Note

A Combinatorial Construction for Products of Linear Transformations over a Finite Field

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Kovacs (J. Combin. Theory, Ser. A [45 (1987), 290–299]) has derived an expression for the number of ordered k-tuples, $(A_k, ..., A_1)$, of $n \times n$ matrices over $GF(q)$ whose product $A_k \cdots A_1$ has prescribed rank. We give a combinatorial construction for this result. © 1990 Academic Press, Inc.

Let $\mathcal{V}$ be a vector space of dimension $n$ over $GF(q)$. We determine the number, $p_k(n, i, j)$, of $k$-tuples of linear operators over $\mathcal{V}$ such that the rank of the restriction of their product to a prescribed $i$-dimensional subspace of $\mathcal{V}$ is equal to $j$.

The set of all linear operators on $\mathcal{V}$ is denoted by $\mathcal{H}$. If $T = (T_k, ..., T_1) \in \mathcal{H}^k$, then we denote $T_k \cdots T_1 \in \mathcal{H}$ by $\hat{T}$. Throughout, $\dim_{GF(q)}$ and $\text{span}_{GF(q)}$ are abbreviated to dim and span. The set of all $i$-dimensional subspaces of $\mathcal{V}$ is denoted by $(\mathcal{V}^i)$. Implicit use is made of the fact that, if $\mathcal{V}_1, \mathcal{V}_2$ are vector spaces, then $\mathcal{V}_1 \cong \mathcal{V}_2$ if and only if $\dim \mathcal{V}_1 - \dim \mathcal{V}_2$, so enumerative quantities associated with vector spaces depend only on dimensions (and, of course, the ground field). It is well known (see, for example, [1, 2]) that $|\binom{\mathcal{V}}{i}| = \prod_{k=1}^{i} \frac{(1 - q^{n-k+1})}{(1 - q^k)}$, the Gaussian coefficient, which is denoted by $\binom{i}{q}$.

We begin with a combinatorial derivation of a linear relationship involving $p_k(n, i, j)$.

**Theorem 1.** For $0 \leq l \leq i \leq n$,

$$\sum_{j=0}^{i} p_k(n, i, j) q^{(i-j)} \binom{i}{j}_q = \binom{i}{l}_q p_k(n, l, l).$$

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Proof. Let $\mathcal{U} \in (\mathcal{U})$ be arbitrary but fixed. We derive two different expressions for $\psi$, the cardinality of the set \[ \{ (\mathcal{X}, T) \in (\mathcal{U}) \times \mathcal{H}^k : r(T|_{\mathcal{X}}) = l \}. \]

First, by summing over $\mathcal{X}$ we see that $\psi = \sum_{\mathcal{X} \in (\mathcal{U})} \{ T \in \mathcal{H}^k : r(T|_{\mathcal{X}}) = l \} = |(\mathcal{U})| \cdot |\{ T \in \mathcal{H}^k : r(T|_{\mathcal{X}_0}) = l \}|$, where $\mathcal{X}_0 \in (\mathcal{U})$ is arbitrary but fixed. Thus

$$\psi = \left( \begin{array}{c} i \\ l \end{array} \right) p_k(n, l, l). \quad (1)$$

Second, by summing over $T$, we see that $\psi = \sum_{j=1}^{\dim \mathcal{U}} \sum_{T \in \mathcal{H}^k} \{ \mathcal{X} \in (\mathcal{U}) : r(T|_{\mathcal{X}}) = l, r(T|_{\mathcal{U}}) = j \}$. We now give a construction for $\mathcal{X}$. Given $T$, let $T' \in (\mathcal{U})$ be arbitrary but fixed such that $\ker T' \oplus T = \mathcal{U}$ where $\dim \mathcal{U} = j$. Thus $r(T'|_{\mathcal{U}}) = j$. Let $\mathcal{U} \in (\mathcal{U})$ have a canonical ordered basis $(y_1, \ldots, y_l)$, and let $c_1, \ldots, c_l \in (\ker f|_{\mathcal{U}})'$. Then

(i) $\text{span}(y_1 + c_1, \ldots, y_l + c_l) \in (\mathcal{U})$,

(ii) $\mathcal{F} \text{span}(y_1 + c_1, \ldots, y_l + c_l) = \mathcal{F}\mathcal{U}$ so $r(T|_{\text{span}(y_1 + c_1, \ldots, y_l + c_l)}) = \dim \mathcal{U} = l$,

(iii) $\text{span}(y_1 + c_1, \ldots, y_l + c_l) = \text{span}(y_1 + d_1, \ldots, y_l + d_l)$ if and only if $c_m = d_m$ for $m = 1, \ldots, l$.

We may therefore suppose that $\mathcal{X} = \text{span}(y_1 + c_1, \ldots, y_l + c_l)$ for some $(y_1, \ldots, y_l)$ and $(c_1, \ldots, c_l)$, so $\{ \mathcal{X} \in (\mathcal{U}) : r(T|_{\mathcal{X}}) = l, r(T|_{\mathcal{U}}) = j \} = \sum_{\mathcal{X} \in (\mathcal{U})} \{ (\ker f|_{\mathcal{U}})' \} = \sum_{\mathcal{X} \in (\mathcal{U})} \{ (\ker f|_{\mathcal{U}})' \} = \sum_{\mathcal{X} \in (\mathcal{U})} \{ (\ker f|_{\mathcal{U}})' \} = \sum_{\mathcal{X} \in (\mathcal{U})} \{ (\ker f|_{\mathcal{U}})' \} = \sum_{\mathcal{X} \in (\mathcal{U})} \{ (\ker f|_{\mathcal{U}})' \}$. Thus

$$\psi = \sum_{j=1}^{\dim \mathcal{U}} \left( \begin{array}{c} i \\ l \end{array} \right) q^{(i-j)l} \{ T \in \mathcal{H}^k : r(T|_{\mathcal{U}}) = j \}$$

$$= \sum_{j=1}^{\dim \mathcal{U}} \left( \begin{array}{c} i \\ l \end{array} \right) q^{(i-j)l} p_k(n, i, j). \quad (2)$$

The result follows by equating (1) and (2).

To evaluate $p_k(n, i, j)$ explicitly, we invert the linear relationship given in Theorem 1, and evaluate $p_k(n, l, l)$ for $0 \leq l \leq n$, using the next two propositions.

**Proposition 2.** Let $f_0, f_1, \ldots, g_0, g_1, \ldots$ be formal Laurent series in the indeterminate $u$. Then

$$f_j = \sum_{l \geq j} \left( \begin{array}{c} j \\ l \end{array} \right) g_l \quad \text{for} \quad j = 0, 1, \ldots$$

if and only if

$$g_l = \sum_{j \geq l} (-1)^{j-l} u^{(j-l)} \left( \begin{array}{c} j \\ l \end{array} \right) f_j \quad \text{for} \quad l = 0, 1, \ldots$$
Proof. Let \( j^!_u \) denote \((1 - u)(1 - u^2) \cdots (1 - u^j)\). The zeta and Möbius functions for the lattice of partitions ordered by refinement (Goldman and Rota \[1\]) are, respectively, \( \zeta(t) = \sum_{k \geq 0} t^k / j^!_u \) and \( \mu(t) = \sum_{k \geq 0} (-1)^k u(\frac{j}{2}) t^k / j^!_u \), and, moreover, \( \zeta(t) \mu(t) = 1 \). Let \( f(t) = \sum_{k \geq 0} f_k t^k / j^!_u \) and \( g(t) = \sum_{k \geq 0} g_k t^k / j^!_u \). Then \( f(t) = \zeta(t^{-1}) g(t) \) if and only if \( g(t) = \mu(t^{-1}) f(t) \), and the result follows by comparing the coefficients in each of these. 

**Proposition 3.** For \( 0 \leq l \leq n \),

\[
p_k(n, l, l) = \left\{ q^{n(n-l)}(q^n - 1)(q^n - q) \cdots (q^n - q^{l-1}) \right\}^k.
\]

Proof. \( p_k(n, l, l) = \left\{ \{ T \in \mathcal{X}^k : r(\hat{T} \upharpoonright _X) = l \} \right\} \), where \( \mathcal{X} \in \binom{I}{k} \) is arbitrary but fixed. Thus \( r(T_s \upharpoonright _X) = l \) for \( s = 1, \ldots, k \) so \( p_k(n, l, l) = p_k^1(n, l, l) \). But \( p_1(n, l, l) = q^{n(n-l)}(q^n - 1)(q^n - q) \cdots (q^n - q^{l-1}) \), since the basis elements of \( \mathcal{X} \) must be mapped into a linearly independent \( l \)-tuple of elements of \( \mathcal{Y}^c \), of which there are clearly \((q^n - 1)(q^n - q) \cdots (q^n - q^{l-1}) \). The remaining \( n-l \) elements in the basis of \( \mathcal{Y}^c \) formed by extending the basis of \( \mathcal{X} \) can be mapped to any of the \( q^n \) elements of \( \mathcal{Y}^c \).

We now complete the evaluation of \( p_k(n, i, j) \).

**Corollary 4.** For \( 0 \leq j \leq i \leq n \),

\[
p_k(n, i, j) = \frac{1}{q^{i(\frac{j}{2}) - (\frac{j}{2})}} \sum_{l=j}^{i} \binom{i-l}{l} \frac{(-1)^{i-j} q^{(\frac{j-l}{2})}}{q} \times \left\{ q^{n(n-l)}(q^n - 1)(q^n - q) \cdots (q^n - q^{l-1}) \right\}^k.
\]

Proof. Multiplying both sides of Theorem 1 by \((-1)^j q^{(\frac{j+1}{2}) - \frac{n}{2}} \) we obtain

\[
\sum_{j \geq l} p_k(n, i, j) q^{-\frac{j}{2}} (-1)^j (-1)^{i-l} q^{(\frac{j-l}{2})} \binom{j}{l} q^{\frac{n}{2}} p_k(n, l, l).
\]

Now let \( g_j = (-1)^j q^{(\frac{j+1}{2}) - \frac{n}{2}} p_k(n, l, l) \) and \( f_j = p_k(n, i, j) q^{-\frac{j}{2}} (-1)^j \) and the result follows from Proposition 2, after substituting the value for \( p_k(n, l, l) \) given by Proposition 3.

Kovacs’ \[3\] expression for the number of ordered \( k \)-tuples of matrices over \( GF(q) \) whose product has rank \( t \) is obtained by setting \( i = n, j = n - t \) in Corollary 4.
Algebraic proofs of Theorem 1 and Corollary 4 can be obtained as follows. Let \( M_k = \left[ p_k(n, i, j) \right]_{0 \leq i, j \leq n} \), \( Q = \left[ \frac{(j)!}{q^{(i-j)}} \right]_{0 \leq i, j \leq n} \), and \( D_k = \text{diag}(p_k(n, 0, 0), \ldots, p_k(n, n, n)) \). The following facts can be verified: \( M_k Q = Q D_k \), \( M_k = M_k^t \), and \( D_k = D_k^t \). Such a \( Q \) exists because \( M_k \) is diagonalizable, since its eigenvalues, \( p_k(n, i, i) \), for \( i = 0, \ldots, n \), are mutually distinct. These results may be combined to give \( M_k Q = Q D_k \), and hence Theorem 1 by comparing the \((i, j)\)-elements of these matrices. Corollary 4 follows by using the fact that \( M_k = Q D_k Q^{-1} \), where \( Q^{-1} = \left[ (-1)^{i-j} \frac{(j)!}{q^{(i-j)}} - \frac{(j)!}{q^{(i-\frac{j}{2})}} - \frac{(j)!}{q^{(i-\frac{j}{2})}} \right]_{0 \leq i, j \leq n} \).

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