1: Let $a$, $n$ and $k$ be positive integers. Suppose that $m \geq 3$ and $\gcd(a, m) = 1$. Show that $a^k + (m-a)^k \equiv 0 \mod m^2$ if and only if $m$ is odd and $k \equiv m \mod 2m$.

2: Find the number of positive integers $k$ such that $k^2 + 2013$ is a square.

3: For each positive integer $n$, let $a_n$ be the first digit in the decimal representation of $2^n$, let $b_n$ be the number of indices $k \leq n$ for which $a_k = 1$, and let $c_n$ be the number of indices $k \leq n$ for which $a_k = 2$. Show that there exists a positive integer $N$ such that for all $n \geq N$ we have $b_n > c_n$.

4: Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers such that $a_n \leq \frac{a_{n-1} + a_{n-2}}{2}$ for all $n \geq 3$. Show that $\{a_n\}$ converges.

5: Let $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$. Suppose that $1 < f(1) < f(f(1)) < f(f(f(1)))$. Show that $a \geq 0$.

6: Let $E$ be an ellipse in $\mathbb{R}^2$ centred at the point $O$. Let $A$ and $B$ be two points on $E$ such that the line $OA$ is perpendicular to the line $OB$. Show that the distance from $O$ to the line through $A$ and $B$ does not depend on the choice of $A$ and $B$. 
1: Find the number of positive integers \( k \) such that \( k^2 + 10! \) is a perfect square.

2: Let \( f : [0, 1] \to \mathbb{R} \) be continuous. Suppose that \( \int_0^x f(t) \, dt \geq f(x) \geq 0 \) for all \( x \in [0, 1] \). Show that \( f(x) = 0 \) for all \( x \in [0, 1] \).

3: For each positive integer \( n \), let \( a_n \) be the first digit in the decimal representation of \( 2^n \), let \( b_n \) be the number of indices \( k \leq n \) for which \( a_k = 1 \), and let \( c_n \) be the number of indices \( k \leq n \) for which \( a_k = 2 \). Show that there exists a positive integer \( N \) such that for all \( n \geq N \) we have \( b_n > c_n \).

4: Let \( p \) be an odd prime. Show that \( \binom{2p}{p} \equiv 2 \mod p^2 \).

5: Let \( V \) be a vector space over \( \mathbb{R} \). Let \( V^* \) be the space of linear maps \( g : V \to \mathbb{R} \). Let \( F \) be a finite subset of \( V^* \). Let \( U = \{ x \in V \,|\, f(x) = 0 \text{ for all } f \in F \} \). Show that for all \( g \in V^* \), if \( g(x) = 0 \) for all \( x \in U \) then \( g \in \text{Span}(F) \).

6: Let \( a, b \) and \( c \) be positive real numbers. Let \( E \) be the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \) in \( \mathbb{R}^3 \). Let \( u, v, w \in E \) be such that the set \( \{u, v, w\} \) is orthogonal. Show that the distance from the origin to the plane through \( u, v \) and \( w \) does not depend on the choice of \( u, v \) and \( w \).