SPECIAL K
Saturday November 9, 2013
10:00 am - 1:00 pm

1: Let a, n and k be positive integers. Suppose that $m \geq 3$ and $\gcd(a, m) = 1$. Show that $a^k + (m-a)^k \equiv 0 \bmod m^2$ if and only if m is odd and $k \equiv m \bmod 2m$.

2: Find the number of positive integers k such that $k^2 + 2013$ is a square.

3: For each positive integer n, let a_n be the first digit in the decimal representation of 2^n, let b_n be the number of indices $k \leq n$ for which $a_k = 1$, and let c_n be the number of indices $k \leq n$ for which $a_k = 2$. Show that there exists a positive integer N such that for all $n \geq N$ we have $b_n > c_n$.

4: Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers such that $a_n \leq \dfrac{a_{n-1} + a_{n-2}}{2}$ for all $n \geq 3$. Show that $\{a_n\}$ converges.

5: Let $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbf{Z}$. Suppose that $1 < f(1) < f\big(f(1)\big) < f\Big(f\big(f(1)\big)\Big)$. Show that $a \geq 0$.

6: Let E be an ellipse in $\mathbf{R}^2$ centred at the point O. Let A and B be two points on E such that the line OA is perpendicular to the line OB. Show that the distance from O to the line through A and B does not depend on the choice of A and B.

BIG E
Saturday November 9, 2013
10:00 am - 1:00 pm

1: Find the number of positive integers k such that $k^2 + 10!$ is a perfect square.

2: Let $f : [0,1] \to \mathbf{R}$ be continuous. Suppose that $\int_0^x f(t)\,dt \geq f(x) \geq 0$ for all $x \in [0,1]$. Show that $f(x) = 0$ for all $x \in [0,1]$.

3: For each positive integer n, let a_n be the first digit in the decimal representation of 2^n, let b_n be the number of indices $k \leq n$ for which $a_k = 1$, and let c_n be the number of indices $k \leq n$ for which $a_k = 2$. Show that there exists a positive integer N such that for all $n \geq N$ we have $b_n > c_n$.

4: Let p be an odd prime. Show that $\binom{2p}{p} \equiv 2 \bmod p^2$.

5: Let V be a vector space over $\mathbf{R}$. Let V^* be the space of linear maps $g : V \to \mathbf{R}$. Let F be a finite subset of V^*. Let $U = \{x \in V \,|\, f(x) = 0 \text{ for all } f \in F\}$. Show that for all $g \in V^*$, if $g(x) = 0$ for all $x \in U$ then $g \in \text{Span}(F)$.

6: Let a, b and c be positive real numbers. Let E be the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in $\mathbf{R}^3$. Let $u, v, w \in E$ be such that the set $\{u, v, w\}$ is orthogonal. Show that the distance from the origin to the plane through u, v and w does not depend on the choice of u, v and w.