AN INVERSION THEOREM FOR CLUSTER DECOMPOSITIONS OF SEQUENCES WITH DISTINGUISHED SUBSEQUENCES

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ABSTRACT

Certain enumeration problems may be expressed in terms of sequences possessing a specified number of subsequences which are elements of a prescribed set of distinguished sequences. We obtain an inversion theorem which expresses the required generating function in terms of one connected with the set of overlapping distinguished sequences called clusters. Techniques are given for determining the cluster generating function both in the general case and in the case in which combinatorial methods are more effective. By specialising the set of distinguished sequences we may solve a number of classical permutation and sequence problems. A number of other examples is also given.

1. *Introduction*

The enumeration of sequences with specified frequencies of occurrences of distinguished subsequences occurs typically when sequences are used for encoding combinatorial configurations. The general problem is solved in an explicit form which takes account of the distinguished sequences and the precise way in which they overlap. In certain instances the distribution of overlaps between distinguished sequences is regular enough to permit the generating function to be expressed as an explicit rational function.

A particular type of sequence, called a cluster, which is a linearly ordered collection of overlapping distinguished sequences, emerges as a useful combinatorial device. The value of the approach lies in the fact that the overlapping structure of distinguished sequences is accounted for in a natural way. Accordingly, once appropriate clusters have been defined in terms of some set of distinguished sequences, the clusters may be enumerated separately, without reference to the manner in which they are distributed in actual sequences. The main counting lemma, given in Section 2, expresses the required generating function in terms of the generating function for clusters. A general expression for the cluster generating function is given in Section 3. In certain cases it is possible to obtain the cluster generating function by combinatorial means. This is discussed in Section 4, where a number of applications is presented.

We solve a number of sequence problems which have been treated already by a variety of methods. The solutions presented here, however, are motivated directly from the common combinatorial structure of the problems. The inversion theorem, which connects the desired generating function with the cluster generating function, rests only on the Principle of Inclusion and Exclusion. Moreover, the cluster generating function often may be written down immediately by purely combinatorial reasoning using elementary arguments. This fact is reflected in the exposition by our deliberate suppression of the routine algebraic details of substituting the cluster generating function into the inversion theorem, since these details may be supplied by the reader and their inclusion merely obfuscates the essential simplicity of the arguments.

2. *Preliminaries*

Throughout this paper $x_1^{k_1} \ldots x_n^{k_n}$ is denoted by $\mathbf{x}^{\mathbf{k}}$ where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{k} = (k_1, \ldots, k_n)$, and $[\mathbf{x}^{\mathbf{k}}](\ldots)$ denotes the coefficient of $\mathbf{x}^{\mathbf{k}}$ in the formal power series $(\ldots)$.

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Let $\mathcal{N} = \{1, 2, ..., n\}$ and let $\mathcal{N}^*$ be the monoid of sequences over $\mathcal{N}$ with concatenation (denoted by juxtaposition). The *type*, $\tau(\sigma)$, of σ is $(k_1, ..., k_n)$ where σ has exactly k_i occurrences of i, $i = 1, ..., n$. Let $\mathcal{N}^+ = \mathcal{N}^* \setminus \{\varepsilon\}$ where ε is the empty sequence. If $\alpha, \beta, \gamma, \delta \in \mathcal{N}^*$ are such that $\alpha = \beta\gamma\delta$ then we shall call γ a *subsequence* of α.

Suppose that $\mathscr{A} = \{A_1, ..., A_p\}$ is a set of distinguished sequences in $\mathcal{N}^+$. If $\mathscr{A}$ has the property that there are no $A, B \in \mathscr{A}$ such that A is a subsequence of B, then it is called a *reduced* set of distinguished sequences. The $\mathscr{A}$-type of a sequence σ is $\kappa(\sigma) = (m_1, ..., m_p)$ where m_i is the number of occurrences of A_i as a subsequence of σ. We next define a q-cluster in terms of a sequence and an ordered set of overlapping elements of $\mathscr{A}$ which cover the sequence.

Definition 2.1.

(1) A *q-cluster* ($q \geq 1$) on the alphabet $\mathcal{N}$, and associated with the reduced set $\mathscr{A}$, is a triple $(\sigma_1 ... \sigma_r, A_{i_1} ... A_{i_q}, (l_1, ..., l_q))$ such that $\sigma_1 ... \sigma_r \in \mathcal{N}^+$ and $(l_1, ..., l_q)$ satisfies the following conditions where r_j is the length of $A_{i_j}, j = 1, ..., q$:

(i) $\sigma_{l_j}\sigma_{l_j+1}...\sigma_{l_j+r_j-1} = A_{i_j},\ \ j = 1, ..., q$

(ii) $0 < l_{j+1} - l_j < r_j,\ \ j = 1, ..., q-1$

(iii) $r = l_q + r_q - 1$ and $l_1 = 1$.

(2) $\mathscr{D}(\mathscr{A})$ is the set of all q-clusters on the alphabet $\mathcal{N}$, and associated with $\mathscr{A}$, for all $q \geq 1$.

We next define the cluster generating function $C(\mathbf{x}, \mathbf{y})$ as an ordinary generating function of the set $\mathscr{D}(\mathscr{A})$.

Definition 2.2. The *cluster generating function* is

$$C(\mathbf{x}, \mathbf{y}) = \sum_{(\mu_1, \mu_2, \mu_3) \in \mathscr{D}(\mathscr{A})} \mathbf{x}^{\tau(\mu_1)} \mathbf{y}^{\alpha(\mu_2)}$$

where $\alpha(\mu_2) = (m_1, ..., m_p)$ and m_i is the number of occurrences of A_i in μ_2, $i = 1, ..., p$.

We remark that there may be many clusters (μ_1, μ_2, μ_3) with the same μ_1 but different (μ_2, μ_3) and for each of these $\mathbf{y}^{\alpha(\mu_2)} | \mathbf{y}^{\kappa(\mu_1)}$. There is, however, a unique (μ_2, μ_3) such that $\alpha(\mu_2) = \kappa(\mu_1)$.

The set $\mathscr{D}(\mathscr{A})$ is often combinatorially convenient since it may be constructed by arranging the elements of $\mathscr{A}$, with duplication permitted, in all possible ways so that adjacent elements of $\mathscr{A}$ overlap (*viz.* conditions (i)–(iii) of Definition 2.1.1 are satisfied).

Occasionally we shall have cause to consider $C(\mathbf{x}, y\mathbf{1})$ and $C(x\mathbf{1}, y\mathbf{1})$ where $\mathbf{1}$ is a unit vector of the appropriate length. To avoid proliferation of notation we permit ourselves the minor abuse of notation of denoting $C(\mathbf{x}, y\mathbf{1})$ by $C(\mathbf{x}, y)$ and $C(x\mathbf{1}, y\mathbf{1})$ by $C(x, y)$, and since this is only in the sections on specific examples, where of course the cluster generating functions are calculated *ab initio*, no confusion arises.

The following lemma gives the connexion between the desired generating function and the cluster generating function.

LEMMA 2.3. *The number of sequences in $\mathcal{N}^*$ of type* $\mathbf{k}$ *and $\mathscr{A}$-type* $\mathbf{m}$ *is* $[\mathbf{x}^{\mathbf{k}}\mathbf{y}^{\mathbf{m}}]\Phi(\mathbf{x}, \mathbf{y})$ *where*

$$\Phi(\mathbf{x}, \mathbf{y}) = \{1 - (x_1 + ... + x_n) - C(\mathbf{x}, \mathbf{y} - \mathbf{1})\}^{-1}.$$

Proof. Consider the set $\mathscr{E}(\mathscr{A})$ consisting of all triples of the form $(\sigma_1 \ldots \sigma_r, A_{i_1} \ldots A_{i_q}, (l_1, \ldots, l_q))$ for all $q \geqslant 1$, or $(\sigma_1 \ldots \sigma_r, \varnothing, \varnothing)$, where

(i) $\sigma_{l_j} \ldots \sigma_{l_j + r_j - 1} = A_{i_j}, j = 1, \ldots, q$, and

(ii) $1 \leqslant l_1 < \ldots < l_q \leqslant r - r_q + 1$

in which $r_j = |A_{i_j}|$ for $j = 1, \ldots, q$. Let

$$\Psi(\mathbf{x}, \mathbf{y}) = \sum_{(\mu_1, \mu_2, \mu_3) \in \mathscr{E}(\mathscr{A})} \mathbf{x}^{\tau(\mu_1)} \mathbf{y}^{\alpha(\mu_2)}$$

where $\alpha(\mu_2) = (m_1, \ldots, m_p)$ and m_i is the number of occurrences of A_i in μ_2, $i = 1, \ldots, p$. Thus, by the Principle of Inclusion and Exclusion we have $\Phi(\mathbf{x}, \mathbf{y}) = \Psi(\mathbf{x}, \mathbf{y} - \mathbf{1})$. However, $\Psi(\mathbf{x}, \mathbf{y})$ may be derived from $C(\mathbf{x}, \mathbf{y})$ by the following construction on $\mathscr{D}(\mathscr{A})$.

Select t elements of $\mathscr{D}(\mathscr{A})$ and order them linearly. There are $t+1$ positions into which sequences from $\mathscr{N}^*$ may be inserted independently. The positions are the $t-1$ gaps between adjacent pairs, the beginning of the sequence of clusters and the end of the sequence of clusters. The generating function for $\mathscr{N}^*$ with respect to sequence type alone is

$$S(\mathbf{x}) = \{1 - (x_1 + \ldots + x_n)\}^{-1}$$

so the generating function for the inserted sequences is S^{t+1}. The generating function for the ordered set of t clusters is C^t so

$$\Psi(\mathbf{x}, \mathbf{y}) = \sum_{t=0}^{\infty} S^{t+1} C^t = S(1 - CS)^{-1}$$

and the result follows.

Our strategy is accordingly to determine the cluster generating function $C(\mathbf{x}, \mathbf{y})$ first and to use Lemma 2.3 to obtain the desired generating function $\Phi(\mathbf{x}, \mathbf{y})$.

3. *The cluster generating function for arbitrary sets*

We now develop a general expression for the cluster generating function. The following definition is required.

Definition 3.1. The connector matrix $\mathbf{W}$ is such that

$$\mathbf{W} = [w_{ij}]_{p \times p} \text{ where } w_{ij} = \sum_{\alpha} \mathbf{x}^{\tau(\alpha)}$$

where the sum is taken over all $\alpha \in \mathscr{N}^+$ such that $A_i = \alpha\beta$ for $\beta \in \mathscr{N}^+$ providing that $A_j = \beta\gamma$ for some $\gamma \in \mathscr{N}^+$.

In other words β is an initial segment of A_j which is identical to a terminal segment of A_i, and α is the portion of A_i which remains when this terminal segment has been detached. Clearly, there may be a number of ways in which A_i and A_j may overlap.

PROPOSITION 3.2. *The cluster generating function is*

$$C(\mathbf{x}, \mathbf{y}) = \text{trace}\, \{\mathbf{I} - \mathbf{YW}\}^{-1} \mathbf{YLJ}$$

in which $\mathbf{Y} = \operatorname{diag}(y_1, \ldots, y_p)$, $\mathbf{L} = \operatorname{diag}(\mathbf{x}^{\tau(\mathscr{A}_1)}, \ldots, \mathbf{x}^{\tau(\mathscr{A}_p)})$ *and* $\mathbf{J}$ *is the* $p \times p$ *matrix all of whose elements are* 1.

Proof. Let $f_i = \sum \mathbf{x}^{\tau(\mu_1)}\mathbf{y}^{\alpha(\mu_2)}$ where the summation is over all $(\mu_1, \mu_2, \mu_3) \in \mathscr{D}(\mathscr{A})$ such that μ_1 has an initial segment A_i, and where $\alpha(\mu_2) = (m_1, \ldots, m_p)$ in which m_i is the number of occurrences of A_i in μ_2 for $i = 1, \ldots, p$. Thus $C(\mathbf{x}, \mathbf{y}) = f_1 + \ldots + f_p$. Then each element of the set enumerated by f_i may be constructed by prefixing μ_1 in an arbitrary cluster (μ_1, μ_2, μ_3) with A_i in such a fashion that a terminal segment of A_i overlaps with an initial segment of μ_1. Accordingly $\mathbf{f} = (f_1, \ldots, f_p)$ satisfies the following system:

$$f_i = y_i \mathbf{x}^{\tau(A_i)} + \sum_{j=1}^{p} y_i w_{ij} f_j, \qquad i = 1, \ldots, p$$

which may be rewritten

$$\mathbf{f}^T = \mathbf{L}\mathbf{y}^T + \mathbf{Y}\mathbf{W}\mathbf{f}^T \text{ where } \mathbf{y} = (y_1, \ldots, y_p).$$

Thus

$$\mathbf{f}^T = (\mathbf{I} - \mathbf{Y}\mathbf{W})^{-1}\mathbf{L}\mathbf{y}^T \text{ since } \mathbf{I} - \mathbf{Y}\mathbf{W} \text{ is non-singular.}$$

So $f_1 + \ldots + f_p = \operatorname{trace}(\mathbf{I} - \mathbf{Y}\mathbf{W})^{-1}\mathbf{L}\mathbf{Y}\mathbf{J}$ and the proposition follows.

COROLLARY 3.3. *The number of sequences in* $\mathscr{N}^*$ *of type* $\mathbf{k}$ *with no distinguished subsequences is*

$$[\mathbf{x}^{\mathbf{k}}]\{1 - (x_1 + \ldots + x_n) + \operatorname{trace}(\mathbf{I} + \mathbf{W})^{-1}\mathbf{L}\mathbf{J}\}^{-1}.$$

Proof. This follows directly from Lemma 2.3 with $\mathbf{y} = \mathbf{0}$, and Proposition 3.2.

COROLLARY 3.4. *The number of sequences in* $\mathscr{N}^*$ *of length* l *with no distinguished subsequences is*

$$[x^l]\{1 - nx + \operatorname{trace}(\chi(x^{-1}))^{-1}\mathbf{J}\}^{-1}$$

where

$$[\chi(x)]_{ij} = \sum_{\beta} x^{|\beta|}$$

in which the sum is over all $\beta \in \mathscr{N}^+$ *such that* $A_i = \alpha\beta$, $\alpha \in \mathscr{N}^*$ *providing that* $A_j = \beta\gamma$ *for some* $\gamma \in \mathscr{N}^*$.

Proof. Clearly $\mathbf{I} + \mathbf{W}|_{\mathbf{x} = x\mathbf{1}} = \Lambda(x)\chi(x^{-1})$ where $\Lambda(x) = \operatorname{diag}(x^{a_1}, \ldots, x^{a_p})$ where $a_i = |A_i|$, $i = 1, \ldots, p$, and the corollary follows from Corollary 3.3 with $\mathbf{x} = x\mathbf{1}$.

The matrix χ occurs in the work of Guibas and Odlyzko [**3, 4**] where it has been termed the correlation matrix (for distinguished sequences). Kim, Putcha and Roush [**8**] give an expression for the number of sequences with no distinguished subsequences.

The method is entirely different from the above.

The next example gives an instance of the use of the general expression for the cluster generating function.

EXAMPLE 3.5. *The number of sequences of type* **l** *with* m_i *blocks of adjacent numbers i of length k, for* $i = 1, 2, \ldots, n$ *is*

$$[\mathbf{x}^{\mathbf{l}}\mathbf{y}^{\mathbf{m}}]\left\{1 - \sum_{i=1}^{n} \{x_i(1-y_ix_i)+(y_i-1)x_i^k\}\{1-y_ix_i+(y_i-1)x_i^k\}^{-1}\right\}^{-1}.$$

Proof. Let $\mathscr{A} = \{111\ldots1, 222\ldots2, nnn\ldots n\}$, each of length k. Then $\mathbf{W} = \sum_{i=1}^{k-1} \mathbf{X}^i$ and $\mathbf{L} = \mathbf{X}^k$ where $\mathbf{X} = \operatorname{diag}(x_1, \ldots, x_n)$.

From Lemma 2.3 and Proposition 3.2 we have

$$\Phi(\mathbf{x},\mathbf{y}) = \left\{1-(x_1+\ldots+x_n)-\operatorname{trace}\left\{\left(\mathbf{I}-(\mathbf{Y}-\mathbf{I})\sum_{i=1}^{k-1}\mathbf{X}^i\right)^{-1}(\mathbf{Y}-\mathbf{I})\mathbf{X}^k\mathbf{J}\right\}\right\}^{-1}$$

$$= \left\{1-(x_1+\ldots+x_n)-\sum_{i=1}^{n} x_i^k(y_i-1)[1-(y_i-1)(x_i-x_i^k)(1-x_i)^{-1}]^{-1}\right\}^{-1}$$

and the corollary follows.

We may use the methods of this section to obtain generating functions which are not necessarily obtained from clusters. The following proposition demonstrates the method for the case of alternating sequences. André [1] considered alternating permutations.

PROPOSITION 3.6. *The number of sequences* $\sigma_1\ldots\sigma_l$ *in* $\mathscr{N}^*$ *of odd length l such that* $\sigma_1 < \sigma_2 \geqslant \sigma_3 < \ldots \geqslant \sigma_l$, *and type* **k** *is*

$$[\mathbf{x}^{\mathbf{k}}]\left\{\sum_{j=0}^{\infty}(-1)^j\gamma_{2j+1}(\mathbf{x})\right\}\left\{\sum_{j=0}^{\infty}(-1)^j\gamma_{2j}(\mathbf{x})\right\}^{-1}$$

where

$$\prod_{i=1}^{n}(1+xx_i) = \sum_{j\geqslant 0}\gamma_j(\mathbf{x})x^j.$$

Proof. Let σ be such a sequence beginning with k. Then $ij\sigma$ is also a sequence of this type providing $i < j$ and $j \geqslant k$. Let y_t be the generating function for all sequences of this type beginning with t. The required generating function is accordingly $\xi = y_1 + \ldots + y_n$. By considering the concatenation of ij and σ we have

$$y_i = x_i + \sum_{\substack{n\geqslant j>i\geqslant 1\\ n\geqslant j\geqslant k\geqslant 1}} x_ix_jy_k.$$

Let $\mathbf{y} = (y_1, \ldots, y_n)^T$, $\mathbf{X} = \mathrm{diag}\,(x_1, \ldots, x_n)$ and let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $n \times n$ matrices such that

$$a_{ij} = \begin{cases} 1 \text{ if } i < j \\ 0 \text{ otherwise} \end{cases} \text{and } b_{ij} = \begin{cases} 1 \text{ if } i \geqslant j \\ 0 \text{ otherwise}. \end{cases}$$

Then

$$\mathbf{y} = \mathbf{x} + \mathbf{XAXBy}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$ so $\xi = \mathrm{trace}\,(\mathbf{I} - \mathbf{XAXB})^{-1}\mathbf{XJ}$ where $\mathbf{J}$ is the $n \times n$ matrix all of whose elements are 1. Thus

$$\xi = \mathrm{trace}\,\{\mathbf{I} - \mathbf{XA}(\mathbf{I} + (\mathbf{XA})^2)^{-1}\mathbf{XJ}\}^{-1}(\mathbf{I} + (\mathbf{XA})^2)^{-1}\mathbf{XJ}.$$

But if $\mathbf{C}$ is a square matrix of rank one, then

$$(\mathbf{I} + \mathbf{C})^{-1} = \mathbf{I} - (1 + \mathrm{trace}\,\mathbf{C})^{-1}\,\mathbf{C}.$$

Thus applying this to the evaluation of ξ we have

$$\begin{aligned} \xi &= \mathrm{trace}\,\{\mathbf{I} + [1 - \mathrm{trace}\,\mathbf{XA}(\mathbf{I} + (\mathbf{XA})^2)^{-1}\mathbf{XJ}]^{-1}\mathbf{XA}(\mathbf{I} + (\mathbf{XA})^2)^{-1}\mathbf{XJ}\} \\ &\qquad\qquad \times (\mathbf{I} + (\mathbf{XA})^2)^{-1}\mathbf{XJ} \\ &= \{1 - \mathrm{trace}\,\mathbf{XA}(\mathbf{I} + (\mathbf{XA})^2)^{-1}\mathbf{XJ}\}^{-1}\,\mathrm{trace}\,(\mathbf{I} + (\mathbf{XA})^2)^{-1}\mathbf{XJ}. \end{aligned}$$

But trace $(\mathbf{XA})^k\mathbf{XJ} = \gamma_{k+1}(\mathbf{x})$ so the result follows.

The generating function obtained in Proposition 3.6 has been given by Carlitz [2].

Throughout §4, $\gamma_k(\mathbf{x})$ is used with the interpretation given in Proposition 3.6. Other definitions of this quantity may be furnished for other combinatorial situations. However, we do not enlarge upon this point here, but further details are given in Jackson and Goulden [7]. Where the context allows we replace $\gamma_k(\mathbf{x})$ by γ_k, the dependence of γ_k on $\mathbf{x}$ being understood.

4. *Combinatorial methods for determining the cluster generating function*

For certain sets $\mathscr{A}$ of distinguished sequences the cluster generating function may be determined by combinatorial means. The first result is for a special case of the open problem of enumerating the number of sequences of length l in $\mathscr{N}^*$ which contain no subsequences of the form $\{w^p | w \in \mathscr{N}^+\}$, for fixed $p > 1$.

COROLLARY 4.1. *Let $\mathscr{B}_p = \{w^p | w \in \mathscr{N}^k\}$ where $k, p \geqslant 1$. Then the number of sequences in $\mathscr{N}^*$ of length l with q subsequences in $\mathscr{B}_p$ is $[x^l y^q]FG^{-1}$ where*

$$F = (1-x)(1-nx) - (y-1)\{x(1-x^{k(p-1)}) - nx^2(1-x^{k(p-1)-1}) - n^k x^{kp}(1-x)\}$$

and

$$G = (1-nx)\{(1-x)(1-nx) - (y-1)[x(1-x^{k(p-1)}) - nx^2(1-x^{k(p-1)-1})]\}.$$

Proof. The set $\mathscr{B}_p$ is a reduced set of sequences. We form the cluster generating function $C(x, y)$ as follows. Let $[x^{pk}]\phi(x)$ be the number of members of $\mathscr{B}_p$ of length pk.

Thus $\phi(x) = n^k x^{pk}$. Let $f(x)$ be such that $[x^m]\phi(x)f(x)$ is the number of sequences of length m which are 2-clusters. Clearly

$$f(x) = x + x^2 + \ldots + x^{k(p-1)} + nx^{k(p-1)+1} + n^2 x^{k(p-1)+2} + \ldots + n^{k-1} x^{kp-1}$$

by considering the overlap of W_1 and W_2 where $W_1, W_2 \in \mathscr{B}_p$. Thus

$$f(x) = (x - x^{k(p-1)})(1-x)^{-1} + (x^{k(p-1)} - n^k x^{kp})(1-nx)^{-1}.$$

Now

$$C(x, y) = y\phi(x) \sum_{i=0}^{\infty} y^i f^i(x) = y\{1 - yf(x)\}^{-1}\phi(x)$$

and from Lemma 2.3 the desired generating function is

$$\{1 - nx - C(x, y-1)\}^{-1}.$$

The result follows immediately on making the above substitutions.

The next result was given in Jackson **[5]** where it was obtained in an entirely different way. The corollary may be specialised to give solutions to the Smirnov problem, the Simon Newcomb problem and various other problems involving partitions and compositions of integers. The details of these specialisations are given in Jackson **[5]**.

COROLLARY 4.2. *The number of sequences in $\mathscr{N}^+$ of type* $\mathbf{m}$ *with i rises, j levels and k falls is*

$$[\mathbf{x}^{\mathbf{m}} r^i l^j f^k] \frac{\prod_{t=1}^{n} \{1-(l-f)x_t\} - \prod_{t=1}^{n} \{1-(l-r)x_t\}}{f\prod_{t=1}^{n} \{1-(l-r)x_t\} - r\prod_{t=1}^{n} \{1-(l-f)x_t\}}.$$

Proof. Let $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$ where

$$\mathscr{A}_1 = \{ij \in \mathscr{N}^2 | i < j\}, \quad \mathscr{A}_2 = \{ii \in \mathscr{N}^2\} \text{ and } \mathscr{A}_3 = \{ij \in \mathscr{N}^2 | i > j\}.$$

Then $\mathscr{A}$ is a reduced set of sequences, $\mathscr{A}_1$ is the set of rises, $\mathscr{A}_2$ is the set of levels and $\mathscr{A}_3$ is the set of falls.

Let r, l and f be indeterminates marking the occurrence of elements of $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ respectively. The cluster generating function for $\mathscr{A}$ is

$$C(\mathbf{x}, r, l) = r^{-1}\left\{\prod_{i=1}^{n} \{1 + rx_i + rlx_i^2 + rl^2x_i^3 + \ldots\} - r(x_1 + \ldots + x_n) - 1\right\}$$

since a cluster is a non-decreasing sequence of length greater than one. Thus, from Lemma 2.3, the number of sequences in $\mathscr{N}^*$ of type m with i rises and j levels is

$$[\mathbf{x}^{\mathbf{m}} r^i l^j]\Psi(\mathbf{x}, r, l) = \psi(\mathbf{m}, i, j)$$

where

$$\Psi(\mathbf{x}, r, l) = \{1-(x_1+\ldots+x_n)-C(\mathbf{x}, r-1, l-1)\}^{-1}$$

$$= \sum_{\mathbf{m},i,j} \psi(\mathbf{m}, i, j) x_1^{m_1}\ldots x_n^{m_n} r^i l^j .$$

But $m_1+\ldots+m_n = i+j+k+1$. Thus the number of sequences in $\mathcal{N}^+$ of type $\mathbf{m}$ with i rises, j levels and k falls, is

$$[\mathbf{x}^{\mathbf{m}} r^i l^j f^k]\Phi(\mathbf{x}, r, l, f)$$

where

$$\Phi(\mathbf{x}, r, l, f) = f^{-1}\{\Psi(f\mathbf{x}, rf^{-1}, lf^{-1})-1\}$$

and the result follows.

In the following example we use clusters to enumerate sequences with a prescribed number of increasing subsequences of length p.

COROLLARY 4.3. *The number of sequences in $\mathcal{N}^+$ of type $\mathbf{k}$ with exactly i strictly increasing subsequences of length p is* $[\mathbf{x}^{\mathbf{k}} u^i]\{1-\gamma_1-F\langle\gamma\rangle\}^{-1}$ *where*

$$F(u, x) = (u-1)x^p\{1-(u-1)(x-x^p)(1-x)^{-1}\}^{-1}$$

and $F\langle\gamma\rangle = \sum_{j=0}^{\infty} F_j(u)\gamma_j$ *where* $F(u, x) = \sum_{j=0}^{\infty} F_j(u)x^j$.

Proof. Let (μ_1, μ_2, μ_3) be a k-cluster on the set $\mathscr{A}$ of all strictly increasing sequences of length p. Let μ_1 have length m, and $\mu_3 = (l_1, \ldots, l_k)$, the set of starting positions in μ_1 for elements of $\mathscr{A}$. Now μ_2 is uniquely specified by (μ_1, μ_3). Let $d_i = l_{i+1}-l_i$ for $i = 1, \ldots, k-1$ where $l_1 = 1$ and $l_k = m-p+1$. Then because of the conditions imposed on μ_3 in Definition 2.1 we have $1 \leqslant d_i \leqslant p-1$ for $i = 1, \ldots, k-1$, and $d_1+\ldots+d_{k-1} = m-p$. But the set of all μ_3 is in [1 : 1] correspondence with the set of all $(d_1, \ldots, d_{k-1})$ satisfying these conditions. The number of such $(d_1, \ldots, d_{k-1})$ is

$$[x^{m-p}](x-x^p)^{(k-1)}(1-x)^{-(k-1)} = [x^m]x^p(x-x^p)^{k-1}(1-x)^{-(k-1)}.$$

Thus $C(\mathbf{x}, u) = \sum_{m\geqslant p}\sum_{k\geqslant 1}\sum_{(\mu_1,\mu_2,\mu_3)} \mathbf{x}^{\tau(\mu_1)}u^k$, from Definition 2.2, where the summation is over all $(\mu_1, \mu_2, \mu_3) \in \mathscr{D}(\mathscr{A})$ such that $|\mu_1| = m$ and $|\mu_2| = |\mu_3| = k$. It follows that

$$C(\mathbf{x}, u) = \sum_{m\geqslant p}\sum_{k\geqslant 1} u^k[x^m]x^p(x-x^p)^{k-1}(1-x)^{-(k-1)} \sum_{\mu_1} \mathbf{x}^{\tau(\mu_1)}$$

where the summation is over all μ_1 such that $|\mu_1| = m$. Thus

$$C(\mathbf{x}, u) = \sum_{m\geqslant p}\sum_{k\geqslant 1} \gamma_m u^k[x^m]x^p(x-x^p)^{k-1}(1-x)^{-(k-1)}$$

$$= \sum_{m\geqslant p} \gamma_m[x^m]ux^p\{1-u(x-x^p)(1-x)^{-1}\}^{-1}$$

and the result follows from Lemma 2.3.

We note that Example 3.5 may be obtained in the same way. The final example concerns the enumeration of sequences with respect to strict maxima.

COROLLARY 4.4. *The number of sequences in $\mathcal{N}^*$ of type* **k** *with l strict maxima is*

$$[\mathbf{x}^{\mathbf{k}}y^l]\left\{1-M\left\{\prod_{i=1}^{n}(1+Mx_i)-\prod_{i=1}^{n}(1-Mx_i)\right\}\left\{\prod_{i=1}^{n}(1+Mx_i)+\prod_{i=1}^{n}(1-Mx_i)\right\}^{-1}\right\}^{-1}$$

where $M = (1-y)^{1/2}$.

Proof. Let $\mathscr{A} = \{ijk \in \mathcal{N}^* | i < j \geqslant k\}$. Then $\mathscr{A}$ is a reduced set and if $ijk \in \mathscr{A}$ then j is a strict maximum. Let y mark the occurrence of a member of $\mathscr{A}$ in a sequence. The clusters are alternating sequences of odd length greater than one. Thus, with the appropriate modification to Proposition 3.6 we obtain the cluster generating function $C(\mathbf{x}, y)$ as

$$C(\mathbf{x}, y) = \left\{\sum_{k\geqslant 0}(-y)^k\gamma_{2k+1}\right\}\left\{\sum_{k\geqslant 0}(-y)^k\gamma_{2k}\right\}^{-1} - (x_1+\ldots+x_n).$$

But we know from Proposition 3.6 that

$$\sum_{k\geqslant 0} z^k\gamma_k = \prod_{i=1}^{n}(1+zx_i)$$

and the result follows from Lemma 2.3.

Although attention has been confined exclusively to sequences, it is nevertheless possible to obtain corresponding results for permutations on $\{1, \ldots, n\}$. The following proposition is required.

PROPOSITION 4.5. *Let* $\gamma_k = [x^k]\prod_{i=1}^{n}(1+xx_i)$. *Let* η *be a function of* $\gamma_0, \gamma_1, \ldots$ *Then* $[x_1x_2\ldots x_n]\eta(\gamma_0, \gamma_1, \ldots) = \left[\frac{x^n}{n!}\right]\eta\left(1, \frac{x}{1!}, \frac{x^2}{2!}, \ldots\right)$.

Proof. See Jackson and Goulden [7].

COROLLARY 4.6. *The number of permutations on* $\{1, \ldots, n\}$ *with p strict maxima is*

$$\left[\frac{x^n}{n!}y^p\right]\{1-M\tanh Mx\}^{-1} \text{ where } M = (1-y)^{1/2}.$$

Proof. This follows directly from Corollary 4.4 and Proposition 4.5.

Corollary 4.6 appears in Jackson and Aleliunas [6], but is included here to demonstrate the specialisation from sequences to permutations. Analogues of

Proposition 4.5 for other combinatorial situations may also be obtained. The details of these are given in Jackson and Goulden [7].

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