is the object obtained from T by interchanging the places in which k and $k+1$ appear – this is a Young tableaux if k and $k+1$ do not appear in the same row or column of T, but otherwise it is NOT a Young tableau. For example, $(3,4)T_1$, $(3,4)T_2$, $(3,4)T_3$ are not Young tableaux, but $(3,4)T_4 = T_5$ and $(3,4)T_5 = T_4$, and $(2,3)T_1 = T_2$, $(2,3)T_2 = T_1$, etc.

Finally, for $a, b \in \{1, \ldots, n\}$ and a Young tableau T with n cells, we define

$$d_T(a, b) = c(y) - c(x), \tag{55}$$

where a appears in cell x, b appears in cell y, and $c(x)$ and $c(y)$ are the *contents* of cells x and y, respectively.

16. Lecture of June 21

Now we define a linear transformation $\rho_\lambda((k,k+1))$ for each $k = 1,\ldots,n-1$ and each basis element T by

(56)
$$\rho^{(\lambda)}((k,k+1))T = \begin{cases} a(T,k)T + b(T,k)T', & \text{if } T' = (k,k+1)T \quad \text{is a Young tableau,} \\ a(T,k)T, & \text{otherwise,} \end{cases}$$

where $a(T,k)$ and $b(T,k)$ are scalars. The following result is due to Alfred Young.

Theorem 16.1. *A representation of $S(n)$ is obtained from* (56) *in the following two cases:*

(A) $a(T,k) = \frac{1}{d_T(k,k+1)}$ *and* $b(T,k) = \sqrt{1 - a(T,k)^2}$,

(B) $a(T,k) = \frac{1}{d_T(k,k+1)}$ *and* $b(T,k) = \begin{cases} 1 - a(T,k)^2, & \text{if } T < T', \\ 1 & \text{otherwise.} \end{cases}$

The representation given in *(A)* above is known as Young's *orthogonal representation*, and the representation given in *(B)* above is known as Young's *seminormal representation*. As an example of *(B)* with $\lambda = (3,2)$ and $k = 2$, we get

$$\rho^{(\lambda)}((2,3))T_1 = \frac{1}{2}T_1 + \frac{3}{4}T_2$$
$$\rho^{(\lambda)}((2,3))T_2 = T_1 - \frac{1}{2}T_2$$
$$\rho^{(\lambda)}((2,3))T_3 = \frac{1}{2}T_3 + \frac{3}{4}T_4$$
$$\rho^{(\lambda)}((2,3))T_4 = T_3 - \frac{1}{2}T_4$$
$$\rho^{(\lambda)}((2,3))T_5 = T_5$$

If we express this as a matrix acting on the ordered basis by matrix multiplication on the *right* of the row vector $(T_1,\ldots,T_{f^{(\lambda)}})$, we get the matrix

$$M^{(\lambda)}((2,3)) = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} & 0 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Of course, this is a single matrix corresponding to the adjacent transposition $(2,3)$ in $S(5)$ (and since this matrix has trace equal to 1, the character evaluated at $(2,3)$ has value 1). To have a matrix representation of $S(5)$, we need one 5×5 matrix for each of the $5! = 120$ permutations in $S(5)$. Of course, we also know that $M^{(\lambda)}(id)$ is the 5×5 identity matrix. Using the construction above, we can also obtain the matrices $M^{(\lambda)}((1,2))$, $M^{(\lambda)}((3,4))$, $M^{(\lambda)}((4,5))$. Then, we can write each remaining permutation σ in $S(5)$ as a product of the adjacent transpositions $(1,2),(2,3),(3,4),(4,5)$, and thus create the 5×5 matrix $M^{(\lambda)}(\sigma)$ as the same product of the corresponding matrices $M^{(\lambda)}((1,2)), M^{(\lambda)}((3,4)), M^{(\lambda)}((3,4)), M^{(\lambda)}((4,5))$. Since we are using matrix multiplication on the right, we will need to write these products of permutations so that the first permutation to be applied is leftmost, proceeding from left to right until the final permutation to be applied is rightmost. Thus for a permutation σ we will write the image of i under the action of σ to be $(i)\sigma$ rather than the more conventional $\sigma(i)$ to emphasize that products act from left to right; hence we write $(i)\sigma\tau$ to mean $((i)\sigma)\tau$.

How do we know that this gives us a representation of $S(5)$? – e.g., what conditions do the matrices representing the adjacent transpositions have to satisfy? The answer is that the $m \times m$ matrices A_k chosen to represent the adjacent transposition $(k, k+1)$, $k = 1, \ldots, n-1$, are subject to necessary and sufficient conditions called the *Coxeter* relations, stated below:

(1) $A_k^2 = I_{m\times m}$, for $k = 1, \ldots, n-1$,
(2) $A_k A_j = A_j A_k$, for $|k-j| \geq 2$,
(3) $A_k A_{k+1} A_k = A_{k+1} A_k A_{k+1}$, for $k = 1, \ldots, n-2$.

Note that the product on both sides of condition (3) above is equal to the transposition $(k, k+2)$.

In Problem 1 of Assignment 3, you will be asked to check that in general Young's seminormal representation does determine a representation of $S(n)$, by checking the Coxeter relations are satisfied for each partition λ.

Now we will construct all of Young's seminormal representations for $S(3)$, by considering the partitions (3), $(1,1,1)$, and $(2,1)$ of 3, in turn. First, for $\lambda = (3)$, there is a single Young tableau of shape (3), namely $T_1 = 1\,2\,3$, so the matrix representation will be of degree 1. Now, we have $d_{T_1}(1,2) = d_{T_1}(2,3) = 1$, which gives $\rho^{(3)}((1,2))T_1 = \rho^{(3)}((2,3))T_1 = T_1$, and hence we have 1×1 matrices (written as scalars) $M^{(3)}((1,2)) = M^{(3)}((2,3)) = 1$, as well as $M^{(3)}(id) = 1$. If we write the 3 remaining permutations in $S(3)$ as products of adjacent transpositions, and then create their matrix as the corresponding product of 1's, we will obtain $M^{(3)}(\sigma) = 1$ and hence $\chi^{(3)}(\sigma) = \operatorname{sgn}(\sigma)$ for all permutations $\sigma \in S(3)$. This is called the *trivial* representation. It is straightforward to check that Young's seminormal representation for $S(n)$ corresponding to the partition (n) determines the trivial representation for all n.

Second, for $\lambda = (1,1,1)$, there is a single Young tableau of shape $(1,1,1)$, namely

$$T_1 = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\ ,$$

so in this case the matrix representation will also be of degree 1. Now, we have $d_{T_1}(1,2) = d_{T_1}(2,3) = -1$, which gives $\rho^{(1,1,1)}((1,2))T_1 = \rho^{(1,1,1)}((2,3))T_1 = -T_1$, and hence we have 1×1 matrices (written as scalars) $M^{(1,1,1)}((1,2)) = M^{(1,1,1)}((2,3)) = -1$, as well as $M^{(1,1,1)}(id) = 1$. If we write the 3 remaining permutations in $S(3)$ as products of adjacent transpositions, and then create their matrix as the corresponding product of -1's, we will obtain $M^{(3)}(\sigma) = \operatorname{sgn}(\sigma)$ and hence $\chi^{(3)}(\sigma) = 1$ for all permutations $\sigma \in S(3)$. This is called the *sign* representation. It is straightforward to check that Young's seminormal representation for $S(n)$ corresponding to the partition $(1, \ldots, 1)$ determines the sign representation for all n.

Third, for $\lambda = (2,1)$, there are two Young tableaux of shape $(2,1)$, and in last-letter order they are given by

$$\begin{array}{cc} 1 & 3 \\ 2 & \end{array} < \begin{array}{cc} 1 & 2 \\ 3 & \end{array},$$

$$T_1 \qquad\qquad T_2$$

so the matrix representation will be of degree 2. Now, neither $(1,2)T_1$ nor $(1,2)T_2$ are Young tableaux, but we do have $(2,3)T_1 = T_2$ and $(2,3)T_2 = T_1$. Moreover, we have $d_{T_1}(1,2) = -1$, $d_{T_1}(2,3) = 2$, $d_{T_2}(1,2) = 1$ and $d_{T_2}(2,3) = -2$, so we obtain

$$\rho^{(2,1)}((1,2))T_1 = -T_1, \qquad \rho^{(2,1)}((1,2))T_2 = T_2,$$

$$\rho^{(2,1)}((2,3))T_1 = \frac{1}{2}T_1 + \frac{3}{4}T_2, \qquad \rho^{(2,1)}((2,3))T_2 = T_1 - \frac{1}{2}T_2,$$

and hence we have 2×2 matrices

$$M^{(2,1)}((1,2)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M^{(2,1)}((2,3)) = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{1}{2} \end{pmatrix},$$

and of course $M^{(2,1)}(id)$ is the 2×2 identity matrix. But (multiplying left to right!) we have $(1,2)(2,3) = (1,3,2)$, $(2,3)(1,2) = (1,2,3)$, and $(1,2)(2,3)(1,2) = (1,3)$, and multiplying the corresponding matrices gives

$$M^{(2,1)}((1,3,2)) = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{4} \\ 1 & -\frac{1}{2} \end{pmatrix}, \qquad M^{(2,1)}((1,2,3)) = \begin{pmatrix} -\frac{1}{2} & \frac{3}{4} \\ -1 & -\frac{1}{2} \end{pmatrix},$$

$$M^{(2,1)}((1,3)) = \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} \\ -1 & -\frac{1}{2} \end{pmatrix},$$

(where, e.g., we have used $(1,2,3)$ to denote the directed cycle σ with $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$). Thus we have character values $\chi^{(2,1)}(id) = 2$, $\chi^{(2,1)}((2,1)) = 0$ for the three transpositions (which belong to the conjugacy class with cycle lengths specified by the parts of the partition $(2,1)$), and $\chi^{(2,1)}((3)) = -1$ for the two 3-cycles (which belong to the conjugacy class with cycle lengths specified by the parts of the partition (3)).

Thus we have constructed three representations of $S(3)$ and determined the values of their characters. Of course, the first two of these representations are irreducible since they are of degree 1. It turns out that the third, which is of degree 2, is also irreducible (and indeed, that all of Young's seminormal representations are irreducible). In general, having constucted a representation and hence the character of that representation, how can we determine if it is irreducbile or not? A device that is convenient for this purpose is the *group inner product*, defined as follows. Let ϕ, ψ be two functions (say, complex valued) defined on a finite group G. Then we define

$$\langle \phi , \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\psi(g^{-1}). \tag{57}$$

Note that $\langle \phi , \psi \rangle = \langle \psi , \phi \rangle$, which follows easily by changing the summation variable above to g^{-1}.

It is a fundamental fact about the irreducible characters $\chi^{(i)}$, $i = 1, \ldots, k$ (recall that in general G has k conjugacy classes) is the orthogonality relation

$$\langle \chi^{(i)} , \chi^{(j)} \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

But for any character Λ, we have $\Lambda = \sum_{i=1}^{k} n_i \chi^{(i)}$, where n_i, $i = 1, \ldots, k$ are nonnegative integers. Applying orthogonality of irreducible characters and bilinearity of the group inner product, we then obtain

$$\langle \Lambda , \Lambda \rangle = \sum_{i=1}^{k} n_i^2,$$

and hence conclude that a representation with character Λ is irreducible if and only if

$$\langle \Lambda , \Lambda \rangle = 1. \tag{58}$$

Now in the symmetric group $S(n)$, g and g^{-1} are in the same conjugacy class, since the cycles of g^{-1} are obtained by reversing the cycles of g. Also, if ϕ and ψ are characters, then their values are constant on conjugacy classes, so in this case we obtain

$$\langle \phi , \psi \rangle = \frac{1}{|n!|} \sum_{\lambda \vdash n} |C^{(\lambda)}| \phi(\lambda)\psi(\lambda) = \frac{1}{|n!|} \sum_{\lambda \vdash n} \frac{n!}{z(\lambda)} \phi(\lambda)\psi(\lambda) = \sum_{\lambda \vdash n} \frac{1}{z(\lambda)} \phi(\lambda)\psi(\lambda). \tag{59}$$

17. Lecture of June 26

Fo example, applying (59) and (58) to the three characters for Young's seminormal representations of $S(3)$ constructed in the previous lecture, we obtain (using $z(1,1,1) = 6$, $z(2,1) = 2$, $z(3) = 3$):

$$\langle \Lambda , \Lambda \rangle = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1,$$

for the trivial representation,

$$\langle \Lambda , \Lambda \rangle = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1,$$

for the sign representation, and

$$\langle \Lambda , \Lambda \rangle = \frac{4}{6} + \frac{0}{2} + \frac{1}{3} = 1,$$

for the representation indexed by the partition $(2, 1)$. We conclude that each of these three representations is irreducible.

It is straightforward to construct other much less complicated group representations, as long as one doesn't worry about irreducibility. As a first example, consider the *natural* representation for the symmetric group $S(n)$ (or indeed any permutation subgroup of $S(n)$). As on pages 45 to 47 of the Course Notes for Young's seminormal representation, we'll consider group elements as acting on the right, and e.g., for a permutation σ will write the image of i under the action of σ to be $(i)\sigma$ rather than the more conventional $\sigma(i)$ to emphasize that products act from left to right; hence we write $(i)\sigma\tau$ to mean $((i)\sigma)\tau$. Using this notation, let $(v_1, \ldots, v_n)$ be an ordered basis for a vector space of dimension n, and, for each $\sigma \in S(n)$, define $\rho(\sigma)$ to be the linear transformation whose action is given by $\rho(\sigma)\, v_i = v_{(i)\sigma}$, for $i = 1, \ldots, n$. The corresponding $n \times n$ matrix M_σ (acting by right multiplication of the row vector $(v_1, \ldots, v_n)$) has a 1 in position $(i, (i)\sigma)$ for each $i = 1, \ldots, n$, and 0's elsewhere - so it is a *permutation* matrix. This is a representation, since $M_{\sigma\pi} = M_\sigma M_\pi$, and of course it has degree n. The character of this representation $\Lambda(\sigma)$, which is given by the trace of M_σ, is the number of fixed points of σ; of course this is constant on conjugacy classes, and is indeed equal to the number of 1's in the partition specifying the cycle lengths for the class. For example, when $n = 3$, we obtain $\Lambda(1, 1, 1) = 3$, $\Lambda(2, 1) = 1$, $\Lambda(3) = 0$, and thus have

$$\langle \Lambda \,,\, \Lambda \rangle = \frac{9}{6} + \frac{1}{2} + \frac{0}{3} = 2$$

for the natural representation of $S(3)$, which is therefore not irreducible. By examining the character values for the conjugacy classes, we find that this representation can be obtained by similarity as a direct sum of 1 copy of the trivial representation, and 1 copy of the irreducible representation of degree 2 (and, of course $1^2 + 1^2 = 2$).

As a second example of a simple construction for a group representation, consider the *right-regular* representation of $S(n)$ (or indeed any group). Let $\{v_\pi : \pi \in S(n)\}$ be a basis for a vector space of dimension $n!$, and, for each $\sigma \in S(n)$, define $\rho(\sigma)$ to be the linear transformation whose action is given by $\rho(\sigma)\, v_\pi = v_{\pi\sigma}$, for $\pi \in S(n)$. The corresponding $n! \times n!$ matrix M_σ (again acting by right multiplication of the row vector listing the v_π, $\pi \in S(n)$ in some arbitrary order) has a 1 in position $(\pi, \pi\sigma)$ for each $\pi \in S(n)$ (indexing the rows and columns in the same, arbitrary, way), and 0's elsewhere - so it is a *permutation* matrix. This is a representation, since $M_{\sigma\tau} = M_\sigma M_\tau$, and of course it has degree $n!$. To determine the character of this representation, note that $\operatorname{trace} M_\sigma$ is equal to the number of $\pi \in S(n)$ such that $\pi\sigma = \pi$. Multiplying this equation on the left by π^{-1}, we obtain $\sigma = id$, so this equation is satisfied for all π when σ is the identity permutation, and it is never satisfied when σ is not the identity permutation. We conclude that the character Λ of this representation is given by $\Lambda(1, \ldots, 1) = n!$, and $\Lambda(\lambda) = 0$ for all other partitions λ of n. For example, when $n = 3$, we obtain $\Lambda(1, 1, 1) = 6$, $\Lambda(2, 1) = 0$, $\Lambda(3) = 0$, and thus have

$$\langle \Lambda \,,\, \Lambda \rangle = \frac{36}{6} + \frac{0}{2} + \frac{0}{3} = 6$$

for the right-regular representation of $S(3)$, which is therefore not irreducible. By examining the character values for the conjugacy classes, we find that this representation can be obtained by similarity as a direct sum of 1 copy of the trivial representation, 1 copy of the sign representation, and 2 copies of the irreducible representation of degree 2 (and of course $1^2 + 1^2 + 2^2 = 6$).

Finally, as a third example of a simple construction, consider *induced* representations. Let G be a finite group, with subgroup H, and a representation B_h, $h \in H$ of degree m. Define B_g to be the $m \times m$ 0 matrix when $g \in G$ is not contained in the subgroup H. Suppose that

$$G = H\, t_1 \cup \ldots \cup H\, t_n$$

is a decomposition of G into right cosets, where $n = |G|/|H| \geq 1$. Define M_g to be the $mn \times mn$ matrix given as in terms of $m \times m$ blocks as

$$M_g = \left(B_{t_i g t_j^{-1}} \right)_{i,j=1,\ldots,n}.$$

Note that this matrix has only a single non-zero block for each i and for each j (e.g., for each i, we have $t_i g \in H\, t_\ell$, or equivalently $t_i g t_j^{-1} \in H$, for some unique ℓ; similarly, for each j, we have $t_j g^{-1} \in H\, t_k$, or equivalently $t_j g t_k^{-1} \in H$, or equivalently its inverse $t_k g t_j^{-1}$ is in H, for some unique k). It is then straightforward to check that M_g, $g \in G$, is a representation of G, and we say that this representation is *induced* by the representation B_h, $h \in H$ of the subgroup H. To determine the character Λ of this induced representation, note that

$$\Lambda(g) = \operatorname{trace} M_g = \sum_{i=1}^{n} \operatorname{trace} B_{t_i g t_i^{-1}}.$$

Now suppose that g is contained in the conjugacy class $C^{(\ell)}$ of G. Then $t_i g t_i^{-1} \in C^{(\ell)}$ for all $i = 1, \ldots, n$, and for a non-zero contribution to the summation above, we need to know when $t_i g t_i^{-1} \in H$. By symmetrizing this sum, it is straightforward to show that

$$\Lambda(g) = \Lambda(\ell) = \frac{|G|}{|H| \cdot |C^{(\ell)}|} \sum_{h \in H \cap C^{(\ell)}} \operatorname{trace} B_h,$$

thus expressing the character of the induced representation M_g, $g \in G$, in terms of the character of the underlying representation B_h, $h \in H$. Now consider what happens for the trivial representation $B_h = 1$, $h \in H$ of H. In this case we simply get

$$\Lambda(\ell) = \frac{|G| \cdot |H \cap C^{(\ell)}|}{|H| \cdot |C^{(\ell)}|}. \tag{60}$$

We now apply this for the case of the symmetric group $G = S(n)$, and the *Young subgroup* H_ν specified by the partition $\nu = (\nu_1, \ldots, \nu_n)$ of n (where some of the ν_i's might be 0). If we let $S(\alpha)$ denote the symmetric group that acts on some set α of positive integers, then we define

$$H_\nu = S(\{1, \ldots, \nu_1\}) \times S(\{\nu_1 + 1, \ldots, \nu_1 + \nu_2\}) \times \cdots \times S(\{\nu_1 + \ldots + \nu_{n-1} + 1, \ldots, \nu_1 + \ldots + \nu_n\}),$$

which means that we permute the first ν_1 elements of $\{1\ldots, n\}$ arbitrarily, then the next ν_2 elements, and so on until we have permuted the final ν_n elements (recalling that $\nu_1 + \ldots + \nu_n = n$ since ν is a partition of n). For example, this means that $|H_\nu| = \nu_1! \cdots \nu_n!$. Now it is an elementary counting exercise to show that for another partition λ of n and the corresponding conjugacy class $C^{(\lambda)}$, we have

$$|H_\nu \cap C^{(\lambda)}| = \frac{\nu_1! \cdots \nu_n!}{z(\lambda)} \big[m_\nu(x_1, \ldots, x_n)\big] p_\lambda(x_1, \ldots, x_n),$$

but here we have $|G| = |S(n)| = n!$, $|H| = |H_\nu| = \nu_1! \cdots \nu_n!$ and $|C^{(\lambda)}| = \frac{n!}{z(\lambda)}$, so from (60) we obtain the value of the character of $S(n)$ induced by the trivial representation of the Young subgroup H_ν as

$$\Lambda(\lambda) = \big[m_\nu(x_1, \ldots, x_n)\big] p_\lambda(x_1, \ldots, x_n).$$

Recall that for each ν, this character is a nonnegative integer linear combination of irreducible characters. Now let $Q(x_1, \ldots, x_n)$ be any homogeneous polynomial of total degree k and with integer coefficients. Then it follows from the result above that for any monomial $x_1^{c_1} \cdots x_n^{c_n}$ of total degree $n + k$, the coefficient

$$\big[x_1^{c_1} \cdots x_n^{c_n}\big] Q(x_1, \ldots, x_n) p_\lambda(x_1, \ldots, x_n) \tag{61}$$

is an integer linear combination of induced characters, and hence an integer linear combination of irreducible characters of $S(n)$. If the corresponding group inner product has value 1, then it the coefficient in (61) must either be an irreducible character, or an irreducible character multiplied by -1. Frobenius considered the choice $Q = \prod_{1 \le i < j \le n}(x_i - x_j)$ and $c_j = \alpha_j + n - j$, $j = 1, \ldots, n$, for any partition $\alpha = (\alpha_1, \ldots, \alpha_n)$ of n. Clearly the coefficient in (61) is equal to $[s_\alpha]p_\lambda = \langle\, s_\alpha\,,\, p_\lambda \,\rangle$. Moreover, in a classical symmetric function calculation, he proved that the corresponding group inner product (using different values of λ for the different conjugacy classes of the symmetric group), has value 1, verifies by checking the coefficient when λ is the partition of all 1's that it cannot be an irreducible character multiplied by -1, and concludes that

$$\langle\, s_\alpha\,,\, p_\lambda \,\rangle$$

is the character of some irreducible representation of $S(n)$ indexed by the conjugacy class $C^{(\alpha)}$, evaluated at elements of the conjugacy class $C^{(\lambda)}$, as described near the top of page 42 of the Course Notes.

How do irreducible group characters arise in combinatorial and other mathematical contexts? Consider the formalism of the *group algebra* of a finite group G over the complex numbers. An element of this group algebra can be written in the form $a = \sum_{g \in G} a_g\, g$, where a_g is a complex number referred to as the *coefficient* of g for each $g \in G$. To multiply a by a complex number z, we define $z\,a = a\,z = \sum_{g \in G}(za_g)g$, where of course the multiplication za_g is in the complex numbers. If $b = \sum_{h \in G} b_h\, h$ is another element of the group algebra, then we define

$$a + b = \sum_{g \in G}(a_g + b_g)g, \qquad a\,b = \sum_{g,h \in G}(a_g b_h)(gh),$$

where the sum $a_g + b_g$ and product $a_g b_h$ are in the complex numbers, and the product gh is in the group G.

Now corresponding to each conjugacy class $C^{(i)}$ of G, define the following element of the group algebra:

$$\mathsf{C}^{(i)} = \sum_{g \in C^{(i)}} g, \quad i = 1, \ldots, k.$$

Then a fundamental fact from representation theory of finite groups is that we can write each $\mathsf{C}^{(i)}$ as a linear combination of *orthogonal idempotents* $\mathsf{E}^{(i)}$, $i = 1, \ldots, k$ with irreducible characters as coefficients (using orthoginality of the irreducible characters). In particular, we have

$$\mathsf{C}^{(i)} = |C^{(i)}| \sum_{j=1}^{k} \frac{\chi^{(j)}(i)}{\chi^{(j)}(id)}\, \mathsf{E}^{(j)}, \qquad i = 1, \ldots, k, \tag{62}$$

where

$$\mathsf{E}^{(i)}\, \mathsf{E}^{(j)} = \delta_{i,j} \mathsf{E}^{(i)}, \qquad i, j = 1, \ldots, k. \tag{63}$$

That is, the product of two different $\mathsf{E}^{(i)}$'s is always zero, and the square of any of the $\mathsf{E}^{(i)}$'s is equal to itself. We can invert the linear equations given in (62), to obtain

$$\mathsf{E}^{(i)} = \frac{\chi^{(i)}(id)}{|G|} \sum_{j=1}^{k} \overline{\chi}^{(i)}(j)\, \mathsf{C}^{(j)}, \qquad i = 1, \ldots, k. \tag{64}$$

In (64), the complex conjugate $\overline{\chi}^{(i)}(j)$ of the character value $\chi^{(i)}(j)$ arises because in an arbitrary group we have $\chi^{(i)}(g^{-1})) = \overline{\chi}^{(i)}(g)$ for any character; in the symmetric group, g and g^{-1} are always conjugates, so we have $\chi^{(i)}(g^{-1})) = \chi^{(i)}(g))$, which implies that all irreducible characters in the symmetric group are real.

18. Lecture of June 28

Also in the symmetric group $S(n)$ the conjugacy classes are indexed by partitions of n, so in the symmetric group, from (62) and (63) considering the product of conjugacy classes in $S(n)$, we obtain

$$\begin{aligned} \mathsf{C}^{(\lambda)}\mathsf{C}^{(\mu)} &= |C^{(\lambda)}||C^{(\mu)}| \sum_{\theta_1 \vdash n} \frac{\chi^{(\theta_1)}(\lambda)}{\chi^{(\theta_1)}(id)} \mathsf{E}^{(\theta_1)} \sum_{\theta_2 \vdash n} \frac{\chi^{(\theta_2)}(\mu)}{\chi^{(\theta_2)}(id)} \mathsf{E}^{(\theta_2)} \\ &= |C^{(\lambda)}||C^{(\mu)}| \sum_{\theta \vdash n} \frac{\chi^{(\theta)}(\lambda)\chi^{(\theta)}(\mu)}{\left(\chi^{(\theta)}(id)\right)^2} \mathsf{E}^{(\theta)}, \end{aligned}$$

and from (64) we then get

$$\left[\mathsf{C}^{(\nu)}\right]\mathsf{C}^{(\lambda)}\mathsf{C}^{(\mu)} = \frac{|C^{(\lambda)}||C^{(\mu)}|}{n!} \sum_{\theta \vdash n} \frac{\chi^{(\theta)}(\lambda)\chi^{(\theta)}(\mu)\chi^{(\theta)}(\nu)}{\chi^{(\theta)}(1^n)}, \tag{65}$$

where we have used the notation 1^n to denote the partition with n parts, all equal to 1.

For an application in which these conjugacy class calculations arise, consider a rooted map in an orientable surface. This is a graph embedded in the surface so that all faces are two-cells (homeomorphic to a disc). One vertex is distinguished, called the root vertex, and one edge incident with the root vertex is distinguished, called the root edge. For example, a rooted map in the torus is drawn in Figure 9.

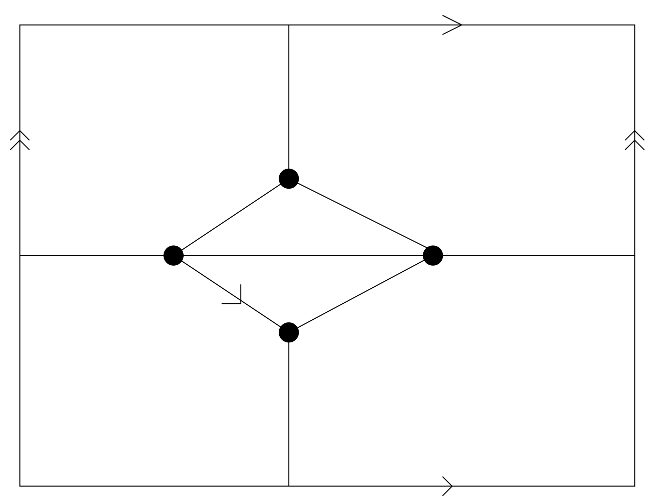

Figure 9. A map embedded in the torus.

There is an arrow on one edge, which identifies that edge as the root edge, and the vertex at the tail of the arrow as the root vertex. The usual parallel line convention is used for the rectangular boundary, instructing us to identify top and bottom, in the same direction, and left and right, again in the same direction. The map has 4 vertices, 7 edges, and 3 faces (two of the faces are triangular, drawn in the middle of the rectangle, the remaining face consists of the four corner regions in the drawing. As a consistency check, note that Euler's formula gives

$$V - E + F = 4 - 7 + 3 = 0 = 2 - 2g,$$

and indeed we have genus $g = 1$ for the torus. Now, for a map with n edges, assign labels $1, 2, \ldots, 2n$ to the (two) ends of the edges, with 1 assigned to the root vertex end of the root edge. Thus there are $(2n-1)!$ possible assignments of labels for each map with n edges. For example, one such assignment for the map given in Figure 9 is given in Figure 10. Now let

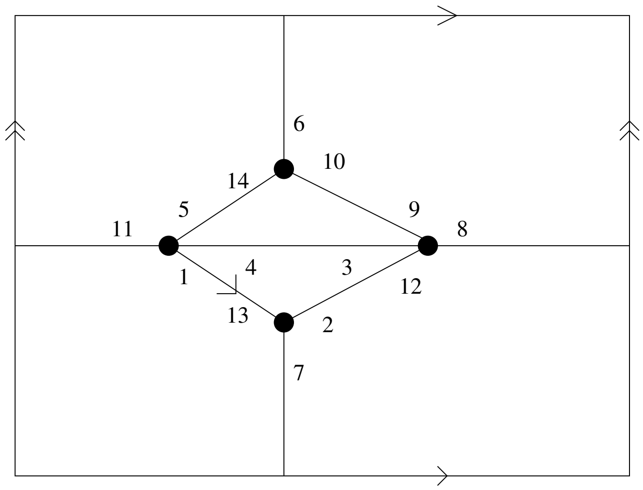

FIGURE 10. An assignment of labels to a map.

σ be the permutation whose disjoint cycles list the labels encountered when moving around each vertex in clockwise order, and let ρ be the permutation whose disjoint cycles (all of length 2) list the pairs of labels for the edges. For example, in Figure 10, we obtain

$$\sigma = (1\,11\,5\,4)(2\,7\,13)(3\,9\,8\,12)(6\,10\,4), \qquad \rho = (1\,13)(2\,12)(3\,4)(5\,14)(6\,7)(8\,11)(9\,10).$$

Now we consider the product $\rho\sigma$ (acting from right to left, so we apply σ first). In the example, this gives

$$\rho\sigma = (1\,8\,2\,6\,9\,11\,14\,7)(3\,10\,5)(4\,13\,12),$$

and note that these three cycles describe the labels found when traversing the three faces of the map in counterclockwise order. Note also that $\sigma \in C_{(4433)}$ and $\rho\sigma \in C_{(833)}$ precisely because the vertex degrees of the map are specified by the parts of the partition (4433), and the face degrees of the map are specified by the parts of the partition (833) (by construction, we always have $\rho \in C_{(2222222)} = C_{(2^7)}$).

In fact, the permutations σ and ρ are enough to uniquely reconstruct the labelled drawing of the map, and this is bijective if we require that the labelled drawing is connected (in terms of the permutations, this is equivalent to requiring that the group generated by σ and ρ acts transitively on $\{1, 2 \ldots, 2n\}$). To summarize, there is a $1 : (2n-1)!$ correspondence between rooted maps and

$$\{(\sigma, \rho) : \rho \in C^{(2^n)}, \langle \sigma, \rho \rangle \text{ acts transitively on } \{1, 2, \ldots, 2n\}\},$$

in which the conjugacy class of σ specifies the vertex degrees, and the conjugacy class of $\sigma\rho$ specifies the face degrees (and the genus can be obtained by Euler's formula). If we simply enumerate pairs of permutations, and put these in a generating function, then taking the logarithm will account for the "connected" (or, "transitive") condition. Of course, to specify that vertex degrees are specified by partition λ and face degrees are specified by partition ν in this generating function, we need to evaluate the combinatorial coefficient given in (65), when $\mu = (2^n)$.

To be specific, let

$$a_{\lambda,\nu} = |\{(\sigma,\rho) \in C^{(2^n)} \times C^{(\lambda)} : \rho\sigma \in C^{(\nu)}\}|,$$

where λ, ν are partitions of $2n$, and $n \geq 1$. Then we immediately have

$$a_{\lambda,\nu} = |C^{(\nu)}| \cdot [\mathsf{C}^{(\nu)}]\mathsf{C}^{(2^n)}\mathsf{C}^{(\lambda)}, \tag{66}$$

and if we define the generating series

$$A(t, x_1, \ldots, y_1, \ldots) = 1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{\lambda,\nu\vdash 2n} a_{\lambda,\nu} p_\lambda(x_1,\ldots)p_\nu(y_1,\ldots),$$

then from (65) and (66), we obtain

$$\begin{aligned} A &= 1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{\mu,\nu\vdash 2n} \frac{|C^{(\nu)}||C^{(2^n)}||C^{(\lambda)}|}{(2n)!} \sum_{\theta\vdash 2n} \frac{\chi^\theta(\nu)\chi^\theta(2^n)\chi^\theta(\lambda)}{\chi^\theta(1^{2n})} p_\lambda(x_1,\ldots)p_\nu(y_1,\ldots) \\ &= \sum_{n\geq 0} t^{2n}|C^{(2^n)}| \sum_{\theta\vdash 2n} \frac{\chi^\theta(2^n)}{\chi^\theta(1^{2n})} s_\theta(x_1,\ldots)s_\theta(y_1,\ldots), \qquad (67) \end{aligned}$$

where the second equality follows from the expression for the Schur function in terms of power sums given in (54) (and using $\langle\, p_\alpha, s_\lambda \,\rangle = \chi^{(\lambda)}(\alpha)$, $|C^\lambda| = \frac{(2n)!}{z(\alpha)}$).

19. Lecture of July 5

Now, suppose we define $m_{\mu,\nu}$, to be the number of rooted maps with vertex degrees specified by the parts of μ, and face degrees specified by the parts of ν, and n edges, where μ, ν are partitions of $2n$, and $n \geq 1$. As a consequence of the $1 : (2n-1)!$ correspondence given in the previous lecture, we have

$$(2n-1)!\, m_{\mu,\nu} = |\{(\sigma,\rho) \in C^{(2^n)} \times C^{(\lambda)} : \rho\sigma \in C^{(\nu)}\,,\ \langle\sigma,\rho\rangle \text{ acts transitively on } \{1,2,\ldots,2n\}\}|,$$

and the theory of exponential generating series then gives

$$(2n-1)!\, m_{\mu,\nu} = [p_\mu(x_1,\ldots)p_\nu(y_1,\ldots)\frac{t^{2n}}{(2n)!}] \log A.$$

Note that in the generating series A above, the partition 2^n arises in the contribution $|C^{(2^n)}|\chi^\theta(2^n)$. But we have

$$|C^{(2^n)}|\chi^\theta(2^n) = \big[p_2(z_1,\ldots)^n\big](2n)!\, s_\theta(z_1,\ldots),$$

which means that a more symmetrical version of (67) is given by the sum

$$\sum_{n\geq 0} t^n n! \sum_{\theta\vdash n} \frac{s_\theta(z_1,\ldots)s_\theta(x_1,\ldots)s_\theta(y_1,\ldots)}{\chi^\theta(1^n)},$$

which can be rewritten via the hook formula for the degree $\chi^\theta(1^n)$ as

$$\sum_{n\geq 0} t^n \sum_{\theta \vdash n} s_\theta(z_1,\ldots)s_\theta(x_1,\ldots)s_\theta(y_1,\ldots)\prod_{\alpha\in\theta} h(\alpha), \tag{68}$$

For an application in which the generating function (68) arises, consider a rooted *hypermap* in an orientable surface, which is a rooted map in which the faces are (properly) two coloured, say black and white, and in which the face that is incident with the root edge, clockwise around the root vertex, is a black face (and we call the associated corner of this white face the *root corner*). For example, In Figure 11, we have a rooted hypermap in the plane, with two black faces, marked by A and B in the drawing, and three white faces. The black faces

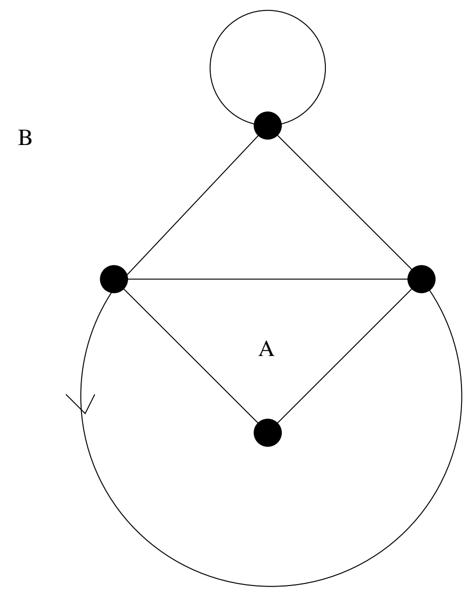

FIGURE 11. A rooted hypermap.

are regarded as hyperedges, and the white faces as hyperfaces. Now for a hypermap with n edges (not hyperedges), assign labels to the corners of the black faces, with 1 assigned to the root corner. Now, the sum of vertex degrees is $2n$, and each vertex has even degree, with the corners of faces alternately black and white as we move around the vertex, so the total number of black corners is n. Thus there are $(n-1)!$ possible assignments of labels for each hypermap with n edges. For example, one such assignment for the hypermap given in Figure 11, is given in Figure 12. Now let σ be the permutation whose disjoint cycles list the labels encountered when moving around each vertex in clockwise order, and let ρ be the permutation whose disjoint cycles list the labels encountered when traversing the black faces in counterclockwise direction around the *interior* of the face (be careful for the outer face), so for the labelling in Figure 12, we obtain

$$\sigma = (1\,3)(2\,5)(4\,7)(6), \qquad \rho = (1\,4\,7\,5)(2\,3\,6).$$

Now we consider the product $\rho\sigma$. In the example, this gives

$$\rho\sigma = (1\,6\,2)(3\,4\,5)(7),$$

and note that these three cycles describe the labels found when traversing the white faces in a clockwise direction (looking at clockwise adjacent corners). In this case, we have a $1:(n-1)!$

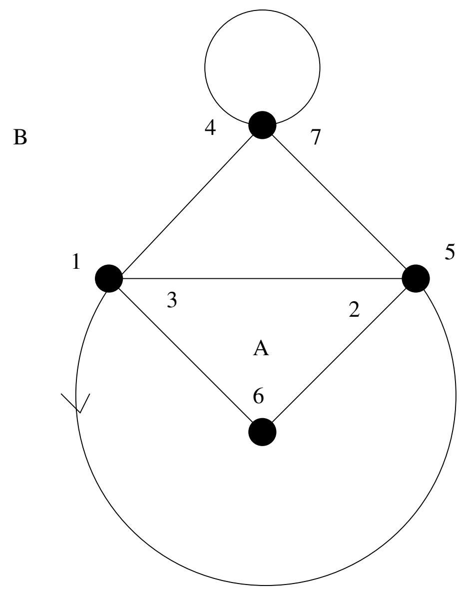

FIGURE 12. As assignment of labels to a hypermap.

correspondence between rooted maps and pairs of arbitrary permutations (σ, ρ), subject only to the same connectivity condition as for maps above. Thus we again enumerate pairs of permutations and put these in a generating function, then take logarithm to account for the "connected" condition. For the generating function, using $p_\nu(y_1, \ldots)$ to specify that vertex degrees are given by partition ν, $p_\lambda(x_1, \ldots)$ to specify that hyperface (white face) degrees by partition λ, and $p_\mu(z_1, \ldots)$ to specify that hyperedge (black face) degrees by partition μ, after some calculations, we obtain precisely (68).

Now we move to considering new classes of symmetric functions. Let

$$H(t; x_1, \ldots) = \prod_{j\geq 1}(1 - x_j t)^{-1} = \sum_{n\geq 0} h_n(x_1, \ldots)t^n,$$

where h_n is the complete symmetric function. Now let

$$s = tH(t; x_1, \ldots), \tag{69}$$

and take the compositional inverse to obtain

$$t = sH^*(s; x_1, \ldots), \tag{70}$$

where $H^*(s; x_1, \ldots) = \sum_{n\geq 0} h^*_n(x_1, \ldots)s^n$. Explicitly, we have $h^*_0 = 1$, and for $n \geq 1$, we apply Lagrange's Implicit Function Theorem to (69) to obtain

$$\begin{aligned} h^*_n &= [s^{n+1}]t = \frac{1}{n+1}[z^n]H(z; x_1, \ldots)^{-(n+1)} \\ &= \frac{1}{n+1}[\lambda^n] \exp\left(-(n+1)\sum_{k\geq 1}\frac{p_k}{k}\lambda^k\right) \\ &= \sum_{\lambda\vdash n}(-1)^{l(\lambda)}(n+1)^{l(\lambda)-1}\frac{|\mathcal{C}_\lambda|}{n!}p_\lambda, \end{aligned}$$

where $l(\lambda)$ is the number of parts in the partition λ. Thus h_n^* is a symmetric function of homogeneous total degree n. Now define

$$\xi : \Lambda \to \Lambda : h_i \mapsto h_i^*, \qquad i \geq 1,$$

which is well-defined since the h_i's are algebraically independent. From the computation of h_n^* above, for $n \geq 1$ this gives

$$\xi(h_n^*) = \frac{1}{n+1}[z^n]H^*(z; x_1, \ldots)^{-(n+1)} = [t^{n+1}]s,$$

where s and t are related by the functional equation (70), by Lagrange's theorem. But this is equivalent to (69), so

$$\xi(h_n^*) = [t^{n+1}]s = [t^n]H(s; x_1, \ldots) = h_n,$$

and we conclude that ξ is an involution on Λ, so it is an *automorphism*. This implies immediately that the h_i^*'s are algebraically independent, that $\{h_\lambda^* : \text{partitions } \lambda\}$ is a basis for Λ, and that $\Lambda = \mathbb{Z}[h_1^*, h_2^*, \ldots]$.

Now define $\{u_\lambda\}$ to be the dual basis to $\{h_\lambda^*\}$, i.e.,

$$\langle u_\lambda, h_\mu^* \rangle = \delta_{\lambda,\mu}.$$

We have the following result for dual bases.

Theorem 19.1. *If $\{a_\lambda\}$ and $\{b_\lambda\}$ are dual bases for $\Lambda_{\mathbb{Q}}$, then*

$$[a_\lambda]a_\mu a_\nu = [b_\mu(y_1, \ldots)b_\nu(z_1, \ldots)]b_\lambda(y_1, \ldots, z_1, \ldots).$$

Proof. We have

$$\begin{aligned}
[a_\lambda]a_\mu a_\nu &= [a_\lambda(x_1, \ldots)]a_\mu(x_1, \ldots)a_\nu(x_1, \ldots) \\
&= [a_\lambda(x_1, \ldots)b_\mu(y_1, \ldots)b_\nu(z_1, \ldots)] \sum_\sigma a_\sigma(x_1, \ldots)b_\sigma(y_1, \ldots) \sum_\rho a_\rho(x_1, \ldots)b_\rho(z_1, \ldots) \\
&= [a_\lambda(x_1, \ldots)b_\mu(y_1, \ldots)b_\nu(z_1, \ldots)] \prod_{i,j\geq 1}(1 - x_i y_j)^{-1}(1 - x_i z_j)^{-1} \\
&= [a_\lambda(x_1, \ldots)b_\mu(y_1, \ldots)b_\nu(z_1, \ldots)] \sum_\alpha a_\alpha(x_1, \ldots)b_\alpha(y_1, \ldots, z_1, \ldots) \\
&= [b_\mu(y_1, \ldots)b_\nu(z_1, \ldots)]b_\lambda(y_1, \ldots, z_1, \ldots),
\end{aligned}$$

giving the result, where the third and fourth equality follow from Theorem 7.1. □

It turns out that the symmetric functions $\{h_\lambda^*\}$ and $\{u_\lambda\}$ have a nice interpretation for products of permutations. On the way to this interpretation, we begin by considering a combinatorial result for a particular connection coefficient for conjugacy classes in the symmetric group.

Theorem 19.2. *Let $n \geq 1$, and $\mu, \nu \vdash n$ with $l(\mu) + l(\nu) = n + 1$. Then $[\mathsf{C}^{(n)}]\mathsf{C}^{(\mu)}\mathsf{C}^{(\nu)}$ is equal to the number of edge-rooted plane trees on n edges, with vertices that are properly two-coloured black and white, and white and black vertex degrees specified by the parts of μ and ν, respectively.*

Proof. We give an explicit bijection between edge-rooted plane trees on n edges, with vertices that are properly two-coloured black and white, and ordered pairs of permutations (σ, ρ) on $\{1, \ldots, n\}$, such that $\sigma\rho = (1\,2\ldots n)$. Consider such a tree. Traverse the outside of the tree in clockwise direction, beginning on the side of the root edge in which the incident black vertex follows the incident white vertex in the clockwise direction. Label the root edge 1, and label the other edges $2, \ldots, n$, in the order that they are encountered in the traversal, when their incident black vertex follows the incident white vertex in the clockwise direction. For example, Figure 13 gives an example of a tree with $n = 12$ edges, and the edges labelled as described above. Let the disjoint cycles of σ be the circular lists of edge labels encountered

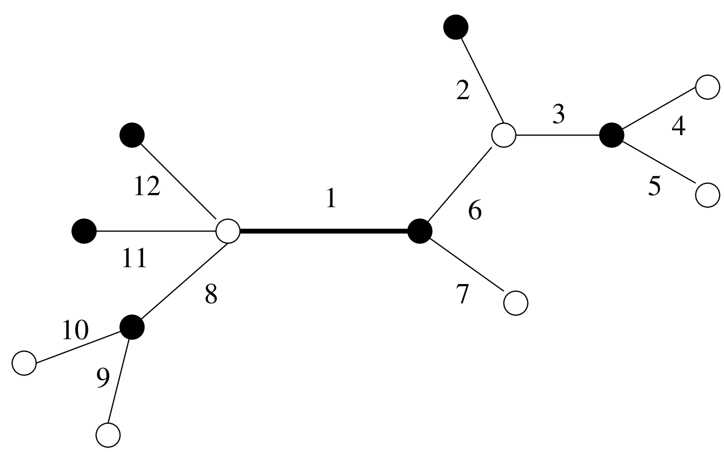

FIGURE 13. Labelling an edge-rooted plane tree.

on incident edges clockwise around the white vertices, and let the disjoint cycles of ρ be the circular lists of edge labels encountered on incident edges clockwise around the black vertices. For example from the labelling of the tree in Figure 13, we obtain

$$\sigma = (1\,8\,11\,12)(2\,3\,6)(4)(5)(7)(9)(10), \qquad \rho = (1\,6\,7)(2)(3\,4\,5)(8\,9\,10)(11)(12).$$

Note in this example that $\sigma\rho = (1\,2\,3\ldots 12)$, and that by construction this always holds, since the labelling rule implies that $\sigma\rho$ must map i to $i+1 \pmod{n}$, for all $i = 1, \ldots, n$. Also note that the disjoint cycle lengths for σ and ρ specify partitions μ and ν that are precisely the degrees of the white and black vertices, respectively. But this implies that the total number of vertices in the trees is equal to $l(\mu) + l(\nu)$, and this must also equal $n + 1$, since a tree with n edges must have $n + 1$ vertices.

This construction is easily seen to be reversible – given (σ, ρ), one creates the labelled plane tree uniquely by first constructing the vertices with labelled incident half-edges, then joining the half-edges with the same label. The result follows. □

We immediately obtain the following corollary, that translates Theorem 19.2 into generating function terms.

Corollary 19.3. *Let $n \geq 1$, and $\mu, \nu \vdash n$ with $l(\mu) + l(\nu) = n + 1$. Then*

$$[\mathsf{C}^{(n)}]\mathsf{C}^{(\mu)}\mathsf{C}^{(\nu)} = [w_\mu b_\nu t^{n+1}]WB, \tag{71}$$

where

$$(72) \qquad B = t\left(b_1 + b_2 W + b_3 W^2 + \ldots\right), \qquad W = t\left(w_1 + w_2 B + w_3 B^2 + \ldots\right),$$

and $w_\mu = w_{\mu_1} w_{\mu_2} \cdots$, $b_\nu = b_{\nu_1} b_{\nu_2} \cdots$, *for* $\mu = (\mu_1, \mu_2, \ldots)$, $\nu = (\nu_1, \nu_2, \ldots)$.

Proof. Consider an edge-rooted plane tree T on at least one edge, with vertices that are properly two-coloured black and white. Then T can be decomposed uniquely into two rooted ordered trees T_1 and T_2, by cutting the root edge in the "middle". These rooted trees have root vertices of opposite colour – let T_1 have black root vertex, and T_2 have white root vertex. Both trees have properly coloured vertices, and we attach a half-edge to the root vertex so that the root vertex degrees in T_1 and T_2 agree with the vertex degrees in T. See Figure 14 for an example of this construction, where T is drawn at the top, with T_1 and T_2

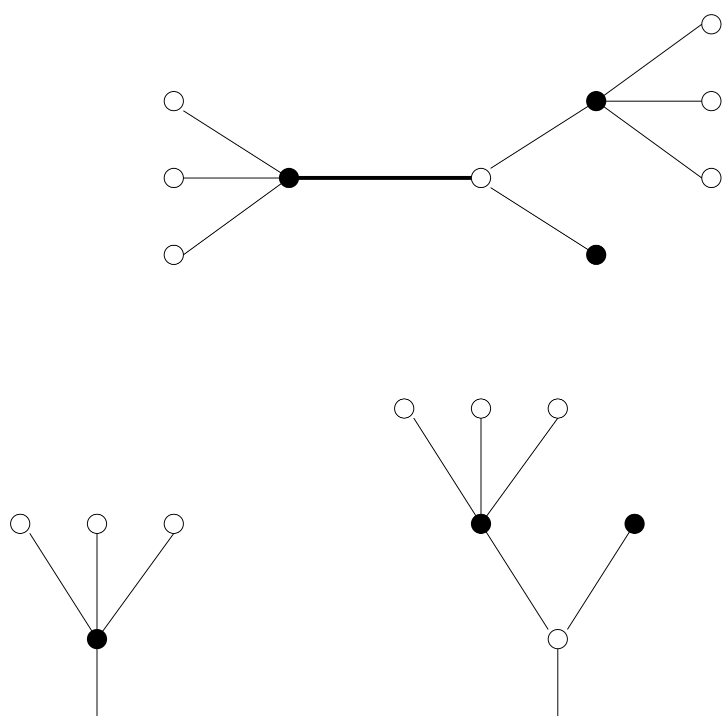

Figure 14. Decomposition of a plane tree.

below. Let B, W be the generating functions for these sets of trees, B when the root vertex is black, and W when the root vertex is white. In both B and W, t is an indeterminate whose exponent gives the number of vertices, b_j is an indeterminate whose exponent gives the number of black vertices of degree j, $j \geq 1$, and w_j is an indeterminate whose exponent gives the number of white vertices of degree j, $j \geq 1$. The generating functions B and W satisfy the functional equations given in (72), since removing the root vertex in these sets of trees gives an ordered list of trees of opposite root vertex colour. The result follows from Theorem 19.2. □

20. Lecture of July 10

Suppose we multiply a permutation σ on $\{1,\ldots,n\}$ by a transposition $(a\,b)$. If a and b are on different cycles in σ, then these cycles are joined into a single cycle in the product $(a\,b)\sigma$, and if a and b are on the same cycle in σ, then that cycle is cut into two cycles in $(a\,b)\sigma$, with a and b in different cycles. This means that if we write a permutation in $C^{(\mu)}$ as a product of k transpositions, then $n-l(\mu)=j-c$, where j of the transpositions are joins, and c are cuts (we view the successive multiplication of transpositions as a sequence of partial products, beginning with the empty product - the identity permutation, which has n cycles, and terminating with the permutation in $C^{(\mu)}$, which has $l(\mu)$ cycles). But $j+c=k$, so $n-l(\mu)=k-2c$, and thus the minimum value of k is $n-l(\mu)$, achieved when $c=0$. This implies that if

$$[\mathsf{C}^{(\lambda)}]\mathsf{C}^{(\mu)}\mathsf{C}^{(\nu)}\neq 0,$$

then we must have

$$(n-l(\mu))+(n-l(\nu))\geq n-l(\lambda). \tag{73}$$

In the extreme case of equality in (73), we refer to the corresponding permutation factorization as *minimal*.

Hence Theorem 19.2 and Corollary 19.3 are results that apply to minimal permutation factorization, since if $l(\lambda)=1$, then equality in (73) becomes

$$(n-l(\mu))+(n-l(\nu))=n-1,$$

which is equivalent to $l(\mu))+l(\nu))=n+1$. For minimal factorizations with arbitrary λ, equality in (73) implies that each cycle in $C^{(\lambda)}$ must be formed by multiplication of disjoint cycles in $C^{(\mu)}$ and $C^{(\nu)}$, and so Corollary 19.3 immediately implies that, if $n\geq 1$ and $(n-l(\mu))+(n-l(\nu))=n-l(\lambda)$, then

$$[\mathsf{C}^{(\lambda)}]\mathsf{C}^{(\mu)}\mathsf{C}^{(\nu)}=[w_\mu b_\nu]\prod_{i\geq 1}\left([t^{\lambda_i+1}]WB\right), \tag{74}$$

where B,W are given by (72).

Now we transform (74) into a more symmetric form. First, we make the substitutions

$$w_i=h^*_{i-1}(y_1,\ldots),\qquad b_i=h^*_{i-1}(z_1,\ldots),\qquad i\geq 1, \tag{75}$$

which is legitimate since the h^*_{i-1} are algebraically independent (for $i=1$, we are fine, since the variable t records total degree, so the exponents on w_1 and b_1 can be recovered despite the fact that $h^*_0=1$). Under this substitution, the functional equations in (72) are transformed to

$$W=tH^*(B;y_1,\ldots),\qquad B=tH^*(W;z_1,\ldots). \tag{76}$$

Now let

$$a=BH^*(B;y_1,\ldots),\qquad a'=WH^*(W;z_1,\ldots), \tag{77}$$

which combined with (76) gives

$$ta=tBH^*(B;y_1,\ldots)=BW=tWH^*(W;z_1,\ldots)=ta',$$

so we have

$$a=a'=\frac{WB}{t}. \tag{78}$$

But, from (77) we obtain

(79) $$B = aH(a; y_1, \ldots), \qquad W = a'H(a'; z_1, \ldots),$$

and multiplying these together, and applying (78), gives

$$WB = aH(a; y_1, \ldots)aH(a; z_1, \ldots) = a^2H(a; y_1, \ldots, z_1, \ldots).$$

Combining this with (78) gives

$$t = \frac{WB}{a} = aH(a; y_1, \ldots, z_1, \ldots),$$

which is equivalent to

$$a = tH^*(t; y_1, \ldots, z_1, \ldots),$$

so, from (78) we obtain

$$WB = ta = t^2H^*(t; y_1, \ldots, z_1, \ldots).$$

This implies that

$$[t^{\lambda_i+1}]WB = [t^{\lambda_i-1}]H^*(t; y_1, \ldots, z_1, \ldots) = h^*_{\lambda_i-1}(y_1, \ldots, z_1, \ldots),$$

so (74) becomes

$$[\mathsf{C}^{(\lambda)}]\mathsf{C}^{(\mu)}\mathsf{C}^{(\nu)} = [h^*_{\mu-1}(y_1, \ldots)h^*_{\nu-1}(z_1, \ldots)]h_{\lambda-1}(y_1, \ldots, z_1, \ldots),$$

where $\lambda - 1$ means that 1 is subtracted from all parts of λ. Finally, together with Theorem 19.1, this equality implies the following result for minimal permutation factorizations.

Theorem 20.1. *If $n \geq 1$ and $(n - l(\mu)) + (n - l(\nu)) = n - l(\lambda)$, then*

$$[\mathsf{C}^{(\lambda)}]\mathsf{C}^{(\mu)}\mathsf{C}^{(\nu)} = [u_{\lambda-1}]u_{\mu-1}u_{\nu-1}.$$

As an immediate corollary, we obtain the following result, also for minimal permutation factorizations.

Corollary 20.2. *If $n, k \geq 1$ and $(n - l(\mu_1)) + \ldots + (n - l(\mu_k)) = n - l(\lambda)$, then*

$$[\mathsf{C}^{(\lambda)}]\mathsf{C}^{(\mu_1)}\cdots\mathsf{C}^{(\mu_k)} = [u_{\lambda-1}]u_{\mu_1-1}\cdots u_{\mu_k-1}.$$

These results say that, for minimal permutation factorizations, we can transform multiplication of conjugacy classes $C^{(\lambda)}$ to an equivalent multiplication of the symmetric functions $u_{\lambda-1}$. To see how this can be an effective transformation, we consider the problem of minimal factorizations of an n-cycle n which all factors are transpositions. In this case, in the notation of Corollary 20.2, we have $\mu_i = (2\,1^{n-2})$ (the partition with a single 2 and $n-2$ 1's as parts), for $i = 1, \ldots, k$, and $\lambda = (n)$. Hence we have $n - l(\mu_i) = 1$ for $i = 1, \ldots, k$, and $n - l(\lambda) = n - 1$, and thus $k = n - 1$. Now from elementary considerations we can determine the number of ways to write any fixed element of $C^{(n)}$, say $(1\,2\ldots n)$, as a product of $n-1$ transpositions $\tau_1 \cdots \tau_{n-1}$ as follows: create the graph with vertices $1, \ldots, n$, and $n-1$ edges, with edge $a\,b$ corresponding to the transposition $\tau_i = (a\,b)$. This graph must be connected, since the product of the transpositions is equal to the permutation $(1\,2\ldots n)$, which acts transitively on $1, \ldots, n$. Hence this graph is s tree. There are n^{n-2} trees on n vertices, and there are $(n-1)!$ ways of ordering the edges (as the factors), and so there are $(n-1)!\,n^{n-2}$ ordered factorizations that can be created by doing this in all ways. But the product is *any* element of $C^{(n)}$, and the $(n-1)!$ different elements of $C^{(n)}$ are created equally often as products of this type. Thus $(1\,2\ldots n)$ can be written as a product of $n-1$ transpositions in exactly n^{n-2} ways.

Now consider counting equivalence classes of these factorizations, where we write

$$\tau_1\ldots\tau_i\tau_{i+1}\ldots\tau_{n-1} \equiv \tau_1\ldots\tau_{i+1}\tau_i\ldots\tau_{n-1}$$

when $\tau_i = (a\,b)$, $\tau_{i+1} = (c\,d)$, and a, b, c, d are 4 different elements of $\{1, \ldots, n\}$. That is, we interchange the order of the factors τ_i and τ_{i+1} when there products $\tau_i\tau_{i+1}$ and $\tau_{i+1}\tau_i$ are equal in $S(n)$.

21. Lecture of July 12

Now we consider a generalization for strings in $\{1, \ldots, n\}^\star$ of the equivalence of strings of transpositions above (for which the alphabet of transpositions if size $\binom{n}{2}$). Let $\mathcal{C}$ be a set of distinct unordered pairs $\{i, j\}$, where $1 \le i < j \le n$. For fixed $\mathcal{C}$, define the following equivalence relation on strings: $s \equiv s'$ if s' can be obtained from s by a sequence of the following allowable moves – if i and j appear consecutively, and $\{i, j\} \in \mathcal{C}$, then interchange i and j. (We say that $\mathcal{C}$ is the set of allowable *commutations.*) For example, if the set of allowable commutations is $\mathcal{C}_1 = \{\{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 6\}, \{5, 6\}\}$, then we have, for example,

$$3454 \equiv 4354 \equiv 4534 \equiv 4543.$$

Define $c_1 = x_1 + \ldots + x_n$ and

$$c_k = \sum x_\alpha, \qquad k \ge 2,$$

where the sum is over all $\alpha \subseteq \{1, \ldots, n\}$ such that all pairs of elements in α are allowable commutations, and $x_\alpha = \prod_{j\in\alpha} x_j$. For example, when the allowable commutations are given by the set $\mathcal{C}_1$ above, we have

$$c_2 = x_2x_3 + x_2x_4 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_6 + x_5x_6, \qquad c_3 = x_2x_3x_4 + x_3x_4x_6 + x_3x_5x_6,$$

and $c_k = 0$ for $k \ge 4$.

We are interested in counting equivalence classes, so let $\mathcal{S}$ denote a set of equivalence class representatives, and for $s = s_1\cdots s_m \in \mathcal{S}$, define $x_s = \prod_{j=1}^m x_{s_j}$.

Theorem 21.1 (Cartier-Foata)**.**

$$\sum_{s\in\mathcal{S}} x_s = \frac{1}{1 - c_1 + c_2 - c_3 + \ldots}.$$

Proof. Crossmultiplying in the result by the denominator on the right hand side, and writing the c_k's as sums over subsets, we have the following equivalent result:

$$\sum_\alpha (-1)^{|\alpha|} x_\alpha \sum_{s\in\mathcal{S}} x_s = 1, \tag{80}$$

where the first summation is over all $\alpha \subseteq \{1, \ldots, n\}$ such that all pairs in α are contained in $\mathcal{C}$.

We define a mapping ψ on all ordered pairs (α, s) of summation indices in (80) except $(\emptyset, \varepsilon)$, as follows: Let α_m be the largest element in α (if α is nonempty), and let $s = s_1\cdots s_k$. Let s_i be the largest symbol in s that commutes with everything in α, and that can appear leftmost in a string that is equivalent to s. Note that s_i doesn't always exist, but must exist when α is empty (if α is empty, then all symbols commute with everything in α).

Case 1: If $\alpha = \emptyset$ or $\alpha_m < s_i$, then let

$$\psi((\alpha, s)) = (\alpha \cup \{s_i\}, t),$$

where $s \equiv s_i t$.

Case 2: Otherwise, let

$$\psi((\alpha, s)) = (\alpha \setminus \{\alpha_m\}, \alpha_m s).$$

For example, with the set $\mathcal{C}_1$ of allowable commutations considered above, we have

$$\psi((\{2,4\}, 543)) = (\{2\}, 4543), \qquad \psi((\{2\}, 543)) = (\{2,3\}, 54),$$

where in both cases we had $s_i = 3$.

We claim that ψ is an involution with no fixed points, and thus a bijection. You are asked to prove this claim in Problem 1 of Assignment 4. Note that, if we define

$$\mathrm{wt}\,((\alpha, s)) = (-1)^{|\alpha|} x_\alpha x_s,$$

then clearly $\mathrm{wt}\,(\psi(\alpha, s)) = -\mathrm{wt}\,((\alpha, s))$. The fact that ψ is an involution with no fixed points implies that the terms on the left hand side of (80) cancel in pairs, except for the single term corresponding to $(\alpha, s) = (\emptyset, \varepsilon)$. But the latter term is 1, and the result follows immediately. $\square$

As an example of Theorem 21.1, we consider equivalence classes of ordered factorizations of $(1\,2 \dots n)$ into $n-1$ transpositions, where the factorizations are regarded as strings of pairs, and equivalence is defined as allowable commutations – we allow $(i\,j)$ and $(k\,l)$ to commute when i, j, k, l are four distinct symbols (so the corresponding transpositions commute in the symmetric group).

We create equivalence class representatives for all ordered products of $n-1$ transpositions in the group algebra of the symmetric group, and then evaluate $[\mathsf{C}^{(n)}]$. Applying Theorem 21.1, in the group algebra we have

$$c_k = \mathsf{C}^{(1^{n-2k}2^k)}, \qquad k \geq 1,$$

where $(1^{n-2k}2^k)$ is the partition with k parts equal to 2, and $n-2k$ parts equal to 1. Each string counted by c_k has k transpositions, so the number we require is

(81) $$b_n = [\mathsf{C}^{(n)} t^{n-1}] \frac{1}{1 + \sum_{k \geq 1} \mathsf{C}^{(1^{n-2k}2^k)} (-t)^k} = [u_{n-1}] \frac{1}{\sum_{k \geq 0} (-1)^k u_{(1^k)}},$$

where the second equality follows from Corollary 20.2.

The following results will allow us to evaluate coefficients in expressions involving the symmetric functions u_λ, as required in (81).

Theorem 21.2. *Let $F(x) = 1 + f_1 x + f_2 x^2 + \dots$. Then*

$$[u_\lambda(x_1, \dots)] \prod_{i \geq 1} F(x_i) = c_\lambda,$$

where $c_\lambda = \prod_{i \geq 1} c_{\lambda_i}$, and

$$c_j = [s^{j+1}] t, \qquad j \geq 1,$$

where $s = tF(t)$.

Proof. Let ϑ be a homomorphism with $\vartheta(h_j(y_1,\ldots)) = f_j$, $j \geq 1$. Then we have

$$\prod_{i\geq 1} F(x_i) = \vartheta \prod_{i\geq 1} H(x_i; y_1,\ldots) = \vartheta \prod_{i,j\geq 1} (1 - x_i y_j)^{-1},$$

so from Theorem 7.1, we obtain

$$[u_\lambda(x_1,\ldots)] \prod_{i\geq 1} F(x_i) = \vartheta [u_\lambda(x_1,\ldots)] \sum_\mu u_\mu(x_1,\ldots) h^*_\mu(y_1,\ldots) = \vartheta h^*_\lambda(y_1,\ldots).$$

Now let $s = tH(t; y_1,\ldots)$, so $t = sH^*(s; y_1,\ldots)$. Then $h^*_j(y_1,\ldots) = [s^{j+1}]t$, and applying ϑ, we obtain

$$\vartheta h^*_j(y_1,\ldots) = [s^{j+1}]t, \qquad s = tF(t),$$

giving the result. □

Corollary 21.3. *Let $A(x) = 1 + a_1 x + a_2 x^2 + \ldots$, and $a_\lambda = \prod_{i\geq 1} a_{\lambda_i}$. Then*

$$[u_\lambda(x_1,\ldots)] \left(\sum_\mu a_\mu u_\mu(x_1,\ldots) \right)^w = c_\lambda,$$

where $c_\lambda = \prod_{i\geq 1} c_{\lambda_i}$, and

$$c_j = [s^{j+1}] vA(v) = [s^j] A(v)^w, \qquad j \geq 1,$$

where $s = vA(v)^{1-w}$.

Proof. Let γ be a homomorphism with $\gamma(h^*_j(y_1,\ldots)) = a_j$, $j \geq 1$. Then from Theorem 7.1, we have

$$\left(\sum_\mu a_\mu u_\mu(x_1,\ldots) \right)^w = \gamma \prod_{i,j\geq 1} (1 - x_i y_j)^{-w} = \gamma \prod_{i\geq 1} H(x_i; y_1,\ldots)^w.$$

Then Lemma 21.2 gives

$$LHS = \gamma c_\lambda(y_1,\ldots),$$

where $c_j(y_1,\ldots) = [s^{j+1}]t$, and $s = tH(t, y_1,\ldots)^w$. Now let

$$v = tH(t; y_1,\ldots), \tag{82}$$

which is equivalent to

$$t = vH^*(v; y_1,\ldots). \tag{83}$$

Eliminating t, we have

$$c_j(y_1,\ldots) = [s^{j+1}] vH^*(v; y_1,\ldots), \qquad s = v \left(H^*(v; y_1,\ldots)\right)^{1-w},$$

and the result follows by applying γ. □

Now we return to the calculation of the number b_n of inequivalent minimal transposition factorizations of $(1\,2\ldots n)$. Continuing from (81), we have

$$b_n = [u_{n-1}] \left(\sum_{k\geq 0} (-1)^k u_{(1^k)} \right)^{-1} = [u_{n-1}] \left(\sum_\mu a_\mu u_\mu \right)^{-1},$$

where $a_1 = -1$, and $a_k = 0$ for $k \geq 2$. Thus we can apply Corollary 21.3 with $\lambda = (n-1)$, $A(x) = 1 - x$, and $w = -1$, which gives

$$b_n = [s^{n-1}](1 - v)^{-1},$$

where $s = v(1-v)^2$. Then from Lagrange's Implicit FunctionTheorem, we obtain, $n \geq 2$,

$$b_n = \frac{1}{n-1}[z^{n-2}](1-z)^{-2}\left((1-z)^{-2}\right)^{n-1} = \frac{1}{n-1}[z^{n-2}](1-z)^{-2n} = \frac{1}{n-1}\binom{3n-3}{n-2},$$

and for $n = 1$, the answer is 1 (a single, empty, factorization). A uniform expression for this answer is

$$b_n = \frac{1}{2n-1}\binom{3n-3}{n-1}, \qquad n \geq 1.$$

For example, when $n = 3$, we have $n^{n-2} = 3^1 = 3$, and $b_3 = \frac{1}{5}\binom{6}{2} = 3$, and indeed there are 3 factorizations of $(1\,2\,3)$, namely

$$(1\,2)(2\,3), \qquad (1\,3)(1\,2), \qquad (2\,3)(1\,3),$$

all of which are inequivalent. When $n = 4$, $n^{n-2} = 4^2 = 16$, and $b_3 = \frac{1}{7}\binom{9}{3} = 12$. The 16 factorizations of $(1\,2\,3\,4)$ are given below, organized into their 12 equivalence classes:

$$(1\,2)(2\,3)(3\,4), \qquad (1\,2)(2\,4)(2\,3), \qquad (1\,4)(1\,2)(2\,3), \qquad (1\,4)(1\,3)(1\,2),$$
$$(2\,3)(1\,3)(3\,4), \qquad (2\,3)(3\,4)(1\,4), \qquad (3\,4)(1\,4)(1\,2), \qquad (3\,4)(2\,4)(1\,4),$$
$$(1\,2)(3\,4)(2\,4) \equiv (3\,4)(1\,2)(2\,4), \qquad (1\,3)(1\,2)(3\,4) \equiv (1\,3)(3\,4)(1\,2),$$
$$(1\,4)(2\,3)(1\,3) \equiv (2\,3)(1\,4)(1\,3), \qquad (2\,4)(1\,4)(2\,3) \equiv (2\,4)(2\,3)(1\,4).$$